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Nail H. Ibragimov

Transformation Groups and Lie Algebras

变换群和李代数

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Preface

The term *transformation group* refers to the following properties of a collection G of invertible transformations $\bar{x} = T(x)$ of certain objects x:

1°. G contains the identity transformation I.
2°. G contains the inverse T^{-1} of any $T \in G$.
3°. G contains the product T_2T_1 of any $T_1, T_2 \in G$.

Note that the identity transformation I is defined by the equation $I(x) = x$. The product T_2T_1 is defined as a successive action of T_1 and T_2, i.e.

$$\big(T_2T_1\big)(x) = T_2\big(T_1(x)\big).$$

Finally, the inverse T^{-1} is defined by the equations $T^{-1}T = TT^{-1} = I$.

The group property of G is closely connected with the *invariance* of sets of the objects x under the transformations $T \in G$. We can formulate the statement in the following form.

Proposition. Let S be a set of objects x and G be the collection of all invertible transformations T defined on S and mapping any $x \in S$ into $T(x) = \bar{x} \in S$. Then G is a group.

Proof. Let us verify that the group properties $1° - 3°$ hold. The validity of the property 1° is obvious because $x \in S$ implies $I(x) = x \in S$. Hence, $I \in G$. Furthermore, $T(x) = \bar{x} \in S$ implies that $T^{-1}(\bar{x}) = x \in S$, and hence $T^{-1} \in G$, i.e. the property 2° is also satisfied. Finally, to verify the property 3°, we note that if $T_1, T_2 \in G$, then the action $T_2\big(T_1(x)\big)$ is defined because $T_1(x) \in S$, and $T_2\big(T_1(x)\big) \in S$ because T_2 maps any element of S into an element of S. Hence, $T_1, T_2 \in G$. This completes the proof.

In particular, if x denotes a solution of a given differential equation $F = 0$ and S is the totality of the solutions of $F = 0$, then the above statement shows that the collection of all transformations mapping any solution of $F = 0$ into a solution of the same differential equation compose a group. It is called the *group admitted by the differential equation,* or the *symmetry group* of the equation in question.

Part I of these notes introduces the reader to the basic concepts of the classical theory of local transformation groups and their Lie algebras. It has been designed for the graduate course on *Transformation groups and Lie algebras* that I have been teaching at Blekinge Institute of Technology, Karlskrona, Sweden, since 2002. The

aim of this course was to augment a preliminary knowledge on symmetries of differential equations obtained by students during the course *Differential equations* based on my book [17], *A practical course in differential equations and mathematical modelling*.

Part II of these notes provides an easy to follow introduction to the new topic. It is based on my talks at various conferences, in particular on the plenary lecture at the International Workshop on "Differential equations and chaos" (University of Witwatersrand, Johannesburg, South Africa, January 1996). The final form of the presentation of this material, used in the present book, was prepared for my lectures "Approximate transformation groups" delivered for MSc students at Blekinge Institute of Technology since 2009.

Each part of the book contains an Assignment provided by detailed solutions of all problems. I hope that these assignments will be useful both for students and teachers.

Nail H. Ibragimov

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Part I
Local Transformation Groups

Calculations show that groups admitted by differential equations involve one or more parameters and depend continuously on these parameters. This circumstance led Lie to the concept of *continuous transformation groups.* Multi-parameter continuous transformation groups are composed by *one-parameter groups* depending on a single continuous parameter. Each one-parameter group is determined by its *infinitesimal transformation* or the corresponding first-order linear differential operator termed the *generator* of the one-parameter group. One-parameter transformation groups and their generators are connected by means of the so-called *Lie equations.* Since the existence of solutions of the Lie equations is guaranteed, in general, only for values of the group parameter in a small neighborhood of its initial value, one arrives at what is called *local groups* of continuous transformations.

The generators of multi-parameter transformation groups form specific linear spaces known as *Lie algebras.* Description of continuous transformation groups in terms of their Lie algebras simplifies the calculation and use of groups admitted by differential equations significantly. Namely, the generators of continuous groups admitted by a given differential equation are defined by solving an over-determined system of linear differential equations known as *determining equations.* The characteristic property of determining equations is that *the totality of their solutions spans a Lie algebra.*

Due to the fundamental role of one-parameter groups in Lie's theory of continuous groups, it is natural to begin the study of the general theory of transformation groups and symmetries of differential equations by considering one-parameter groups and their generators.

Chapter 1
Preliminaries

This chapter introduces the reader to a general idea of transformations and exhibits a variety of transformation groups. The duality between changes of frames of reference and point transformations is useful in group analysis. We discuss the idea of the duality in this chapter and will employ it in the next chapter for the prolongation of point transformation groups to derivatives.

1.1 Changes of frames of reference and point transformations

1.1.1 Translations

Consider, in the (x,y) plane, a point P having the coordinates (x,y) in the rectangular Cartesian reference frame with the axes Ox, Oy. Let $e = (e_1, e_2)$ be a fixed unit vector. Consider a new pair of rectangular axes $\overline{O}\bar{x}, \overline{O}\bar{y}$ parallel to the former axes such that $\overline{O}$ has the coordinates $(-ae_1, -be_2)$ with respect to the original frame of reference, where a is an arbitrary real parameter. Then the coordinates $(\bar{x},\bar{y})$ of the point P in the new frame of reference are given by

$$\bar{x} = x + ae_1, \quad \bar{y} = y + be_2. \tag{1.1.1}$$

An alternative interpretation of Eqs. (1.1.1) is as follows. One ignores the new axes $\overline{O}\bar{x}, \overline{O}\bar{y}$ and regards (x,y) and $(\bar{x},\bar{y})$ as the coordinates of points P and $\overline{P}$, respectively, each referred to the original frame Ox, Oy. Then Eqs. (1.1.1) define a transformation of the point $P(x,y)$ into the new position $\overline{P}(\bar{x},\bar{y})$ in the (x,y) plane. Accordingly, equations (1.1.1) determine the displacement (translation) of all points P of the plane through the distance a in the direction of the vector e.

1.1.2 Rotations

Consider again the rectangular Cartesian reference frame with the axes Ox, Oy. Let $O\bar{x}, O\bar{y}$ be the new pair of axes obtained by rotating the original axes round the origin

O counter-clockwise through an angle a. Let (x,y) and $(\bar{x},\bar{y})$ be the coordinates of a point P referred to the axes Ox, Oy and $O\bar{x}, O\bar{y}$, respectively. Then we have

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a. \tag{1.1.2}$$

Indeed, in the polar coordinates (r,θ), connected with the Cartesian coordinates by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1.1.3}$$

the rotation by the angle a about the origin clockwise is written

$$\bar{r} = r, \quad \bar{\theta} = \theta - a. \tag{1.1.4}$$

Equations (1.1.3), (1.1.4) yield the following transformation:

$$\bar{x} = \bar{r}\cos\bar{\theta} = r\cos(\theta - a), \quad \bar{y} = \bar{r}\sin\bar{\theta} = r\sin(\theta - a).$$

Expanding $\cos(\theta - a)$ and $\sin(\theta - a)$ and substituting $r\cos\theta = x$, $r\sin\theta = y$, one arrives at Eqs. (1.1.2).

An alternative interpretation of Eqs. (1.1.2) is as follows. We regard (x,y) and $(\bar{x},\bar{y})$ as the coordinates of the points P and $\overline{P}$, respectively, each referred to the same axes Ox, Oy. Then Eqs. (1.1.2) accomplish the rotation of all points of the plane about O clockwise through the angle a.

1.1.3 Galilean transformation

Everyone travelling by train can observe the duality between uniform motions of his local frame of reference (a train) and outside points (people or other objects on a depot). This remarkable exhibition of the duality, when one cannot determine who is actually moving, is known in the classical mechanics as *Galileo's relativity principle.* It is equivalent to the invariance of equations of motion of mechanical systems under the transformation

$$\bar{t} = t, \quad \overline{\boldsymbol{x}} = \boldsymbol{x} + t\boldsymbol{V}, \tag{1.1.5}$$

where $\boldsymbol{V}$ is the constant velocity. Differentiation of $\overline{\boldsymbol{x}}$ with respect to $\bar{t} = t$ yields

$$\overline{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{V}. \tag{1.1.6}$$

The transformation (1.1.6) of the velocity is a mathematical expression of Galileo's relativity principle. The transformation (1.1.5) is known as the Galilean transformation and lies at the core of the *Galilean group* which is one of the most important groups in non-relativistic physics.

1.2 Introduction of transformation groups

1.2.1 Definitions and examples

We will consider invertible transformations in an n-dimensional Euclidean space $\mathbb{R}^n$ defined, in coordinates, by equations of the form

$$\bar{x}^i = f^i(x), \quad i = 1, \ldots, n, \tag{1.2.1}$$

where the vector-function $f = (f^1, \ldots, f^n)$ is continuous together with its derivatives involved in further discussions. Since the transformation (1.2.1) is invertible, there exists the inverse transformation

$$x^i = (f^{-1})^i(\bar{x}), \quad i = 1, \ldots, n. \tag{1.2.2}$$

Let us denote the transformation (1.2.1) by T and its inverse (1.2.2) by T^{-1}. Thus, T carries any point

$$x = (x^1, \ldots, x^n) \in \mathbb{R}^n$$

into a new position

$$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \in \mathbb{R}^n,$$

and T^{-1} returns $\bar{x}$ into the original position x. It is assumed that the coordinates x^i and $\bar{x}^i$ of points x and $\bar{x}$, respectively, are referred to one and the same coordinate system. The identical transformation

$$\bar{x}^i = x^i, \quad i = 1, \ldots, n, \tag{1.2.3}$$

will be denoted by I.

Let T_1 and T_2 be two transformations of the form (1.2.1) with functions f_1^i and f_2^i, respectively. Their *product* T_2T_1 (termed also *composition* and denoted by $T_2 \circ T_1$) is defined as the consecutive application of these transformations and is given by

$$\bar{\bar{x}}^i = f_2^i(\bar{x}) = f_2^i(f_1(x)), \quad i = 1, \ldots, n. \tag{1.2.4}$$

The geometric interpretation of the product is as follows. Since T_1 carries the point x to the point $\bar{x} = T_1(x)$, which T_2 carries to the new position $\bar{\bar{x}} = T_2(\bar{x})$, the effect of the product T_2T_1 is to carry x directly to its final location $\bar{\bar{x}}$, without a stopover at $\bar{x}$. Thus, equation (1.2.4) means that

$$\bar{\bar{x}} \stackrel{\text{def}}{=} T_2(\bar{x}) = T_2T_1(x). \tag{1.2.5}$$

In this notation, the definition of the inverse transformation (1.2.2) means

$$TT^{-1} = T^{-1}T = I. \tag{1.2.6}$$

Definition 1.1. A set G of transformations (1.2.1) in $\mathbb{R}^n$ containing the identity I is called a transformation group if it contains the inverse T^{-1} of every transformation $T \in G$ and the product T_1T_2 of any transformations $T_1, T_2 \in G$. Thus, the attributes of the group G are:

$$I \in G, \quad \text{and} \quad T^{-1} \in G, \quad T_1T_2 \in G \quad \text{whenever } T, T_1, T_2 \in G. \tag{1.2.7}$$

Example 1.1. The set $G = \{I, T_1, \ldots, T_5\}$ of the transformations

$$\begin{aligned} &I : \bar{x} = x, \qquad T_1 : \bar{x} = 1 - x, \qquad T_2 : \bar{x} = \frac{1}{x}, \\ &T_3 : \bar{x} = \frac{1}{1-x}, \quad T_4 : \bar{x} = \frac{x}{x-1}, \quad T_5 : \bar{x} = \frac{x-1}{x} \end{aligned} \tag{1.2.8}$$

on the straight line is a group containing six elements (see, e.g., [6], §9). The group properties (1.2.7) can be verified by computing the inverses and products of the transformations (1.2.8), e.g.

$$\begin{aligned} &T_1^{-1} = T_1, \; T_2^{-1} = T_2, \; T_3^{-1} = T_5, \; T_4^{-1} = T_4, \; T_5^{-1} = T_3, \\ &T_1^2 = I, \quad T_2^2 = I, \quad T_3^2 = T_5, \quad T_4^2 = I, \quad T_5^2 = T_3, \\ &T_2T_1 = T_3, \quad T_1T_2 = T_5, \quad T_3T_1 = T_2, \quad T_1T_3 = T_4. \end{aligned} \tag{1.2.9}$$

Example 1.2. Consider the set G of all translations (displacements) T_a :

$$\bar{x} = x + a \tag{1.2.10}$$

on the straight line. Since $\bar{x} = x$ when $a = 0$, the set G contains the identity $I = T_0$. Furthermore, the combined effect of two translations, T_a and T_b, acting in succession, is to displace x through the distance $a + b$. Hence,

$$T_bT_a = T_{a+b}. \tag{1.2.11}$$

Equation (1.2.11) shows that

$$T_a^{-1} = T_{-a}.$$

Thus, the transformations (1.2.10) obey the group properties (1.2.7), and hence define a *one-parameter group* G, i.e. a group containing one arbitrary parameter a. This group is known as the *translation group* and provides one of the simplest illustrations to the following definition.

Definition 1.2. A set G of transformations T_a in $\mathbb{R}^n$ depending continuously on a parameter a, where a ranges over all real numbers from a given interval $U \subset R$, is called a *one-parameter group* if there is a unique value $a = a_0$ in U providing the identical transformation, $T_{a_0} = I$, and the following conditions hold for all $a, b \in U$:

$$T_a^{-1} = T_{a^{-1}} \in G, \quad T_bT_a = T_c \in G, \tag{1.2.12}$$

where $a^{-1}, c \in U$ and $c = \phi(a,b)$ is a continuous function.

Definition 1.3. A group containing a finite number of parameters and depending continuously on these parameters was termed by Lie a *finite continuous group*. A continuous group which depends on $r < \infty$ essential parameters is called an r-parameter group and is denoted by G_r.

Remark 1.1. A continuous group of a different type is provided by invertible transformations $\bar{x} = f(x)$ of a straight line. The set of all these transformations, where $f(x)$ ranges over all continuously differentiable functions satisfying the invertibility condition $f'(x) \neq 0$, is a group. Continuous groups involving arbitrary functions are called *infinite continuous groups*. Thus, $\bar{x} = f(x)$ is an example of an infinite continuous group involving one arbitrary function $f(x)$.

Example 1.3. The set G of the transformations

$$\bar{x} = a_1 + a_2 x, \quad a_2 \neq 0 \tag{1.2.13}$$

provides an example of a *two-parameter group*, i.e. a group involving two parameters, a_1, a_2. This group is known as the *general linear group*.

Let us verify that the group properties (1.2.7) are satisfied. We will introduce the vector valued parameter

$$a = (a_1, a_2)$$

and denote the transformation (1.2.13) by T_a. Thus,

$$T_a: \quad \bar{x} = a_1 + a_2 x. \tag{1.2.14}$$

The identity transformation is obtained by letting

$$a_1 = 0, \quad a_2 = 1. \tag{1.2.15}$$

Furthermore, the transformation T_b with the parameter $b = (b_1, b_2)$ maps the point $\bar{x}$ into $\bar{\bar{x}}$ defined as follows:

$$T_b: \quad \bar{\bar{x}} = b_1 + b_2 \bar{x} = b_1 + b_2(a_1 + a_2 x). \tag{1.2.16}$$

Hence, the combined effect of the transformations T_a and T_b acting in succession is

$$T_b T_a: \quad \bar{\bar{x}} = b_1 + b_2 a_1 + b_2 a_2 x = c_1 + c_2 x.$$

It means that

$$T_b T_a = T_c, \tag{1.2.17}$$

where $c = (c_1, c_2)$ is the vector valued parameter with the components

$$c_1 = b_1 + b_2 a_1, \quad c_2 = b_2 a_2. \tag{1.2.18}$$

Let us find the inverse transformation to (1.2.14). Comparing Eqs. (1.2.6), (1.2.17) and using Eqs. (1.2.15), (1.2.18), we see that the inverse transformation $T_b = T_a^{-1}$

to T_a is found by solving the equations

$$b_1 + b_2 a_1 = 0, \quad b_2 a_2 = 1,$$

whence

$$b_2 = \frac{1}{a_2}, \quad b_1 = -\frac{a_1}{a_2}.$$

One can verify by substituting these values of b_1, b_2 in Eqs. (1.2.15) that one has indeed $\bar{\bar{x}} = x$. Thus, $T_a^{-1} = T_{a^{-1}}$, where a^{-1} is the vector valued parameter

$$a^{-1} = \left(-\frac{a_1}{a_2}, \quad \frac{1}{a_2}\right).$$

Definition 1.4. Let G be a transformation group. Its subset $H \subset G$ is called a subgroup of G, if H possesses all group properties (1.2.7), i.e. $I \in H$ and $T^{-1} \in H$, $T_1 T_2 \in H$ whenever $T, T_1, T_2 \in H$.

Example 1.4. The set

$$H = \{I, T_1\}$$

is a subgroup of the group $G = \{I, T_1, \ldots, T_5\}$ of the transformations (1.2.8). Indeed, H contains the identity transformation I. Furthermore,

$$T_1^{-1}, T_1^2 \in H$$

because $T_1^2 = I$, and hence $T_1^{-1} = T_1$.

Example 1.5. The two-parameter group (1.2.13) has two one-parameter subgroups, namely, the translation group (1.2.10) with $a = a_1$,

$$\bar{x} = x + a,$$

and the dilation group

$$T_a: \quad \bar{x} = ax, \quad a \neq 0, \tag{1.2.19}$$

with the parameter $a = a_2$. The group properties for the dilation (1.2.19) can be examined as above. Namely, one can readily verify that the multiplication of the dilations T_a and T_b yields (compare with Eq. (1.2.11))

$$T_b T_a = T_{ab}. \tag{1.2.20}$$

Equation (1.2.20) shows that the inverse transformation to T_a is the dilation with the parameter $a^{-1} = 1/a$:

$$T_a^{-1} = T_{a^{-1}}.$$

The identity transformation is obtained by letting $a = 1$.

Definition 1.5. Two transformation groups are said to be *similar* if one can be obtained from another by an appropriate change of the variables x^i.

Example 1.6. The set G of transformations

$$\bar{x} = x + \alpha_1 - \ln(1 + \alpha_2 e^x)$$

involving two parameters, α_1, α_2, where $|\alpha_2|$ is sufficiently small, defines a two-parameter group ([14], Section 6.3.1). Indeed, let us rewrite the above transformations in the form

$$\bar{x} = -\ln\left[e^{-\alpha_1}\left(e^{-x} + \alpha_2\right)\right] \tag{1.2.21}$$

and denote by T_α and T_β two transformations (1.2.21) with the vector parameters $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, respectively. Thus,

$$T_\alpha : \bar{x} = -\ln\left[e^{-\alpha_1}\left(e^{-x} + \alpha_2\right)\right],$$

$$T_\beta : \bar{x} = -\ln\left[e^{-\beta_1}\left(e^{-x} + \beta_2\right)\right].$$

It is clear from (1.2.21) that G contains the identical transformation and that it obtained by setting

$$\alpha_1 = \alpha_2 = 0. \tag{1.2.22}$$

Furthermore, we have

$$T_\beta T_\alpha : \bar{\bar{x}} = -\ln\left[e^{-\beta_1}\left(e^{-\bar{x}} + \beta_2\right)\right]$$

$$= -\ln\left[e^{-\beta_1}\left(e^{\ln[(e^{-x}+\alpha_2)e^{-\alpha_1}]} + \beta_2\right)\right].$$

The result can be written

$$\bar{\bar{x}} = -\ln\left[e^{-(\alpha_1+\beta_1)}\left(e^{-x} + \alpha_2 + \beta_2 e^{\alpha_1}\right)\right] = -\ln\left[e^{-\gamma_1}\left(e^{-x} + \gamma_2\right)\right],$$

where

$$\gamma_1 = \alpha_1 + \beta_1, \quad \gamma_2 = \alpha_2 + \beta_2 e^{\alpha_1}. \tag{1.2.23}$$

Hence

$$T_\beta T_\alpha = T_\gamma \in G, \tag{1.2.24}$$

where $\gamma = (\gamma_1, \gamma_2)$ is defined by Eqs. (1.2.23). Equations (1.2.23) and (1.2.24) show that the parameter β of the inverse transformation $T_\beta = T_\alpha^{-1}$ to T_α is obtained from Eqs. (1.2.23) by letting $\gamma_1 = \gamma_2 = 0$. Hence,

$$T_\alpha^{-1} = T_{\alpha^{-1}} \in G, \quad \text{where} \quad \alpha^{-1} = (-\alpha_1, -\alpha_2 e^{-\alpha_1}). \tag{1.2.25}$$

Thus, G is a group. It is similar to the general linear group (1.2.13). Namely, after the substitution $y = e^{-x}$ the transformation (1.2.21) is written in the form (1.2.13):

$$\bar{y} = a_1 + a_2 y,$$

where

$$a_1 = \alpha_2 e^{-\alpha_1}, \quad a_2 = e^{-\alpha_1}.$$

1.2.2 *Different types of groups*

Continuous, discontinuous and mixed groups

Definition 1.6. A group G of transformations in $\mathbb{R}^n$ is said to be *continuous,* if any two transformations $T_1, T_2 \in G$ can be connected via a continuous set of elements within the group. In other words, T_1 can be continuously deformed into T_2 within the group G.

One-parameter groups G (Definition 1.2) are manifestly continuous in the above sense. It means geometrically that any point $x \in \mathbb{R}^n$ is carried by the group transformations into the points $\bar{x} = T_a(x)$ whose locus is a continuous curve (passing through x) and is called a *path curve* of the group G. The group property means that *any point of a path curve is carried by G into points of the same curve.* The locus of the images $T_a(x)$ is also termed the *G-orbit* of the point x and denoted by $G(x)$.

In the case of continuous r-parameter groups G_r (Definition 1.3), orbits $G_r(x)$ are continuous r-dimensional manifolds.

Definition 1.7. A group of transformations in $\mathbb{R}^n$ is said to be *discontinuous* if it contains no transformations whose effects on $x \in \mathbb{R}^n$ differ infinitesimally.

Example 1.7. The group of transformations (1.2.8) from Example 1.1 is discontinuous.

There are also groups that are neither continuous nor discontinuous (see [14], Section 6.3.4). An example of such a *mixed group* is offered by symmetry transformations of the rectangular coordinate axes on the plane, i.e. by the following transformations leaving invariant the equation $xy = 0$:

$$T: \bar{x} = f(x), \quad \bar{y} = g(y); \qquad R: \bar{x} = p(y), \quad \bar{y} = q(x), \tag{1.2.26}$$

where $f(x)$, $g(y)$, $p(y)$, and $q(x)$ are arbitrary functions such that

$$f(0) = g(0) = p(0) = q(0) = 0,$$
$$f'(0) = g'(0) = p'(0) = q'(0) = 1.$$

The set G of all transformations (1.2.26) is a group. The group G is infinite since it involves arbitrary functions. But it is not continuous because transformations of type T cannot be continuously deformed into transformations of type R, and vice versa. However, G contains an infinite continuous subgroup composed of the transformations (1.2.26) of type T.

Global and local groups

Example 1.8. Consider transformation T_a :

$$\bar{x} = \frac{x}{1 - ax} \tag{1.2.27}$$

known as a *projective transformation*. It depends on the continuous parameter a and defines a one-parameter set of transformations G.

We have $I \in G$ when $a = 0$. The reckoning shows that the consecutive application of two transformations (1.2.27) T_a, T_b yields

$$\bar{\bar{x}} = \frac{x}{1 - (a+b)x}. \tag{1.2.28}$$

It follows from (1.2.28) that

$$T_a^{-1} = T_{-a} \in G$$

and

$$T_b T_a = T_{a+b} \in G.$$

One naturally may conclude that G is a group. However, this conclusion presumes that the following expressions do not vanish:

$$1 - ax, \quad 1 - bx, \quad 1 - (a+b)x.$$

This simple observation is of fundamental significance for the theory of Lie group analysis.

To clarify the situation, let us consider a fixed point $x_0 > 0$ and move along the path curve through x_0 when the parameter a ranges over all real numbers from a certain interval $0 \le a \le \varepsilon$. Two transformations (1.2.27), T_a and T_b, are well defined in the interval

$$[0,\ 1/x_0),$$

i.e. $1 - ax_0$ and $1 - bx_0$ do not vanish in this interval, but their consecutive application may give the prohibited value

$$a + b = 1/x_0$$

when the product (1.2.28) is not determined. This is the case, e.g. for $a = 1/(3x_0)$ and $b = 2/(3x_0)$. Therefore, let us assume that a and b are taken from a closer vicinity of $a = 0$, e.g. from the interval

$$[0,\ 1/(2x_0)).$$

Then

$$a + b < 1/x_0$$

and the product (1.2.28) is well defined. However, a further multiplication may again result in an unacceptable value of the parameter. For example, when

$$a = b = 1/(3x_0)$$

one has

$$a + b = 2/(3x_0).$$

Hence

$$T_a T_b = T_{a+b}$$

is determined, but

$$T_a T_{a+b} = T_{2a+b}$$

is not. It can be readily seen that iterated multiplication of the projective transformations (1.2.27) inevitably leads to the prohibited value of the group parameter. One cannot solve the problem by merely fixing a "tiny" interval $0 \leq a \leq \varepsilon$, the true nature of the problem being that the orbit of any point x (excluding the isolated point $x = 0$) has a singularity when $a = 1/x$.

The projective group (1.2.27) provides an example of what is called a *local group,* meaning that the composition is defined only for transformations that are sufficiently close to the identity. The vicinity of the identical transformation, where the composition is determined, may depend upon a transformed point x. An alternative definition of a local group G is that path curves of G have singularities.

Example 1.9. *The rotation group* is given by Eqs. (1.1.2):

$$\begin{aligned} \bar{x} &= x\cos a + y\sin a, \\ \bar{y} &= y\cos a - x\sin a, \end{aligned} \tag{1.2.29}$$

and provides another simple example of a local group. Here again the identity transformation corresponds to $a = 0$. The consecutive application of two rotations (1.2.29) through the angles a and b yields:

$$\begin{aligned} \bar{\bar{x}} &= x\cos(a+b) + y\sin(a+b), \\ \bar{\bar{y}} &= y\cos(a+b) - x\sin(a+b). \end{aligned} \tag{1.2.30}$$

Equations (1.2.30) show that the rotation (1.2.29) does not satisfy Definition 1.2 of a one-parameter group. Namely, it does not satisfy the requirement on existence of an interval $U \subset R$ such that there is a unique value $a = a_0$ in U providing the identical transformation $T_{a_0} = I$. To illustrate this statement, let us take, e.g. the interval

$$U = \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right),$$

where a_0 providing the identical transformation is unique, namely $a_0 = 0$. But if we perform the consecutive rotations, say by the angles

$$a = \pi \in U, \quad b = \frac{5}{4}\pi \in U,$$

then the parameter

$$a+b=\frac{9}{4}\pi$$

of resulting rotation (1.2.30) will be outside of the interval U. If we extend the initial interval U in order to include the above value of $a+b$, we lose the uniqueness of a_0 because an extended interval will have at least two values, $a_0=0$ and $a_0=2\pi$, providing the identical transformation (1.2.29).

A transformation group is called a *global group* if the composition of any transformations is defined simultaneously at all generic points x. A common representative of a global group is the translation group (1.2.10). Other global groups of a physical significance are provided by the Galilean transformation (1.1.5) with the vector valued group parameter $\boldsymbol{V}$ and the Lorentz transformation

$$\begin{aligned}\bar{x}&=x\cosh a+y\sinh a,\\ \bar{y}&=y\cosh a+x\sinh a,\end{aligned}\tag{1.2.31}$$

considered in Section 1.3.2, Example 1.13, using a different notation.

Remark 1.2. In what follows, we will apply the nomenclature *group* both to global and local groups.

1.3 Some useful groups

1.3.1 Finite continuous groups on the straight line

Example 1.10. *The linear fractional transformation*

$$\bar{x}=\frac{a_1+a_2x}{a_3x+a_4},\quad a_1a_3-a_2a_4\neq 0\tag{1.3.1}$$

forms a group. It involves four arbitrary constants a_i subject to the invertibility condition, $a_1a_3-a_2a_4\neq 0$. But one of these constants can be eliminated by dividing the right-hand side of Eq. (1.3.1) by any non-vanishing coefficient a_i. If we deal with the transformations (1.3.1) in the vicinity of the identical transformation (i.e. near $a_1=a_3=0, a_2=a_4=1$), then $a_4\neq 0$. Therefore dividing by a_4 one arrives at what is called the *projective group* on the straight line:

$$\bar{x}=\frac{a_1+a_2x}{1+a_3x},\quad a_2\neq a_1a_3.\tag{1.3.2}$$

The group of the general linear fractional transformations (1.3.1) contains, as its subgroup, the general linear group (1.2.13) which itself has the translation (1.2.10) and dilation (1.2.19) as its subgroups. The projective group (1.3.2) is a three-

parameter group (Definition 1.3), while Eqs. (1.2.13) and (1.2.19) provide its two- and one-parameter subgroups, respectively.

The following theorem states that the projective group and its subgroups are the only types of finite continuous groups on the straight line.

Theorem 1.1. Any finite continuous group G_r on the straight line contains at most three essential parameters, i.e. $r \leq 3$. The group G_r is similar to the three-parameter projective group (1.3.2) if $r = 3$, to the general linear group (1.2.13) if $r = 2$, and to the translation group (1.2.10) if $r = 1$.

This important result was first published by Lie in 1874. Its detailed proof is provided in [20], § 1, pp. 2–6. One can also find in Chapter 1 of the book [20] Lie's enumeration of continuous groups on the plane which exhibits that the plane, unlike the straight line, contains a large variety of different types of groups. Consider examples of groups on the plane.

1.3.2 Groups on the plane

Example 1.11. The general translation group on the plane is a two-parameter group given by the transformation

$$\bar{x} = x + a_1, \quad \bar{y} = y + a_2. \tag{1.3.3}$$

Two independent translations in the directions of the x axis,

$$\bar{x} = x + a_1, \quad \bar{y} = y,$$

and the y axis,

$$\bar{x} = x, \quad \bar{y} = y + a_2,$$

provide one-parameter subgroups and compose the two-parameter group (1.3.3). Furthermore, the translations (1.1.1) in the direction of any vector $\boldsymbol{e} = (e_1, e_2)$ form a one-parameter subgroup of the group (1.3.3).

Example 1.12. The composition of the rotation (1.2.29) and the translation (1.3.3),

$$\bar{x} = x\cos a + y\sin a + a_1, \quad \bar{y} = y\cos a - x\sin a + a_2, \tag{1.3.4}$$

provides the *group of isometric motions* on the plane. It is the largest continuous group that changes only the location and orientation of geometric figures (triangles, circles, etc.), but not their magnitude and shape. In other words, the isometric motions (1.3.4) transpose geometric figures as rigid bodies. Consequently, the transformation (1.3.4) (and its generalization to three dimensions) is a mainstay of Euclidean geometry and is known as the group of Euclidean motions, or briefly the *Euclidean group.*

Example 1.13. The *Poincaré group* (known also as the *non-homogeneous Lorentz group*) of the special theory of relativity comprises (in the case of one spatial variable x) translations of t and x and the Lorentz transformation:

$$\begin{aligned} \bar{t} &= t\cosh(a/c) + (x/c)\sinh(a/c), \\ \bar{x} &= x\cosh(a/c) + ct\sinh(a/c), \end{aligned} \tag{1.3.5}$$

where a is a group parameter and $c = 2.99793 \times 10^{10}$ cm/s is the velocity of light in vacuum. Introducing the new parameter $V = c\tanh(a/c)$, one can rewrite the transformation (1.3.5) in the form similar to the Galilean transformation (1.1.5):

$$\bar{t} = \frac{t + (xV/c^2)}{\sqrt{1-(V^2/c^2)}}, \quad \bar{x} = \frac{x+Vt}{\sqrt{1-(V^2/c^2)}}. \tag{1.3.6}$$

The physical meaning of the transformation (1.3.6) and of its parameter V can be illustrated by considering the motion of a particle in a direction parallel to the x-axis. It follows from (1.3.6) that if an observer at rest detects the velocity v of a particle, then an observer moving with the velocity V along the x-axis will detect the velocity

$$\bar{v} = \frac{v+V}{1+(vV/c^2)}. \tag{1.3.7}$$

Consequently, the transformation (1.3.6) is known in the physical literature as the Lorentz boost. In the limit of classical mechanics, when $V/c \to 0$, the Lorentz boost takes the form of the Galilean transformation,

$$\bar{t} = t, \quad \bar{x} = x + Vt,$$

and the formula (1.3.7) reduces to Galileo's relativity principle (1.1.6),

$$\bar{v} = v + V.$$

Example 1.14. The general *projective group* on the plane is given by the linear fractional transformation

$$\bar{x} = \frac{a_{11}x + a_{12}y + a_1}{b_1x + b_2y + b_3}, \quad \bar{y} = \frac{a_{21}x + a_{22}y + a_2}{b_1x + b_2y + b_3}. \tag{1.3.8}$$

The identity transformation is obtained by setting

$$a_{11} = a_{22} = b_3 = 1, \quad a_{12} = a_{21} = a_1 = a_2 = b_1 = b_2 = 0. \tag{1.3.9}$$

In the vicinity of these values of the parameters the transformation (1.3.8) is invertible and, upon dividing by $b_3 \neq 0$, defines the eight-parameter group. The main geometric significance of the projective group (1.3.8) is that it maps any straight line on the plane again into a straight line. Let us verify this property. Since the transformation (1.3.8) is invertible, it suffices to show that if the transformed points

$(\bar{x},\bar{y})$ are located on a straight line, the original points (x,y) also lie on a straight line. Thus, let us assume that

$$A\bar{x}+B\bar{y}+C=0.$$

The substitution of (1.3.8) yields

$$kx+ly+m=0,$$

where

$$\begin{aligned} k &= Aa_{11}+Ba_{21}+Cb_1,\\ l &= Aa_{12}+Ba_{22}+Cb_2,\\ m &= Aa_1+Ba_2+Cb_3. \end{aligned}$$

Hence, the image of any straight line is again a straight line.

Remark 1.3. The well-known subgroups of the general projective group are the six-parameter *linear group*:

$$\bar{x}=a_{11}x+a_{12}y+a_1,\quad \bar{y}=a_{21}x+a_{22}y+a_2;$$

the four-parameter *linear homogeneous group*:

$$\bar{x}=a_{11}x+a_{12}y,\quad \bar{y}=a_{21}x+a_{22}y;$$

and the one-parameter *special projective group*:

$$\bar{x}=\frac{x}{1-ax},\quad \bar{y}=\frac{y}{1-ax}. \tag{1.3.10}$$

Example 1.15. The transformation S:

$$\bar{x}=R^2\frac{x}{r^2},\quad \bar{y}=R^2\frac{y}{r^2}, \tag{1.3.11}$$

where $R\neq 0$ is any constant and $r^2=x^2+y^2$, is known as the *inversion* with respect to a circle of radius R centered at O. The point $O=(0,0)$ and the positive number R are called the center and the radius of inversion (1.3.11). It follows from (1.3.11) that $r\bar{r}=R^2$, where $\bar{r}^2=\bar{x}^2+\bar{y}^2$. Therefore the inversion is also known as the *transformation of reciprocal radii*. We have:

$$S^2=1. \tag{1.3.12}$$

It follows that $S^{-1}=S$, and hence $G=\{I,S\}$ is a group containing two elements. Note that one can transpose the center of inversion to any point as well as to choose any positive number (or $R=\infty$) as a radius of inversion.

Proposition 1.1. Inversion (1.3.11) (i) leaves unaltered any straight line passing through the center O of inversion, (ii) maps any circle passing through O into a straight line which does not pass through O, (iii) maps any straight line that does not pass through O into a circle passing through O, (iv) maps any circle not passing through O again into a circle that does not pass through O.

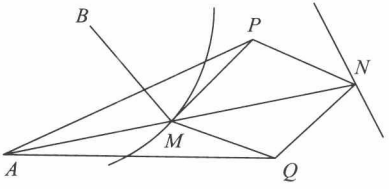

Fig. 1.1 Geometry of inversion.

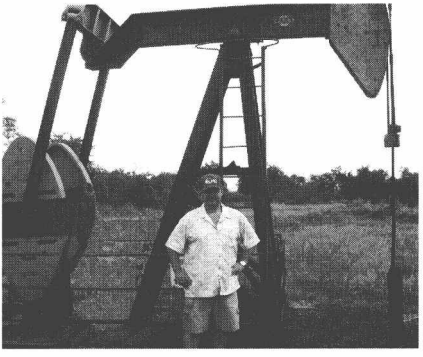

Fig. 1.2 Does the oil recovery pump perform an inversion?

The geometric properties (ii) and (iii) of inversion, formulated in Proposition 1.1, explain the theoretical principles of mechanical constructions for transforming rectilinear motions into circular ones (e.g. in steam engines) and vice versa. Let us dwell on this question by considering the construction given in Figure 1.1 (it is discussed in [14], Note 6.6). One takes seven rigid rods $AP, AQ, BM, PM, PN, QM, QN$ such that

$$AP = AQ, PM = PN = QM = QN.$$

The rods are joined together by hinges as shown in Figure 1.1. The points A and B are fixed so that $AB = BM$. Then elementary geometry shows that AMN is a straight line and that the product $AM \cdot AN$ is constant, namely

$$AM \cdot AN = AP^2 - PM^2.$$

Hence the points M and N are connected by the inversion (1.3.10) with the center A and

$$R^2 = AP^2 - PM^2.$$

Thus, when M moves along the circle with the center B, the point N moves along a straight line. Oil recovery pumps transform circular motions into rectilinear ones. Figure 1.2 invites the reader to discuss if this transformation is inversion.

Remark 1.4. The inversion (1.3.11) is a *conformal transformation* on the plane. Indeed it satisfies the Cauchy–Riemann equations:

$$\frac{\partial \bar{x}}{\partial x}+\frac{\partial \bar{y}}{\partial y}=0, \quad \frac{\partial \bar{x}}{\partial y}-\frac{\partial \bar{y}}{\partial x}=0.$$

Example 1.16. Using Remark 1.4, we can obtain a one-parameter group of conformal transformations as follows. Consider the transformation

$$C_a = S_1 T_a S_1, \tag{1.3.13}$$

where S_1 is the inversion with respect to the circle of unit radius, and T_a is the translation group along the x-axis:

$$\bar{x} = x + a, \quad \bar{y} = y.$$

The combined transformation $S_1 T_a S_1$ acts as follows: S_1 carries (x,y) to the point (x_1,y_1),

$$x_1 = \frac{x}{r^2}, \quad y_1 = \frac{y}{r^2},$$

then T_a carries it to the new position

$$x_2 = x_1 + a = \frac{x+ar^2}{r^2}, \quad y_2 = y_1 = \frac{y}{r^2},$$

after that S_1 brings (x_2,y_2) to the final location $(\bar{x},\bar{y})$ of the initial point (x,y):

$$\bar{x} = \frac{x_2}{r_2^2}, \quad \bar{y} = \frac{y_2}{r_2^2}.$$

Substituting the above values of x_2, y_2 into $r_2^2 = x_2^2 + y_2^2$, one obtains

$$r_2^2 = \frac{(x+ar^2)^2+y^2}{r^4} = \frac{1+2ax+a^2r^2}{r^2}.$$

Hence, the transformation $C_a = S_1 T_a S_1$ has the form:

$$\bar{x} = \frac{x+ar^2}{1+2ax+a^2r^2}, \quad \bar{y} = \frac{y}{1+2ax+a^2r^2}, \tag{1.3.14}$$

where $r^2 = x^2 + y^2$. Let us demonstrate that the set G of transformations (1.3.14) with varying a defines a one-parameter group. Writing (1.3.14) in the form (1.3.13) we have

$$I = C_0 \in G.$$

Furthermore, the composition of two transformations is written

$$C_b C_a = S_1 T_b S_1 S_1 T_a S_1.$$

Invoking the equation $S_1 S_1 = I$ and the group property $T_b T_a = T_{a+b}$ of the translation group, we have

$$C_b C_a = S_1 T_b T_a S_1 = S_1 T_{a+b} S_1 = C_{a+b} \in G.$$

1.3.3 Groups in $\mathbb{R}^n$

Example 1.17. The *Euclidean group* in $\mathbb{R}^n$ comprises n translations

$$\bar{x}^i = x^i + a^i, \quad i = 1, \ldots, n, \tag{1.3.15}$$

and $n(n-1)/2$ rotations in all coordinate planes (x^i, x^j) :

$$\bar{x}^i = x^i \cos a + x^j \sin a, \quad \bar{x}^j = x^j \cos a - x^i \sin a. \tag{1.3.16}$$

Hence, the group of isometric motions in the n-dimensional Euclidean space $\mathbb{R}^n$ contains $r = n(n+1)/2$ essential parameters. For example, solid geometry is based on the six-parameter group G_6 of isometric motions containing three independent rotations and three translations. Continuous and discontinuous subgroups of the six-parameter group of isometric motions in $\mathbb{R}^3$ serve in *crystallography* to characterize symmetries of crystalline substances. Accordingly, group theory furnishes a proper tool for a rigorous mathematical approach to crystallography.

Example 1.18. The general linear group in $\mathbb{R}^n$ has the form

$$\bar{x}^i = a^i_k x^k + a^i, \quad i = 1, \ldots, n,$$

where k is the summation index. A compact form of the above group is

$$\bar{x} = Ax + a, \tag{1.3.17}$$

where x, $\bar{x}$, a are n-dimensional vectors, and $A = (a^i_k)$ is an $n \times n$ non-singular matrix (i.e. $\det A \neq 0$). The general linear group depends on $n(n+1)$ essential parameters and contains the $n(n+1)/2$-parameter Euclidean group as a subgroup.

Example 1.19. The linear homogeneous group is a subgroup of the general linear group. It contains n^2 parameters and has the form:

$$\bar{x} = Ax, \quad \det A \neq 0. \tag{1.3.18}$$

The composition of $\bar{x} = Ax$ and $\bar{\bar{x}} = B\bar{x}$ yields

$$\bar{\bar{x}} = B\bar{x} = BAx,$$

where BA is the product of the matrices A and B. Hence, the composition of transformations (1.3.18) is represented by the usual multiplication of matrices. Furthermore, the inverse to the transformation (1.3.18) is

$$x = A^{-1}\bar{x},$$

where A^{-1} is the inverse matrix. Thus, the linear homogeneous group (1.3.18) can be regarded as the *matrix group* consisting of all non-singular matrices A, where the group composition law is the usual multiplication of matrices. The identity element

of the group is the unit matrix $I = (\delta_k^i)$, where δ_k^i denote the Kronecker symbols: $\delta_k^i = 1$ if $i = k$, and $\delta_k^i = 0$ if $i \neq k$.

Definition 1.8. A *conformal mapping* in $\mathbb{R}^n$ is a transformation preserving the angles, but not necessarily the magnitudes. For example, a conformal mapping transforms a sphere into a sphere, but the latter may have a radius and a center different from those of the original sphere. The set of all conformal mappings is a group called the *conformal group* in $\mathbb{R}^n$. The group of isometric motions is a subgroup of the conformal group.

Example 1.20. Inversion in $\mathbb{R}^3$ with respect to a sphere of radius R centered at $O = (0,0,0)$ is given by

$$\bar{x} = R^2 \frac{x}{r^2}, \quad \bar{y} = R^2 \frac{y}{r^2}, \quad \bar{z} = R^2 \frac{z}{r^2}, \tag{1.3.19}$$

where $r^2 = x^2 + y^2 + z^2$. The radii r and $\bar{r}$ of a point (x,y,z) and of its image $(\bar{x},\bar{y},\bar{z})$ are in the relation of reciprocal radii (cf. Example 1.15), $r\bar{r} = R^2$. The inversion provides an example of a non-trivial (i.e. different from an isometric motion) conformal mapping.

Example 1.21. The continuous conformal group in $\mathbb{R}^3$ is a finite group. It can be obtained as follows. Proceeding as in Example 1.16 and extending it to the y- and z-directions, one arrives at the following three one-parameter groups of conformal mappings:

$$\bar{x} = \frac{x + a_1 r^2}{1 + 2a_1 x + a_1^2 r^2}, \quad \bar{y} = \frac{y}{1 + 2a_1 x + a_1^2 r^2}, \quad \bar{z} = \frac{z}{1 + 2a_1 x + a_1^2 r^2},$$

$$\bar{x} = \frac{x}{1 + 2a_2 y + a_2^2 r^2}, \quad \bar{y} = \frac{y + a_2 r^2}{1 + 2a_2 y + a_2^2 r^2}, \quad \bar{z} = \frac{z}{1 + 2a_2 y + a_2^2 r^2},$$

$$\bar{x} = \frac{x}{1 + 2a_3 z + a_3^2 r^2}, \quad \bar{y} = \frac{y}{1 + 2a_3 z + a_3^2 r^2}, \quad \bar{z} = \frac{z + a_3 r^2}{1 + 2a_3 z + a_3^2 r^2}.$$

J. Liouville obtained in 1847 the following result stating that any non-trivial conformal mapping in $\mathbb{R}^3$ is based on inversion.

Theorem 1.2. The maximal continuous conformal group in $\mathbb{R}^3$ contains 10 essential parameters. It is composed by the six-parameter group of Euclidean motions (i.e. translations and rotations of x,y,z), three one-parameter groups given in Example 1.21 and the one-parameter dilation group $\bar{x} = ax, \bar{y} = ay, \bar{z} = az$.

Remark 1.5. W. Thomson (Lord Kelvin) showed in 1847 (he published his paper in the same volume of *J. Math. Pures et Appl.* as Liouville's paper) that the inversion (1.3.19) leaves invariant the Laplace equation

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = 0$$

provided that the dependent variable u undergoes the transformation

$$\bar{u} = ru.$$

This combined transformation is known in the literature as *Kelvin's transformation* and used in mathematical physics, e.g. in the theory of potentials for constructing Green's function.

Example 1.22. The Galilean group in $\mathbb{R}^n$ is the $(n+1)(n+2)/2$-parameter group comprising the Euclidean group (1.3.15)–(1.3.16), the time translation $\bar{t} = t + b$, and the Galilean transformation (1.1.5) with $V = (b^1, \ldots, b^n)$:

$$\bar{t} = t, \quad \bar{x}^i = x^i + tb^i, \ i = 1, \ldots, n.$$

Example 1.23. The Poincaré group (non-homogeneous Lorentz group) is the group of isometric motions in the Minkowski space-time of special relativity. It is an n-parameter group and comprises the Euclidean group (1.3.15)–(1.3.16), the time translation $\bar{t} = t + b$ and the Lorentz transformation (1.3.5) in n different planes (x^i, t), $i = 1, \ldots, n$, namely:

$$\begin{aligned} \bar{t} &= t \cosh(a/c) + (x^i/c) \sinh(a/c), \\ \bar{x}^i &= x^i \cosh(a/c) + ct \sinh(a/c). \end{aligned} \tag{1.3.20}$$

Each one-parameter group (1.3.20) has its own group parameter a.

Exercises to Chapter 1

Exercise 1.1. Complete the multiplication table (1.2.9), namely, compute

$$T_1T_4, \quad T_4T_1, \quad T_1T_5, \quad T_5T_1, \quad T_2T_3, \quad T_3T_2, \ldots, \quad T_4T_5, \quad T_5T_4$$

and verify that the group properties (1.2.7) are satisfied for all transformations (1.2.8).

Exercise 1.2. Compute the inverses and the products of the following transformations on the straight line:

$$I : \bar{x} = x, \quad T_1 : \bar{x} = -x, \quad T_2 : \bar{x} = \frac{1}{x}, \quad T_2 : \bar{x} = -\frac{1}{x}. \tag{1.3.1}$$

Demonstrate that the set $G = \{I, T_1, T_2, T_3\}$ is a discontinuous group containing four elements.

Exercise 1.3. Show that each of the sets $H_1 = \{I, T_1\}$, $H_2 = \{I, T_2\}$ and $H_3 = \{I, T_3\}$ is a subgroup of the group G from Exercise 1.2.

Exercise 1.4. Prove that the group G from Exercise 1.2 does not have subgroups containing three different elements.

Exercise 1.5. Derive the group property (1.2.28) for the projective transformation (1.2.27),

$$\bar{x} = \frac{x}{1 - ax}.$$

Exercise 1.6. In the polar coordinates (r, θ), the rotation by the angle a about the origin clockwise is naturally written

$$\bar{r} = r, \quad \bar{\theta} = \theta - a.$$

Show that this translation group is written in the rectangular Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

in the form (1.2.29),

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a.$$

Exercise 1.7. Derive the group property (1.2.30) for the rotation (1.2.29).

Exercise 1.8. Demonstrate that the Lorentz transformation (1.2.31),

$$\bar{x} = x\cosh a + y\sinh a, \quad \bar{y} = y\cosh a + x\sinh a,$$

satisfies the group properties (1.2.7) of Definition 1.1. Show that it is a global group.

Exercise 1.9. Read [20], pp. 2–6, and present the proof of Theorem 1.1.

Exercise 1.10. Derive the relativistic formula (1.3.7) for the velocity in moving frame,

$$\bar{v} = \frac{v + V}{1 + (vV/c^2)},$$

by using the representation (1.3.6) of the Lorentz transformation.

Exercise 1.11. Show that the linear fractional transformations

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ay}$$

define a one-parameter group and that this group does not convert an arbitrary straight line into a straight line.

Exercise 1.12. Prove the property (1.3.12) of the inversion (1.3.11).

Chapter 2
One-parameter groups and their invariants

This chapter contains the infinitesimal description of one-parameter local transformation groups, their invariants and invariant equations. The connection between one-parameter groups and their infinitesimal generators is provided by the Lie equations and the exponential map.

2.1 Local groups of transformations

2.1.1 Notation and definition

We will consider invertible transformations T_a in $\mathbb{R}^n$ given by the equation

$$\bar{x} = f(x,a) \tag{2.1.1}$$

with certain functions f depending on a real parameter a. Here $x \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. We assume that the functions f satisfy the condition that (2.1.1) is the identical transformation for a certain value a_0 of the parameter, i.e.

$$f(x,a_0) = x, \tag{2.1.2}$$

and that there are no other values of a in a vicinity of a_0 reducing T_a to the identity.

In coordinates x^i, equations (2.1.1) and (2.1.2) are written:

$$\bar{x}^i = f^i(x,a), \quad f^i(x,a_0) = x^i, \qquad i = 1,\ldots,n. \tag{2.1.3}$$

It is assumed that, given a generic point x, there exists an open interval $U \subset R$ containing a_0 such that the functions $f^i(x,a)$ satisfy the regularity conditions discussed in Section 1.2.1 in a vicinity of x, whenever $a \in U$. Furthermore, it is supposed that a_0 is the only value of $a \in U$ for which Eq. (2.1.1) reduces to the

identical transformation. In short, equation (2.1.1) defines single-valued transformations $T_a: \mathbb{R}^n \to \mathbb{R}^n$ for all a from the interval U.

Transformations occurring in practical applications often satisfy Definition 1.2 of a one-parameter group only when a and b in Eqs. (1.2.12) are restricted to sufficiently small numerical values of the group parameter. Then we have a *local one-parameter group* defined as follows.

Definition 2.1. A set G of transformations T_a in $\mathbb{R}^n$ given by (2.1.3) is called a *one-parameter local group* if there exists a subinterval $U' \subset U$ containing a_0 such that the functions $f^i(x,a)$ satisfy the *composition rule*

$$f^i(f(x,a),b) = f^i(x,c), \quad i = 1,\ldots,n, \tag{2.1.4}$$

for all values $a,b \in U'$. Here $c \in U$ is a smooth function, $c = \phi(a,b)$, of two variables $a,b \in U'$ such that the equation

$$\phi(a,b) = a_0 \tag{2.1.5}$$

has a unique solution b for any $a \in U'$. Given a, the solution b of Eq. (2.1.5) is denoted by a^{-1}. Hence, the inverse transformation T_a^{-1} is given by

$$f^i(\bar{x},a^{-1}) = x^i, \quad i = 1,\ldots,n. \tag{2.1.6}$$

For brevity, local groups are also termed groups (see Examples 1.8, 1.9 and Remark 1.2).

The function

$$c = \phi(a,b) \tag{2.1.7}$$

defined by the composition rule (2.1.4) is termed a *group composition law.* It follows from the *initial conditions* $f^i(x,a_0) = x^i$ and (2.1.4) that

$$\phi(a,a_0) = a, \quad \phi(a_0,b) = b. \tag{2.1.8}$$

According to the above definition, a local group G contains the (unique) identity $I = T_{a_0}$. The composition rule (2.1.4) means that any two transformations $T_a, T_b \in G$ with $a,b \in U'$ carried out one after the other result in a transformation $T_c \in G$, where c belongs to the interval U but may not belong to the subinterval U'. Consequently, one cannot obtain a global group merely by assigning U' as a basic interval instead of U. To draw a parallel between global and local one-parameter groups, equations (2.1.4) and (2.1.6) can be written in the form similar to Eqs. (1.2.12):

$$T_a^{-1} = T_{a^{-1}} \in G, \quad T_b T_a = T_c \in G, \quad \text{where} \quad a,b \in U',\ c \in U.$$

Note that the symbol a^{-1} for the parameter of the inverse transformation (2.1.6) does not signify, in general, $1/a$ (see Eq. (2.1.13) and Exercise 2.1).

2.1.2 Groups written in a canonical parameter

The composition law (2.1.7) depends on a choice of the group parameter (see, e.g. Exercise 2.1). The composition law assumes the simplest form in canonical parameters defined as follows.

Definition 2.2. A transformation group

$$\bar{x}^i = f^i(x,a), \quad i = 1,\dots,n, \tag{2.1.9}$$

where the functions $f^i(x,a)$ are defined in a neighborhood of $a = 0$ and satisfy the conditions

$$f^i(x,0) = x^i, \quad i = 1,\dots,n, \tag{2.1.10}$$

is said to be written in a *canonical* parameter a if the composition rule (2.1.4) has the form

$$f^i(f(x,a),b) = f^i(x,a+b), \quad i = 1,\dots,n. \tag{2.1.11}$$

Here, a and b are numerical values of the group parameter taken from a neighborhood of $a = 0$.

It is manifest that in a canonical parameter the composition law (2.1.7) is written

$$\phi(a,b) = a + b. \tag{2.1.12}$$

It will be shown in Section 2.1.6 that a canonical parameter can be introduced in any local one-parameter group by a suitable choice of the group parameter.

It follows from Eqs. (2.1.11) and (2.1.10) that

$$a^{-1} = -a. \tag{2.1.13}$$

In other words, the inverse transformation (2.1.6) is obtained merely by changing the sign of a :

$$x^i = f^i(\bar{x}, -a), \quad i = 1,\dots,n. \tag{2.1.14}$$

Hence, in a canonical parameter, local groups are defined by the *initial conditions* (2.1.10) and the composition rule (2.1.11).

2.1.3 Infinitesimal transformations and generators

We will assume that a one-parameter group G of transformations (2.1.9) is given in a canonical parameter a and expand the functions $f^i(x,a)$ into the Taylor series in the group parameter a in a neighborhood of $a = 0$. Neglecting the terms of the order $o(a)$ and using the initial conditions (2.1.10), we arrive at the following *infinitesimal transformation* of the group G :

$$\bar{x}^i \approx x^i + a\xi^i(x), \quad i = 1,\dots,n, \tag{2.1.15}$$

where

$$\xi^i(x) = \frac{\partial f^i(x,a)}{\partial a}\Big|_{a=0}. \tag{2.1.16}$$

Geometrically, the infinitesimal transformation (2.1.15)–(2.1.16) defines the tangent vector

$$\xi(x) = (\xi^1(x), \ldots, \xi^n(x))$$

at the point x to the G-orbit of x (see Section 1.2.2). Therefore, ξ is called the *tangent vector field* of the group G. The tangent vector field is often denoted by ξ^i and is written as a first-order linear differential operator[1]

$$X = \xi^i(x)\frac{\partial}{\partial x^i}. \tag{2.1.17}$$

The operator X is called the *infinitesimal generator* of the one-parameter group (2.1.9), or simply the group *generator*.

The following statement shows that the differential operator (2.1.17) behaves, unlike the *vector* ξ^i, as a *scalar* under any change of variables.

Theorem 2.1. Consider a change of variables $\tilde{x} = \varphi(x)$, or in coordinates

$$\tilde{x}^i = \varphi^i(x), \quad i = 1, \ldots, n. \tag{2.1.18}$$

Let $x(\tilde{x}) = \varphi^{-1}(\tilde{x})$ be the inverse to Eq. (2.1.18). The following equation holds for any function $F(x)$:

$$X\big(F(x)\big) = \widetilde{X}\big(F(x(\tilde{x}))\big). \tag{2.1.19}$$

Here $\widetilde{X}$ is the differential operator obtained from the operator (2.1.17) by rewriting it in the new variables as follows:

$$\widetilde{X} = X(\varphi^1)\frac{\partial}{\partial \tilde{x}^1} + \cdots + X(\varphi^n)\frac{\partial}{\partial \tilde{x}^n} \equiv X(\varphi^k)\frac{\partial}{\partial \tilde{x}^k}, \tag{2.1.20}$$

where

$$X(\varphi^k) = \xi^i\frac{\partial \varphi^k(x)}{\partial x^i}.$$

Proof. Using the chain rule for the partial derivatives,

$$\frac{\partial}{\partial x^i} = \sum_{k=1}^{n}\frac{\partial \varphi^k}{\partial x^i}\frac{\partial}{\partial \tilde{x}^k},$$

we have:

$$\begin{aligned} X\big(F(x)\big) &= \xi^i\frac{\partial F(x)}{\partial x^i} = \xi^i\frac{\partial \varphi^k(x)}{\partial x^i}\,\frac{\partial F(x(\tilde{x}))}{\partial \tilde{x}^k} \\ &= X(\varphi^k)\,\frac{\partial F(x(\tilde{x}))}{\partial \tilde{x}^k} = \widetilde{X}\big(F(x(\tilde{x}))\big), \end{aligned} \tag{2.1.21}$$

[1] We use the summation convention, i.e. here $\xi^i(x)\frac{\partial}{\partial x^i} \equiv \sum_{i=1}^{n}\xi^i(x)\frac{\partial}{\partial x^i}$.

where $\widetilde{X}$ is defined by Eq. (2.1.20).

Example 2.1. Consider the rotation group (1.2.29),

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a.$$

Denoting $x^1 = x$, $x^2 = y$, we obtain from Eqs. (2.1.16):

$$\xi^1 = \frac{\partial(x\cos a + y\sin a)}{\partial a}\Big|_{a=0} = (-x\sin a + y\cos a)\big|_{a=0} = y,$$

$$\xi^2 = \frac{\partial(y\cos a - x\sin a)}{\partial a}\Big|_{a=0} = -(y\sin a + x\cos a)\big|_{a=0} = -x.$$

Hence the infinitesimal transformation is written

$$\bar{x} \approx x + ya, \quad \bar{y} \approx y - xa$$

and yields the following generator of the rotation group:

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{2.1.22}$$

2.1.4 Lie equations

The following theorem, due to Lie, asserts that one-parameter local groups are determined by their infinitesimal transformations.

Theorem 2.2. Let G be a local group (2.1.9) with functions f^i obeying (2.1.10) and the composition rule (2.1.11). Let Eq. (2.1.17) be the infinitesimal generator of the group G. Then the functions $\bar{x}^i = f^i(x,a)$ solve the system of first-order ordinary differential equations (known as the *Lie equations*):

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), \quad i = 1,\ldots,n, \tag{2.1.23}$$

with the initial conditions $\bar{x}^i\big|_{a=0} = x^i$, or in the compact form

$$\frac{d\bar{x}}{da} = \xi(\bar{x}), \quad \bar{x}\big|_{a=0} = x. \tag{2.1.24}$$

In other words, the G-orbit of a point $x \in \mathbb{R}^n$ is an integral curve of the Lie equations (2.1.23) passing through the point x.

Conversely, given an infinitesimal transformation (2.1.15), or an operator (2.1.17), where $\xi^i(x)$ are continuously differentiable, the initial value problem (2.1.24) has a unique solution $\bar{x} = f(x,a)$ in a neighborhood of $a = 0$. This solution satisfies the group property (2.1.11). In other words, the solution of Lie's equations provides a one-parameter local group with a given infinitesimal generator (2.1.17).

Proof. To prove the direct statement, let us set $b = \Delta a$ in Eq. (2.1.11):

$$f^i(x, a+\Delta a) = f^i(f(x,a), \Delta a), \quad i = 1, \ldots, n, \tag{2.1.25}$$

then expand both sides of Eq. (2.1.25) in powers of Δa and single out the leading terms:

$$f^i(x, a+\Delta a) \approx f^i(x,a) + \frac{\partial}{\partial a}\left[f^i(x,a)\right] \cdot \Delta a,$$

$$f^i(f(x,a), \Delta a) \approx f^i(x,a) + \frac{\partial}{\partial \Delta a}\left[f^i\Big(f(x,a), \Delta a\Big)\right]_{\Delta a = 0} \cdot \Delta a.$$

Invoking Eq. (2.1.16), one obtains

$$\frac{\partial}{\partial \Delta a}\left[f^i\Big(f(x,a), \Delta a\Big)\right]_{\Delta a = 0} = \xi^i(f(x,a)).$$

Now we substitute the above expressions in Eq. (2.1.25) and obtain the Lie equations (2.1.23):

$$\frac{\partial f^i(x,a)}{\partial a} = \xi^i\big(f(x,a)\big), \quad i = 1, \ldots, n.$$

To prove the second part of the theorem, it is necessary to verify the group property (2.1.11) alone. Indeed, existence and uniqueness of the local solution to the problem (2.1.24) is guaranteed by the classical theorem on existence and uniqueness of the solution to the Cauchy problem. Thus, let us prove that the solution $\bar{x} = f(x,a)$ of Eqs. (2.1.24) satisfies Eq. (2.1.11). Consider, for a fixed value a, the functions $u = (u^1, \ldots, u^n)$ and $v = (v^1, \ldots, v^n)$ of a variable b are defined as follows:

$$u(b) = f(\bar{x}, b) \equiv f(f(x,a), b), \quad v(b) = f(x, a+b).$$

To prove the group property (2.1.11), it suffices to show that $u(b) = v(b)$ in a neighborhood of $b = 0$. Since $f(x,a)$ solves Eqs. (2.1.24), one has:

$$\frac{\mathrm{d}u}{\mathrm{d}b} \equiv \frac{\mathrm{d}f(\bar{x}, b)}{\mathrm{d}b} = \xi(u), \quad u\big|_{b=0} = f(x,a);$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}b} \equiv \frac{\mathrm{d}f(x, a+b)}{\mathrm{d}b} = \xi(v), \quad v\big|_{b=0} = f(x,a).$$

Hence, both $z = u(b)$ and $z = v(b)$ solve the initial value problem

$$\frac{\mathrm{d}z}{\mathrm{d}b} = \xi(z), \quad z\big|_{b=0} = f(x,a).$$

One concludes from the uniqueness theorem that $u(b) = v(b)$ in a neighborhood of $b = 0$, thus completing the proof.

Example 2.2. Let us solve the Lie equations for the operator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Using the notation of Example 2.1, we write Eqs. (2.1.23) in the form

$$\frac{d\bar{x}}{da} = \bar{y}, \quad \frac{d\bar{y}}{da} = -\bar{x}. \tag{2.1.26}$$

The initial conditions are written

$$\bar{x}\big|_{a=0} = x, \quad \bar{y}\big|_{a=0} = y. \tag{2.1.27}$$

A simple way to solve the system (2.1.26) is to reduce it to a single second-order equation. Namely, differentiating the first equation of (2.1.26) and substituting $d\bar{y}/da$ from the second equation of (2.1.26), we obtain

$$\frac{d^2\bar{x}}{da^2} + \bar{x} = 0.$$

The general solution of this second-order equation is well-known (see, e.g. [17], Example 3.3.2) and has the form

$$\bar{x} = C_1 \cos a + C_2 \sin a.$$

Differentiating $\bar{x}$ and invoking the first equation of (2.1.26) we get

$$\bar{y} = C_2 \cos a - C_1 \sin a.$$

Substituting the above expressions for $\bar{x}$ and $\bar{y}$ in Eqs. (2.1.27) for the initial conditions we obtain $C_1 = x$, $C_2 = y$, and hence arrive at the rotation group:

$$\bar{x} = x \cos a + y \sin a, \quad \bar{y} = y \cos a - x \sin a.$$

2.1.5 *Exponential map*

If the coefficients $\xi^i(x)$ of the generator (2.1.17) are analytic functions of $x = (x^1, \ldots, x^n)$, the corresponding group transformation (2.1.9) can be obtained in the form of an infinite series known as the exponential map. This result is given by the following statement.

Theorem 2.3. If $\xi^i(x)$ are analytic functions, the solution to the initial value problem (2.1.24) for the Lie equations (2.1.23) is given by the *exponential map*:

$$\bar{x}^i = e^{aX}(x^i), \quad i = 1, \ldots, n, \tag{2.1.28}$$

where e^{aX} is the operator given by the infinite series

$$e^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \tag{2.1.29}$$

Proof. Let us denote $y^i = \bar{x}^i$ and write Eqs. (2.1.24) in the form

$$\frac{dy^i}{da} = \xi^i(y), \quad y^i|_{t=0} = x^i, \quad i = 1, \ldots, n. \tag{2.1.30}$$

It is known that the solution to a Cauchy problem with analytic data is an analytic function. Therefore we can look for the solution of Eqs. (2.1.30) in the form of an converging Taylor series

$$y^i = x^i + \frac{a}{1!}\left(\frac{dy^i}{da}\right)_0 + \frac{a^2}{2!}\left(\frac{d^2y^i}{da^2}\right)_0 + \cdots + \frac{a^s}{s!}\left(\frac{d^sy^i}{da^s}\right)_0 + \cdots, \tag{2.1.31}$$

where the zero means evaluated at $a = 0$. For any function $h = h(y)$, the chain rule of differentiation and the differential equations (2.1.30) yield

$$\left(\frac{dh}{da}\right)_0 = \left(\frac{\partial h}{\partial y^k}\right)_0 \left(\frac{dy^k}{da}\right)_0 = \xi^k \frac{\partial h}{\partial x^k},$$

or

$$\left(\frac{dh}{da}\right)_0 = X(h),$$

where X is the operator (2.1.17),

$$X = \xi^k(x)\frac{\partial}{\partial x^k}.$$

It is manifest that

$$X(x^i) = \xi^i(x).$$

Therefore the repeated differentiations of Eqs. (2.1.30) yield

$$\left(\frac{dy^i}{da}\right)_0 = \xi^i(x) = X(x^i), \ \left(\frac{d^2y^i}{da^2}\right)_0 = \left(\frac{d\xi^i}{da}\right)_0 = X(\xi^i) = X^2(x^i),$$

and hence

$$\left(\frac{d^sy^i}{da^s}\right)_0 = X^s(x^i), \quad s = 1, 2, 3, \ldots.$$

Accordingly, the development (2.1.31) takes the form

$$y^i = x^i + \frac{a}{1!}X(x^i) + \frac{a^2}{2!}X^2(x^i) + \cdots + \frac{a^s}{s!}X^s(x^i) + \cdots. \tag{2.1.32}$$

This completes the proof of the theorem.

Example 2.3. Let us construct the exponential map for the operator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Using the notation of Example 2.1, we write Eqs. (2.1.28) in the form

$$\begin{aligned}\bar{x} &= e^{aX}(x) = x + \frac{a}{1!}X(x) + \frac{a^2}{2!}X^2(x) + \cdots + \frac{a^s}{s!}X^s(x) + \cdots,\\ \bar{y} &= e^{aX}(y) = y + \frac{a}{1!}X(y) + \frac{a^2}{2!}X^2(y) + \cdots + \frac{a^s}{s!}X^s(y) + \cdots.\end{aligned} \tag{2.1.33}$$

In our case we have:

$$X(x) = y, \quad X(y) = -x.$$

Using these equations, we obtain:

$$X^2(x) = X\big(X(x)\big) = X(y) = -x,$$

and

$$X^2(y) = X\big(X(y)\big) = X(-x) = -y.$$

The iteration gives:

$$\begin{aligned}X^{2s-1}(x) &= (-1)^{s-1}y, \quad X^{2s}(x) = (-1)^s x, \quad s = 1,2,\ldots,\\ X^{2s-1}(y) &= (-1)^s x, \quad X^{2s}(y) = (-1)^s y, \quad s = 1,2,\ldots.\end{aligned}$$

Thus,

$$\begin{aligned}&X(x) = y,\ X^2(x) = -x,\ X^3(x) = -y,\ X^4(x) = x,\ldots,\\ &X(y) = -x,\ X^2(y) = -y,\ X^3(y) = x,\ X^4(y) = y,\ldots.\end{aligned} \tag{2.1.34}$$

Substituting Eq. (2.1.34) in Eq. (2.1.33) we obtain

$$\begin{aligned}\bar{x} &= x\Big(1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \cdots\Big) + y\Big(a - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots\Big),\\ \bar{y} &= y\Big(1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \cdots\Big) - x\Big(a - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots\Big),\end{aligned}$$

and hence arrive again at the rotation group:

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a.$$

Remark 2.1. The exponential map (2.1.28) can be extended to any analytic functions $F(x)$ as follows:

$$F(\bar{x}) = e^{aX}F(x) \equiv \left(1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots\right)F(x). \tag{2.1.35}$$

For the proof, see, e.g. [14], Problem 7.6*.

2.1.6 Determination of a canonical parameter

Recall the definition of a canonical parameter introduced in Section 2.1.2.

Definition 2.3. The group parameter is said to be *canonical* if the composition law (2.1.7) reduces to the addition, $\phi(a,b) = a+b$, i.e. if the group property (2.1.4) is written in the form (2.1.11), $T_b T_a = T_{a+b}$. Furthermore, equation (2.1.10) holds, i.e. $I = T_0$.

Theorem 2.4. In any one-parameter group with an arbitrary composition law (2.1.7), $c = \phi(a,b)$, there exists a canonical parameter $\tilde{a}$. It is defined by

$$\tilde{a} = \int_{a_0}^{a} \frac{\mathrm{d}s}{w(s)}, \quad \text{where} \quad w(s) = \left.\frac{\partial \phi(s,b)}{\partial b}\right|_{b=a_0}. \tag{2.1.36}$$

Proof. Let G be a local one-parameter group with an arbitrary composition law (2.1.7). Let us rewrite Eqs. (2.1.4) in the form

$$f(\bar{x},b) = f(x,c), \quad c = \phi(a,b),$$

and differentiate with respect to b to obtain:

$$\frac{\partial f(\bar{x},b)}{\partial b} = \frac{\partial f(x,c)}{\partial c}\,\frac{\partial \phi(a,b)}{\partial b}.$$

Letting here $b = a_0$ and invoking Eq. (2.1.8), we get

$$\left.\frac{\partial f(\bar{x},b)}{\partial b}\right|_{b=a_0} = \frac{\partial f(x,a)}{\partial a}\left(\frac{\partial \phi(a,b)}{\partial b}\right)_{b=a_0}. \tag{2.1.37}$$

Let us denote the second factor in the right-hand side of Eq. (2.1.37) by $w(a)$. It follows from the second equation of (2.1.8) that $w(a_0) = 1$ and, by continuity, $w(a) \neq 0$ in a neighborhood of $a = a_0$.

If we denote (cf. (2.1.16))

$$\xi(x) = \left.\frac{\partial f(x,a)}{\partial a}\right|_{a=a_0},$$

Equation (2.1.37) will be written

$$\xi(\bar{x}) = w(a)\,\frac{\mathrm{d}\bar{x}}{\mathrm{d}a}.$$

Introducing a new parameter $\tilde{a} = \tilde{a}(a)$ we rewrite this equation in the form

$$\xi(\bar{x}) = w(a)\frac{\mathrm{d}\tilde{a}}{\mathrm{d}a}\frac{\mathrm{d}\bar{x}}{\mathrm{d}a}. \tag{2.1.38}$$

Equation. (2.1.38) coincides with the Lie equation (2.1.23),

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tilde{a}} = \xi(\bar{x}),$$

if we let

$$\frac{\mathrm{d}\tilde{a}}{\mathrm{d}a} = \frac{1}{w(a)}. \tag{2.1.39}$$

Integrating Eq. (2.1.39) and requiring that $\tilde{a} = 0$ when $a = a_0$, we obtain Eq. (2.1.36). Since now Eq. (2.1.38) coincides with the Lie equation (2.1.23), it defines by Theorem 2.2 a one-parameter group with the composition rule (2.1.11) with respect to the parameter $\tilde{a}$. Hence, the composition law (2.1.12) holds:

$$\phi(\tilde{a},\tilde{b}) = \tilde{a}+\tilde{b}.$$

Furthermore, the identical transformation corresponds to $\tilde{a} = 0$. Thus, $\tilde{a}$ defined by Eq. (2.1.36) is a canonical parameter.

Example 2.4. The transformation

$$\bar{x} = x + ax, \quad |a| < 1, \tag{2.1.40}$$

defines a one-parameter group with the composition law

$$\phi(a,b) = a+b+ab$$

and with $a_0 = 0$. In this case, the formula (2.1.36) provides the function $w(s) = 1+s$, and hence the canonical parameter is $\tilde{a} = \int_0^a (1+s)^{-1}\mathrm{d}s$, i.e.

$$\tilde{a} = \ln(1+a).$$

Remark 2.2. Note that the Lie equations written in the form (2.1.23) establish a correspondence between a one-parameter group and its infinitesimal transformation (2.1.15) for groups written in a canonical parameter but cannot be used, in general, if a group parameter is not canonical. Consider, e.g., the one-parameter group from Example 2.4. Here the group transformation (2.1.40) is identical with its infinitesimal transformation, so that $\xi(x) = x$. Therefore Eqs. (2.1.24) are written

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}a} = \bar{x}, \quad \bar{x}\big|_{a=0} = x.$$

The solution of this initial value problem is given by $\bar{x} = xe^a$. It does not coincide with the initial group transformation (2.1.40).

Remark 2.3. In what follows we shall adopt the canonical parameter when referring to one-parameter groups unless otherwise stipulated.

2.2 Invariants

2.2.1 *Definition and infinitesimal test*

Definition 2.4. A function $F(x)$ is called an invariant of a group G of transformations (2.1.9) if $F(\bar{x}) = F(x)$, i.e. if the equation

$$F(f(x,a)) = F(x)$$

is satisfied identically in x and a in a neighborhood of $a = 0$.

Theorem 2.5. A function $F(x)$ is an invariant of the group G with the generator X (2.1.17) if and only if it solves the homogeneous linear partial differential equation

$$X(F) \equiv \xi^i(x)\frac{\partial F(x)}{\partial x^i} = 0. \tag{2.2.1}$$

Proof. Let $F(x)$ be an invariant. Since

$$F(f(x,a)) \approx F(x + a\xi(x)) \approx F(x) + a\,\xi^i(x)\frac{\partial F(x)}{\partial x^i},$$

the invariance condition $F(f(x,a)) = F(x)$ leads to Eq. (2.2.1).

Conversely, let $F(x)$ be any solution of the differential equation (2.2.1). Since Eq. (2.2.1) holds at any point one can consider it at $\bar{x} = f(x,a)$:

$$\xi^i(\bar{x})\frac{\partial F(\bar{x})}{\partial \bar{x}^i} = 0.$$

Hence, invoking the Lie equations (2.1.23) we have

$$\frac{\mathrm{d}F(f(x,a))}{\mathrm{d}a} = \frac{\partial F(\bar{x})}{\partial \bar{x}^i}\frac{\mathrm{d}f^i(x,a)}{\mathrm{d}a} = \xi^i(\bar{x})\frac{\partial F(\bar{x})}{\partial \bar{x}^i} = 0.$$

Since $F(f(x,0)) = F(x)$ (see (2.1.10)), we conclude that

$$u(a) = F(f(x,a))$$

solves the initial value problem

$$\frac{\mathrm{d}u}{\mathrm{d}a} = 0, \quad u\big|_{a=0} = F(x).$$

The solution to the latter problem is unique and is obviously given by $u = F(x)$. Hence, the two solutions, $u = F(f(x,a))$ and $u = F(x)$, are identical for any x. This is precisely the invariance condition, $F(f(x,a)) = F(x)$.

Alternative proof. If $F(x)$ is an analytic function, one can prove the second part of Theorem 2.5 by using the extension (2.1.35) of the exponential map. Indeed, the

equation $X(F) = 0$ implies

$$X^2(F) = X(X(F)) = 0, \ldots, X^s(F) = 0.$$

Consequently, equation (2.1.35) yields $F(\bar{x}) = F(x)$.

Corollary 2.1. If $F(x)$ is an invariant then any function $\Phi(F(x))$ is also an invariant.

Proof. Since $F(x)$ is an invariant, we have $X(F) = 0$. Consequently,

$$X\big(\Phi(F)\big) = \frac{\mathrm{d}\Phi}{\mathrm{d}F} X(F) = 0.$$

Combining the above Theorem 2.5 with Theorem 4.2.1 from [17] we get the following result.

Theorem 2.6. A one-parameter group G of transformations in $\mathbb{R}^n$ has precisely $n-1$ functionally independent invariants. Any set of independent invariants, $\psi_1(x), \ldots, \psi_{n-1}(x)$, is termed a *basis of invariants* of G. Basis is not unique. One can take, as basic invariants, the left-hand sides of $n-1$ first integrals

$$\psi_1(x) = C_1, \ldots, \psi_{n-1}(x) = C_{n-1} \tag{2.2.2}$$

of the *characteristic system* for Eq. (2.2.1):

$$\frac{\mathrm{d}x^1}{\xi^1(x)} = \cdots = \frac{\mathrm{d}x^n}{\xi^n(x)}. \tag{2.2.3}$$

An arbitrary invariant $F(x)$ of G is given by the formula

$$F = \Phi\big(\psi_1(x), \ldots, \psi_{n-1}(x)\big). \tag{2.2.4}$$

Example 2.5. Consider the group of non-uniform dilations in $\mathbb{R}^3$:

$$\bar{x} = x\mathrm{e}^a, \quad \bar{y} = y\mathrm{e}^{2a}, \quad \bar{z} = z\mathrm{e}^{-2a},$$

with the generator

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}.$$

The system (2.2.3) has the form

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{2y} = -\frac{\mathrm{d}z}{2z}.$$

Integrating, e.g. the equations

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{2y}$$

and

$$\frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}z}{2z},$$

one obtains two first integrals, $yx^{-2} = C_1$ and $zx^2 = C_2$. Hence,

$$\psi_1 = yx^{-2}, \quad \psi_2 = zx^2 \tag{2.2.5}$$

provide a basis of invariants so that an arbitrary invariant is

$$F = \Phi(yx^{-2}, zx^2).$$

On the other hand, if one integrates the equations

$$\frac{dy}{y} = -\frac{dz}{z}$$

and

$$\frac{dx}{x} = -\frac{dz}{2z},$$

one obtains another basis of invariants, $\psi_1' = yz$ and $\psi_2' = zx^2$. Then an arbitrary invariant is represented in the form

$$F = \Phi(yz, zx^2).$$

2.2.2 *Canonical variables*

Introduction of canonical variables is one of the basic methods for solving ordinary differential equations with known symmetries.

Recall that in new variables x'^i given by (2.1.18),

$$x'^i = \varphi^i(x), \quad i = 1, \ldots, n, \tag{2.2.6}$$

the operator X (2.1.17) is written in the form (2.1.20):

$$\overline{X} = X(\varphi^i)\frac{\partial}{\partial x'^i}, \tag{2.2.7}$$

where the coefficients

$$X(\varphi^i) = \xi^j(x)\frac{\partial \varphi^i(x)}{\partial x^j}$$

of $\overline{X}$ should be expressed in terms of the new variables $x' = (x'^1, \ldots, x'^n)$ by solving Eqs. (2.2.6) with respect to x^i.

Definition 2.5. *Canonical variables* x'^i for group G of transformations (2.1.9) are defined by the condition that the transformations (2.1.9) are written in these variables as a translation, e.g. in the direction of x'^1 :

$$\bar{x}'^1 = x'^1 + a, \; \bar{x}'^2 = x'^2, \ldots, \; \bar{x}'^n = x'^n. \tag{2.2.8}$$

Theorem 2.7. Canonical variables exist for any one-parameter group.

Proof. Given a group G with the generator X (2.1.17), canonical variables x'^i for G are furnished by the change of variables:

$$x'^1 = \varphi(x),\ x'^2 = \psi_1(x), \ldots,\ x'^n = \psi_{n-1}(x), \tag{2.2.9}$$

where $\varphi(x)$ is any solution to the non-homogeneous linear equation

$$X(\varphi) \equiv \xi^i(x)\frac{\partial \varphi(x)}{\partial x^i} = 1, \tag{2.2.10}$$

and $\psi_1(x), \ldots, \psi_{n-1}(x)$ is a basis of invariants of the group G. Indeed, by definition of basic invariants, the functions $\psi_1(x), \ldots, \psi_{n-1}(x)$ are functionally independent and solve the equation $X(\psi_s) = 0$. Furthermore, $\varphi(x)$ is functionally independent of $\psi_s(x)$, otherwise it would solve the equation $X(\varphi) = 0$ instead of Eq. (2.2.10). Hence, the change of variables (2.2.9) is well defined and, according to Eq. (2.2.7), reduces the operator X to the form

$$\overline{X} = \frac{\partial}{\partial x'^1}.$$

For the above generator $\overline{X}$, the Lie equations (2.1.23) are written

$$\frac{\mathrm{d}\bar{x}'^1}{\mathrm{d}a} = 1, \quad \frac{\mathrm{d}\bar{x}'^s}{\mathrm{d}a} = 0 \quad (s = 2, \ldots, n-1). \tag{2.2.11}$$

Solving these equations and using the initial conditions $\bar{x}'^i\big|_{a=0} = x'^i$, one arrives at the translation group (2.2.8).

Example 2.6. Consider the dilation group with the generator

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}. \tag{2.2.12}$$

Let us take a basis of invariants (2.2.5) from Example 2.5, i.e.

$$\psi_1 = yx^{-2}, \quad \psi_2 = zx^2.$$

One can easily solve Eq. (2.2.10), e.g. by following the procedure given in [17], Section 4.3. Namely, letting $\varphi = \varphi(x)$, one reduces the equation $X(\varphi) = 1$ to

$$x\frac{\mathrm{d}\varphi}{\mathrm{d}x} = 1$$

and ignoring the constant of integration, one obtains $\varphi = \ln|x|$. Hence, the canonical variables (2.2.9) are

$$u = yx^{-2}, \quad v = zx^2, \quad w = \ln|x|. \tag{2.2.13}$$

Example 2.7. Let us find canonical variables for the projective group with the generator

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{2.2.14}$$

The characteristic equation

$$\frac{dx}{x} = \frac{dy}{y}$$

provides the first integral

$$\frac{x}{y} = C,$$

and hence the invariant

$$u = \frac{x}{y}.$$

We easily solve the equation $X(t) = 1$ as in the previous example, i.e. letting $t = t(x)$, and obtain

$$t = -\frac{1}{x}.$$

Thus, we have the canonical variables

$$u = \frac{x}{y}, \quad t = -\frac{1}{x}. \tag{2.2.15}$$

In these variables the operator (2.2.14) is written

$$X = \frac{\partial}{\partial t}$$

and yields the group transformation

$$\bar{t} = t + a, \quad \bar{u} = u. \tag{2.2.16}$$

2.2.3 Construction of groups using canonical variables

Group transformations (2.1.9) with given infinitesimal generators are usually constructed either by solving the Lie equations (2.1.24) or by using the exponential map (2.1.28). In certain complicated cases, the calculations can be simplified by adding the third method based on introducing canonical variables (see [14], Section 7.1.9). Consider the following examples.

Example 2.8. Let us construct the group with the generator

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{2.2.17}$$

We know from Example 2.7 the canonical variables (2.2.15) for the operator X and the group transformation (2.2.16) in the canonical variables. Now we substitute in Eq. (2.2.16) the expressions (2.2.15) for t, u and obtain:

$$-\frac{1}{\bar{x}} = -\frac{1}{x} + a, \quad \frac{\bar{x}}{\bar{y}} = \frac{x}{y}.$$

Solving these equations for $\bar{x}$, $\bar{y}$ we obtain the following transformation group generated by the operator (2.2.17):

$$\bar{x} = \frac{x}{1-ax}, \quad \bar{y} = \frac{y}{1-ax}. \tag{2.2.18}$$

Example 2.9. Let us find the group generated by the following operator:

$$X = (1+t^2)\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} + (r - tv_r)\frac{\partial}{\partial v_r} - tv_\theta\frac{\partial}{\partial v_\theta} - 2t\rho\frac{\partial}{\partial \rho} - 4tp\frac{\partial}{\partial p}. \tag{2.2.19}$$

This group describes a specific symmetry of the so-called shallow-water equations (see [14], Section 7.1.9). Here t is time, r and θ are the polar coordinates on the plane, v_r and v_θ are the components of velocity in the polar coordinates, ρ and p are the density and pressure, respectively.

Solving the partial differential equations (see also Assignment to Part I, Problem 9)

$$X(s) = 1,\ X(\lambda) = 0,\ X(U) = 0,\ X(V) = 0,$$
$$X(R) = 0,\ X(P) = 0$$

we obtain the following canonical variables:

$$\begin{aligned} &s = \arctan t, \quad \lambda = \frac{r}{\sqrt{1+t^2}}, \quad U = rv_r - \frac{tr^2}{1+t^2}, \\ &V = rv_\theta, \quad R = (1+t^2)\rho, \quad P = (1+t^2)^2 p. \end{aligned} \tag{2.2.20}$$

In these variables the operator X becomes $X = \partial/\partial s$ and yields the transformations

$$\bar{s} = s + a, \quad \bar{\lambda} = \lambda, \quad \bar{U} = U, \quad \bar{V} = V, \quad \bar{R} = R, \quad \bar{P} = P.$$

Substituting here the expressions for $s, \ldots, P$ and the similar expressions for the transformed variables,

$$\bar{s} = \arctan \bar{t}, \ \ldots, \ \bar{P} = (1+\bar{t}^2)^2 \bar{p},$$

we obtain the equations

$$\arctan \bar{t} = \arctan t + a, \quad \frac{\bar{r}}{\sqrt{1+\bar{t}^2}} = \frac{r}{\sqrt{1+t^2}},$$

$$\bar{r}\overline{v_r} - \frac{\bar{t}\bar{r}^2}{1+\bar{t}^2} = rv_r - \frac{tr^2}{1+t^2}, \quad \overline{rv_\theta} = rv_\theta,$$

$$(1+\bar{t}^2)\bar{\rho} = (1+t^2)\rho, \quad (1+\bar{t}^2)^2\bar{p} = (1+t^2)^2 p.$$

Solving them for $\bar{t}, \ldots, \bar{p}$, we arrive at the following group transformations:

$$\bar{t} = \frac{\sin a + t\cos a}{\cos a - t\sin a}, \qquad \bar{r} = \frac{r}{\cos a - t\sin a},$$

$$\overline{v_r} = (\cos a - t\sin a)v_r + r\sin a, \quad \overline{v_\theta} = (\cos a - t\sin a)v_\theta,$$

$$\bar{\rho} = (\cos a - t\sin a)^2\rho, \qquad \bar{p} = (\cos a - t\sin a)^4 p.$$

2.2.4 *Frequently used groups in the plane*

Transformations	Generators	Invariants $\psi(x,y)$	Canonical t,u so that $X = \partial/\partial t$
Translations			
along x : $\bar{x} = x+a$, $\bar{y} = y$	$X = \dfrac{\partial}{\partial x}$	$\psi = y$	$t = x$, $u = \psi$
along y : $\bar{x} = x$, $\bar{y} = y+a$	$X = \dfrac{\partial}{\partial y}$	$\psi = x$	$t = y$, $u = \psi$
along $kx+ly = 0$: $\bar{x} = x+la$, $\bar{y} = y-ka$	$X = l\dfrac{\partial}{\partial x} - k\dfrac{\partial}{\partial y}$	$\psi = kx+ly$	$t = x/l, u = \psi$
Rotation $\bar{x} = x\cos a + y\sin a$, $\bar{y} = y\cos a - x\sin a$	$X = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$	$\psi = x^2+y^2$	$t = \arctan(x/y)$, $u = \psi$
Lorentz transformation $\bar{x} = x\cosh a + y\sinh a$, $\bar{y} = y\cosh a + x\sinh a$	$X = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$	$\psi = y^2 - x^2$	$t = \ln\lvert y+x\rvert$, $u = \psi$
Galilean transformation $\bar{x} = x+ay$, $\bar{y} = y$	$X = y\dfrac{\partial}{\partial x}$	$\psi = y$	$t = x/y$, $u = \psi$
Uniform dilation $\bar{x} = xe^a$, $\bar{y} = ye^a$	$X = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$	$\psi = y/x$	$t = \ln\lvert x\rvert$, $u = \psi$
Non-uniform dilation $\bar{x} = xe^a$, $\bar{y} = ye^{ka}$	$X = x\dfrac{\partial}{\partial x} + ky\dfrac{\partial}{\partial y}$	$\psi = y/x^k$	$t = \ln\lvert x\rvert$, $u = \psi$

Continued

Transformations	Generators	Invariants $\psi(x,y)$	Canonical t,u so that $X=\partial/\partial t$
Projective transformations $\bar{x}=\dfrac{x}{1-ax}$, $\bar{y}=\dfrac{y}{1-ax}$,	$X=x^2\dfrac{\partial}{\partial x}+xy\dfrac{\partial}{\partial y}$	$\psi=y/x$	$t=-1/x$, $u=\psi$
$\bar{x}=\dfrac{x}{1-ay}$, $\bar{y}=\dfrac{y}{1-ay}$	$X=xy\dfrac{\partial}{\partial x}+y^2\dfrac{\partial}{\partial y}$	$\psi=y/x$	$t=-1/y$, $u=\psi$
Linear fractional $\bar{x}=\dfrac{x}{1-ax}$, $\bar{y}=\dfrac{y}{1-ay}$	$X=x^2\dfrac{\partial}{\partial x}+y^2\dfrac{\partial}{\partial y}$	$\psi=\dfrac{1}{y}-\dfrac{1}{x}$	$t=-1/x$, $u=\psi$
Conformal transformation $\bar{x}=\dfrac{x+ar^2}{1+2ax+a^2r^2}$, $\bar{y}=\dfrac{y}{1+2ax+a^2r^2}$	$X=(y^2-x^2)\dfrac{\partial}{\partial x}$ $-2xy\dfrac{\partial}{\partial y}$	$\psi=y/r^2$	$t=x/r^2$, $u=\psi$

2.3 Invariant equations

2.3.1 Definition and infinitesimal test

Consider a system of s equations in $\mathbb{R}^n$:

$$F_\sigma(x)=0, \quad \sigma=1,\dots,s, \tag{2.3.1}$$

where $x\in\mathbb{R}^n$ and $s<n$. We impose the condition that the Jacobian matrix of the functions $F_\sigma(x)$ has the rank s,

$$\operatorname{rank}\left(\frac{\partial F_\sigma(x)}{\partial x^i}\right)=s, \tag{2.3.2}$$

at all points x satisfying Eqs. (2.3.1). The locus of solutions x of the system of Eqs. (2.3.1) is an $(n-s)$-dimensional manifold (surface) $M\subset\mathbb{R}^n$.

Definition 2.6. We say that Eqs. (2.3.1) are *invariant* with respect to a group G of transformations $\bar{x}=f(x,a)$, or that Eqs. (2.3.1) *admit* G, if $\bar{x}$ satisfies Eqs. (2.3.1) whenever x solves Eqs. (2.3.1). In other words:

$$F_\sigma(\bar{x})\Big|_{(2.3.1)}=0, \quad \sigma=1,\dots,s. \tag{2.3.3}$$

Geometrically, it means that transformations of the group G carry any point of the $(n-s)$-dimensional surface M along this surface. In other words, the path curve of the group G passing through any point $x\in M$ lies in M. Therefore M is also termed an *invariant manifold* for G.

We will employ further the generalization of the following simple observation. Using the Maclaurin expansion of $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

we can write the function $\sin x$ near $x = 0$ in the form

$$\sin x = xh(x),$$

where

$$h(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

is a bounded function near $x = 0$. It is obvious that this example can be extended to any analytic function. Namely, if $f(x)$ is an analytic function near $x = 0$ and vanishes at $x = 0$ then its Maclaurin expansion has the form

$$f(x) = f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

and hence

$$f(x) = xh(x), \tag{2.3.4}$$

where

$$h(x) = f'(0) + \frac{f''(0)}{2!}x + \frac{f'''(0)}{3!}x^2 + \cdots.$$

Lemma 2.1. Let $g(x)$ be an analytic function such that $g'(x) \neq 0$ at points x satisfying the equation $g(x) = 0$. If $f(y)$ is an analytic function satisfying the condition $f(0) = 0$, then there exists a regular function $h(x)$ such that

$$f(g(x)) = h(x)g(x). \tag{2.3.5}$$

Proof. Using the conditions for the function $g(x)$ we introduce the new variable $y = g(x)$ and write Eq. (2.3.4), $f(y) = y\tilde{h}(y)$, or

$$f(g(x)) = g(x)\tilde{h}(g(x)).$$

Denoting here $\tilde{h}(g(x)) = h(x)$ we arrive at Eq. (2.3.5).

We now return to invariant equations. Let G be a one-parameter group with a generator X (2.1.17). The *infinitesimal test* for a system of Eqs. (2.3.1) to be invariant under the group G is given by the following theorem.

Theorem 2.8. The system of Eqs. (2.3.1) is invariant under the group G with the infinitesimal generator X if and only if

$$XF_\sigma(x)\Big|_{(2.3.1)} = 0, \qquad \sigma = 1, \ldots, s, \tag{2.3.6}$$

where the symbol $\big|_{(2.3.1)}$ means evaluated on the manifold M (2.3.1).

Proof. Let the system of Eqs. (2.3.1) be invariant. One readily arrives at Eqs. (2.3.6) by substituting in Eq. (2.3.3) the infinitesimal expression for $F_\sigma(\bar{x})$:

$$F_\sigma(\bar{x}) \approx F_\sigma(x) + aXF_\sigma(x).$$

Conversely, let us suppose that Eqs. (2.3.6) hold and derive the invariance conditions (2.3.3). We will assume that $F_\sigma(x)$ and $XF_\sigma(x)$ are analytic in a neighborhood of the manifold M (2.3.1). Using a multidimensional version of Lemma 2.1, we will write Eqs. (2.3.6) in the form

$$XF_\sigma(x) = \lambda_\sigma^\nu(x)F_\nu(x), \quad \sigma = 1,\ldots,s, \tag{2.3.7}$$

where the coefficients $\lambda_\sigma^\nu(x)$ are bounded in a neighborhood of the variety M. It follows from Eqs. (2.3.7) that

$$X^2F_\sigma = X(\lambda_\sigma^\nu)F_\nu + \lambda_\sigma^\nu X(F_\nu) = \left(X(\lambda_\sigma^\mu) + \lambda_\sigma^\nu\lambda_\nu^\mu\right)F_\mu.$$

Iteration and substitution into the exponential map (2.1.35) yields

$$F_\sigma(\bar{x}) = \Lambda_\sigma^\nu(x)F_\nu(x). \tag{2.3.8}$$

Equations (2.3.8) represent an equivalent form of Eqs. (2.3.3).

Remark 2.4. If one rewrites Eqs. (2.3.6) in the form

$$(\xi\cdot\nabla F_\sigma)\big|_M = 0,$$

it becomes evident that Eqs. (2.3.6) is the condition for the vector field

$$\xi = (\xi^1,\ldots,\xi^n)$$

to be tangent to the manifold M.

2.3.2 Invariant representation of invariant manifolds

Given a group G, invariant manifolds for G can be equivalently represented by many different systems of Eqs. (2.3.1). To develop a general procedure for constructing invariant manifolds, it is advantageous to distinguish *singular* and *regular* (non-singular) invariant manifolds.

Definition 2.7. Let M be an invariant manifold for a group G. Since G transports points $x \in M$ along M, the action of the group G confined to M is also a group. It is termed a group induced on M by G, or briefly an *induced group*. The induced group, its generator (termed an *induced generator*) and invariants will be denoted $\widetilde{G}$, $\widetilde{X}$ and $\widetilde{\psi}$, respectively.

Definition 2.8. Let G be a one-parameter group with the generator (2.1.17),

$$X = \xi^i(x)\frac{\partial}{\partial x^i}\cdot$$

An invariant manifold M for G is said to be *regular* (with respect to G) if the induced generator $\widetilde{X}$ does not vanish, and *singular* if $\widetilde{X} = 0$. In other words, M is a regular invariant manifold if at least one of the coefficients $\xi^i(x)$ does not vanish on M, and it is singular if all coefficients $\xi^i(x)$ of the generator X vanish identically on M. Since manifolds are considered locally near generic points and since all functions under consideration are supposed to be continuous, these two possibilities cover all cases.

Example 2.10. Consider the group G of transformations $\bar{x} = xe^a, \bar{y} = y$ with the generator $X = x\partial/\partial x$. The equation $x = 0$ is invariant under G, and hence the y-axis is an invariant manifold for G. It is singular since $X = 0$ when $x = 0$.

Theorem 2.9. Let a system of Eqs. (2.3.1) admit a group G. Let the invariant manifold M defined by the system of Eqs. (2.3.1) be regular with respect to G. Then M can be represented by a system whose equations are given by invariant functions with respect to G, i.e. in the form (cf. formula (2.2.4)):

$$\Phi_\sigma(\psi_1(x),\dots,\psi_{n-1}(x)) = 0, \quad \sigma = 1,\dots,s, \tag{2.3.9}$$

where

$$\psi_1(x),\dots,\psi_{n-1}(x) \tag{2.3.10}$$

is a basis of invariants of G. Hence, equations (2.3.9) with arbitrary functions Φ_σ of $n-1$ variables furnish the general form of regular invariant manifolds for G.

2.3.3 *Proof of Theorem 2.9*

The proof given here is due to L.V. Ovsyannikov [22], Section 8.7 (see also [14], Section 7.2.2).

Let M be a regular invariant manifold given by Eqs. (2.3.1). Due to the condition (2.3.2), the following change of variables is well defined:

$$x'^1 = F_1(x),\dots,x'^s = F_s(x),\ x'^{s+1} = \varphi_{s+1}(x),\dots,x'^n = \varphi_n(x). \tag{2.3.11}$$

In the new variables the manifold M is given by the equations

$$x'^\sigma = 0, \quad \sigma = 1,\dots,s. \tag{2.3.12}$$

Then the induced group $\widetilde{G}$ acts in the $(n-s)$-dimensional space $\mathbb{R}^{n-s}$ of variables $\vec{x}' = (x'^{s+1},\dots,x'^n)$. Since M is regular, the induced generator $\widetilde{X} \neq 0$. Hence, the induced group has precisely $n-s-1$ independent invariants. In particular,

$$\widetilde{\psi}_1(\vec{x}'),\dots,\widetilde{\psi}_{n-1}(\vec{x}') \tag{2.3.13}$$

obtained from (2.3.10) by restricting them to M, are invariants of $\widetilde{G}$. Some of these $n-1$ invariants will be functionally dependent, since the number of independent invariants is not greater than $n-s-1$. Let $n-s'-1$ of the invariants (2.3.13) be independent. Since $n-s'-1 \leq n-s-1$, there exist $s' \geq s$ relations:

$$\Phi_\sigma\left(\widetilde{\psi}_1(\tilde{x}'),\ldots,\widetilde{\psi}_{n-1}(\tilde{x}')\right)=0, \quad \sigma=1,\ldots,s'. \tag{2.3.14}$$

Assuming that the relations (2.3.14) (i.e. the functions Φ_σ therein) are known, define an $(n-s')$-dimensional manifold $M' \subset \mathbb{R}^n$ by the equations:

$$\Phi_\sigma\left(\psi_1(x),\ldots,\psi_{n-1}(x)\right)=0, \quad \sigma=1,\ldots,s'. \tag{2.3.15}$$

If $x \in M$, equations (2.3.15) reduce to relations (2.3.14). Hence, $M \subset M'$. Consequently, $\dim M' \geq \dim M$, i.e. $n-s' \geq n-s$. It follows that $s' \leq s$. However, $s' \geq s$ by definition of relations (2.3.14). Hence, $s'=s$. It follows from $\dim M' = \dim M$ and $M \subset M'$ that, locally, $M'=M$. Thus, equations (2.3.15) furnish an invariant representation (2.3.9) of the manifold M. This completes the proof of the theorem.

Remark 2.5. When $n=2$, the relation (2.3.14) reduces to

$$\widetilde{\psi}_1(\tilde{x}')=C$$

with an arbitrary constant C.

2.3.4 Examples on Theorem 2.9

Example 2.11. It is shown in Example 2.10 that the y-axis given by the equation $x=0$ is a singular invariant manifold for the group G of dilations $\bar{x}=xe^a, \bar{y}=y$. Accordingly, the y-axis cannot be represented via invariants of the group G. Indeed, the general form of invariants of G is $\psi(y)$, so that the most general equation of the form (2.3.9) provides $y=$ const., i.e. the straight lines parallel to the x-axis.

Example 2.12. Consider, in the (x,y,z) space, the paraboloid given by

$$F(x,y,z) \equiv x^2+y^2-z=0. \tag{2.3.16}$$

Let G be the group of dilations

$$\bar{x}=xe^a, \quad \bar{y}=ye^a, \quad \bar{z}=ze^{2a}$$

with the generator

$$X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2z\frac{\partial}{\partial z}.$$

We have

$$F(\bar{x},\bar{y},\bar{z})=e^{2a}F(x,y,z),$$

or in infinitesimals,

$$XF = 2F.$$

It follows that Eq. (2.3.16), $F = 0$, is invariant under the group G, but the function F is not an invariant of this group.

The generator X vanishes at the point $O = (0,0,0)$. Let us exclude this singular point from the paraboloid (2.3.16), denote by M the resulting regular invariant surface and find its invariant representation (2.3.9). In our example, a basis of invariants (2.3.10) is provided, e.g. by

$$\psi_1 = \frac{x^2}{z}, \quad \psi_2 = \frac{y^2}{z}.$$

Let us take the change of variables (2.3.11) in the form

$$z' = z - x^2 - y^2, \quad x' = x, \quad y' = y.$$

Then the manifold M is given by $z' = 0$, and the induced invariants are written

$$\widetilde{\psi}_1 = \frac{x'^2}{r^2}, \quad \widetilde{\psi}_2 = \frac{y'^2}{r^2},$$

where $r^2 = x'^2 + y'^2$. It is manifest that

$$\widetilde{\psi}_1 + \widetilde{\psi}_2 = 1.$$

This is the relation (2.3.14). Hence, the invariant representation (2.3.9) of the paraboloid (2.3.16) with the origin O excluded has the form

$$\psi_1 + \psi_2 = 1,$$

i.e.

$$\frac{x^2 + y^2}{z} - 1 = 0. \tag{2.3.17}$$

Example 2.13. Consider Newton's equations for Kepler's problem in two dimensions (see Eqs. (2.2.8) in [17]):

$$w^i + \alpha \frac{x^i}{r^3} = 0, \qquad i = 1,2, \tag{2.3.18}$$

where α is a positive constant and

$$w^i = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}.$$

Let G be the group of simultaneous rotations of the vectors $\boldsymbol{x} = (x^1, x^2)$ and $\boldsymbol{w} = (w^1, w^2)$, i.e. the group with the generator

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + w^2 \frac{\partial}{\partial w^1} - w^1 \frac{\partial}{\partial w^2}. \tag{2.3.19}$$

Geometry provides the following four invariants:

$$\psi_1 = |\boldsymbol{x}| \equiv \sqrt{(x^1)^2 + (x^2)^2}, \quad \psi_2 = |\boldsymbol{w}| \equiv \sqrt{(w^1)^2 + (w^2)^2},$$

$$\psi_3 = |\boldsymbol{x} \times \boldsymbol{w}| \equiv |x^1 w^2 - x^2 w^1|, \quad \psi_4 = \boldsymbol{x} \cdot \boldsymbol{w} \equiv x^1 w^1 + x^2 w^2.$$

But only three of them are independent. Indeed, they are connected by the relation

$$\psi_3^2 + \psi_4^2 = \psi_1^2 \psi_2^2.$$

This relation follows from the definition of scalar and vector products,

$$\boldsymbol{x} \cdot \boldsymbol{w} = |\boldsymbol{x}||\boldsymbol{w}| \cos \omega, \quad |\boldsymbol{x} \times \boldsymbol{w}| = |\boldsymbol{x}||\boldsymbol{w}| \sin \omega,$$

where ω is the angle between $\boldsymbol{x}$ and $\boldsymbol{w}$. A basis of invariants is given by

$$\psi_1 = |\boldsymbol{x}|, \quad \psi_2 = |\boldsymbol{w}|, \quad \psi_3 = |\boldsymbol{x} \times \boldsymbol{w}|. \tag{2.3.20}$$

The induced invariants are

$$\widetilde{\psi}_1 = |\boldsymbol{x}|, \quad \widetilde{\psi}_2 = \alpha |\boldsymbol{x}|^{-2}, \quad \widetilde{\psi}_3 = 0.$$

Hence, two relations of the form (2.3.14) are

$$\widetilde{\psi}_1^2 \widetilde{\psi}_2 = \alpha, \quad \widetilde{\psi}_3 = 0.$$

Therefore, the invariant representation (2.3.9) of Newton's equations (2.3.18) can be written in the form

$$\psi_1^2 \psi_2 = \alpha, \quad \psi_3 = 0,$$

or invoking Eqs. (2.3.20), in the form

$$|\boldsymbol{x}|^2 |\boldsymbol{w}| = \alpha, \quad |\boldsymbol{x} \times \boldsymbol{w}| = 0. \tag{2.3.21}$$

Exercises to Chapter 2

Exercise 2.1. Find the group composition law (2.1.7) and the parameter a^{-1} of the inverse transformation (2.1.6) for each of following two representations of the dilation group: (i) $\bar{x} = ax$, (ii) $\bar{x} = \mathrm{e}^a x$.

Exercise 2.2. Find the group composition law (2.1.7) and the infinitesimal generator (2.1.17) for the Lorentz transformation group (1.2.31),

$$\bar{x} = x \cosh a + y \sinh a, \quad \bar{y} = y \cosh a + x \sinh a.$$

Exercise 2.3. Find the group composition law (2.1.7) and the infinitesimal generator (2.1.17) for the *spiral transformation group*

$$\bar{x} = e^a(x\cos a - y\sin a), \quad \bar{y} = e^a(y\cos a + x\sin a).$$

Exercise 2.4. Find the generator (2.1.17) of the conformal group (1.3.14),

$$\bar{x} = \frac{x + ar^2}{1 + 2ax + a^2r^2}, \quad \bar{y} = \frac{y}{1 + 2ax + a^2r^2}, \quad (r^2 = x^2 + y^2).$$

Exercise 2.5. Find the generator of the conformal group in the three-dimensional space (see Example 1.21),

$$\bar{x} = \frac{x + ar^2}{1 + 2ax + a^2r^2}, \quad \bar{y} = \frac{y}{1 + 2ax + a^2r^2},$$

$$\bar{z} = \frac{z}{1 + 2ax + a^2r^2},$$

where $r^2 = x^2 + y^2 + z^2$.

Exercise 2.6. Explain the major difference between global and local groups and provide your own example of a local group.

Exercise 2.7. Construct the transformation group with the generator (2.2.12),

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z},$$

by two methods: solving the Lie equations and using the exponential map. Then verify that u, v, w given by Eqs. (2.2.13),

$$u = yx^{-2}, \quad v = zx^2, \quad w = \ln|x|,$$

are canonical variables for this group.

Exercise 2.8. Construct the transformation group with the generator X from Exercise 2.7 by using the canonical variables u, v, w given there.

Exercise 2.9. In the (t, x, u)-space, find the one-parameter group with the generator

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u},$$

by using all three methods: Lie equations, exponential map and canonical variables. Is this group local or global?

Exercise 2.10. Using the generator of the spiral transformation group found in Exercise 2.3, check if the equation

$$\sqrt{x^2+y^2} = e^{\arctan(y/x)}$$

is invariant with respect to the spiral group.

Exercise 2.11. Recall that the polar coordinates r, θ and the rectangular Cartesian coordinates x, y in the plane are connected by the equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

or

$$r = \sqrt{x^2+y^2}, \quad \theta = \arctan(y/x).$$

Rewrite the generator of the spiral transformation group found in Exercise 2.3 in the polar coordinates r, θ. Using this generator, construct the spiral transformation group in the polar coordinates.

Exercise 2.12. Solve Exercise 2.10 using the polar coordinates.

Exercise 2.13. Provide the calculations leading to the canonical variables (2.2.20) for the operator (2.2.19).

Chapter 3
Groups admitted by differential equations

This chapter provides a simple introduction to definition and calculation of groups admitted by differential equations. An admitted group is called also a symmetry group. The generators of symmetry groups are often termed infinitesimal symmetries or briefly *symmetries* of differential equations.

3.1 Preliminaries

The concept of a symmetry group of a differential equation can be specified either in terms of its solutions or its frame. Both definitions are useful in group analysis of differential equations.

The first definition treats a symmetry group of a differential equation as a group of transformations mapping every solution of the equation into its solution. It does not need the theory of prolongations of transformations. This definition dealing with the totality of solutions of a given differential equation is natural, but does not give a practical method for finding *all symmetries* of an equation. See appropriate examples in Section 3.4.

The second definition treats a symmetry group of a differential equation as a group of transformations whose *prolongation*, i.e. extension to the derivatives involved in a differential equation in question, leaves invariant the *frame* of the differential equation. This differential algebraic definition does not assume knowledge of solutions. Moreover, it is crucial for the infinitesimal formulation of the invariance of differential equations and provide a practical method for calculating symmetries by solving so-called *determining equations.*

3.1.1 Differential variables and functions

In the differential algebraic approach to symmetries we use the following infinite number of variables (see [14], Chapter 8, or [17], Section 1.4):

$$x=\{x^i\},\ u=\{u^\alpha\},\ u_{(1)}=\{u^\alpha_i\},$$
$$u_{(2)}=\{u^\alpha_{i_1 i_2}\},\ u_{(3)}=\{u^\alpha_{i_1 i_2 i_3}\},\dots, \tag{3.1.1}$$

where $\alpha=1,\dots,m$; $i,i_1,\dots=1,\dots,n$. The variables $u^\alpha_{i_1 i_2}, u^\alpha_{i_1 i_2 i_3},\dots$ are symmetric in the subscripts $i_1 i_2,\ i_1 i_2 i_3,\dots$.

The main operation in the calculus of differential algebra is the *total differentiation* given by the following formal infinite sums:

$$D_i=\frac{\partial}{\partial x^i}+u^\alpha_i\frac{\partial}{\partial u^\alpha}+u^\alpha_{ii_1}\frac{\partial}{\partial u^\alpha_{i_1}}+u^\alpha_{ii_1 i_2}\frac{\partial}{\partial u^\alpha_{i_1 i_2}}+\cdots,\quad i=1,\dots,n. \tag{3.1.2}$$

We avoid the question on convergence by using the action of total derivatives D_i on functions involving only finite number of variables (3.1.1), e.g.

$$D_i\left(f(x,u,u_{(1)})\right)=\frac{\partial f}{\partial x^i}+u^\alpha_i\frac{\partial f}{\partial u^\alpha}+u^\alpha_{ii_1}\frac{\partial f}{\partial u^\alpha_{i_1}}.$$

In particular, letting $f=u^\alpha,\ f=u^\alpha_j,\dots$, one obtains:

$$u^\alpha_i=D_i(u^\alpha),\ u^\alpha_{ij}=D_i(u^\alpha_j)=D_iD_j(u^\alpha),\dots. \tag{3.1.3}$$

In view of Eqs. (3.1.3), the variables u^α are called *differential variables,* and $u_{(1)},u_{(2)},\dots$ are called their successive *derivatives* with respect to the independent variables x^i.

Definition 3.1. A *differential function* $f=f(x,u,u_{(1)},\dots)$ is a locally analytic function (i.e. locally expandable in a Taylor series with respect to all arguments) of a finite number of variables (3.1.1). The highest order of derivatives appearing in f is called the order of the differential function and is denoted by $\mathrm{ord}(f)$, e.g. if $f=f(x,u,u_{(1)},\dots,u_{(s)})$ then $\mathrm{ord}(f)=s$. The set of all differential functions of finite order is denoted by $\mathscr{A}$.

It follows from this definition that the set $\mathscr{A}$ is a vector space endowed with the usual multiplication of functions. In other words, if $f,g\in\mathscr{A}$ and α and β are any constants then

$$\alpha f+\beta g\in\mathscr{A},\quad \mathrm{ord}(\alpha f+\beta g)\le\max\{\mathrm{ord}(f),\mathrm{ord}(g)\},$$
$$fg\in\mathscr{A},\quad \mathrm{ord}(fg)=\max\{\mathrm{ord}(f),\mathrm{ord}(g)\}.$$

Furthermore, the space $\mathscr{A}$ is closed under the total differentiation: if $f\in\mathscr{A}$, then

$$D_i(f)\in\mathscr{A},\quad \mathrm{ord}\,(D_i(f))=\mathrm{ord}(f)+1.$$

Thus, if $f\in\mathscr{A}$ is a differential function of order s, its successive derivatives

$$D_i(f),\quad D_jD_i(f)=D_j(D_i(f)),\dots$$

are differential functions of respective orders $s+1, s+2, \ldots$. Furthermore, one can prove using the symmetry property $u^{\alpha}_{ij} = u^{\alpha}_{ji}, \ldots$, that the successive action of total differentiations D_i and D_j does not depend on the order of differentiation:

$$D_j D_i(f) = D_i D_j(f).$$

3.1.2 Point transformations

Let G be a one-parameter group of transformations (see (2.1.9)) of independent variables $x = (x^1, \ldots, x^n)$ and differential variables $u = (u^1, \ldots, u^m)$:

$$\overline{x}^i = f^i(x,u,a), \quad f^i\big|_{a=0} = x^i, \tag{3.1.4}$$

$$\overline{u}^{\alpha} = \varphi^{\alpha}(x,u,a), \quad \varphi^{\alpha}\big|_{a=0} = u^{\alpha}. \tag{3.1.5}$$

The generator of the group G is written in the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \tag{3.1.6}$$

where

$$\xi^i(x,u) = \left.\frac{\partial f^i(x,u,a)}{\partial a}\right|_{a=0}, \quad \eta^{\alpha}(x,u) = \left.\frac{\partial \varphi^{\alpha}(x,u,a)}{\partial a}\right|_{a=0}. \tag{3.1.7}$$

3.1.3 Frame of differential equations

Consider the equations

$$F_{\sigma}(x,u,u_{(1)},\ldots,u_{(k)}) = 0, \quad \sigma = 1,\ldots,s, \tag{3.1.8}$$

where $F_{\sigma} \in \mathscr{A}$, $\sigma = 1,\ldots,s$, are any differential functions.

In mathematical analysis, the differential variables u^{α} are treated as functions $u^{\alpha}(x)$ and consequently the variables u^{α}_i, $u^{\alpha}_{ij}, \ldots$ are identified with the classical derivatives

$$u^{\alpha}_i = \frac{\partial u^{\alpha}(x)}{\partial x^i}, \quad u^{\alpha}_{ij} = \frac{\partial^2 u^{\alpha}(x)}{\partial x^i \partial x^j}, \ldots.$$

In this way we obtain the classical definition of a system of kth-order differential equations

$$F_{\sigma}\left(x, u^{\alpha}(x), \frac{\partial u^{\alpha}(x)}{\partial x^i}, \ldots, \frac{\partial^k u^{\alpha}(x)}{\partial x^{i_1}\cdots\partial x^{i_k}}\right) = 0, \quad \sigma = 1,\ldots,s. \tag{3.1.9}$$

Definition 3.2. The system of Eqs. (3.1.8), where $x, u, u_{(1)}, \ldots, u_{(k)}$ are identified with the variables (3.1.1), is called the *frame* of the system of the differential equations (3.1.9).

3.2 Prolongation of group transformations

3.2.1 One-dimensional case

Let us begin with the case of one independent variable x and one differential variable which will be denoted by y. The successive derivatives are denoted by $y', y'', \ldots, y^{(s)}, \ldots$. Then the total differentiation (3.1.2) is written:

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots + y^{(s+1)} \frac{\partial}{\partial y^{(s)}} + \cdots . \tag{3.2.1}$$

We will write the transformation (3.1.4)–(3.1.5) of a one-parameter group G on the (x, y) in the following form:

$$\bar{x} = f(x, y, a), \tag{3.2.2}$$

$$\bar{y} = \varphi(x, y, a). \tag{3.2.3}$$

Let us denote by $\overline{D}_x$ the total differentiation with respect to the new independent variable $\bar{x}$ defined by Eq. (3.2.2), $\bar{x} = f(x, y, a)$. The chain rule provides the following relation between the total differentiations in the old and new independent variables:

$$D_x = D_x(f)\overline{D}_x, \tag{3.2.4}$$

where

$$D_x(f) \equiv D_x(f(x, y, a)) = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y}.$$

Now we differentiate Eq. (3.2.3) using the respective sides of Eq. (3.2.4):

$$D_x(f)\overline{D}_x(\bar{y}) = D_x\big(\varphi(x, y, a)\big).$$

Since $\overline{D}_x(\bar{y}) = \bar{y}'$ by definition of the total differentiation, the above equation is written:

$$D_x(f)\bar{y}' = D_x\big(\varphi(x, y, a)\big). \tag{3.2.5}$$

Dividing Eq. (3.2.5) by $D_x(f)$, we obtain the following transformation of y':

$$\bar{y}' \equiv \overline{D}_x(\bar{y}) = \frac{D_x(\varphi)}{D_x(f)}. \tag{3.2.6}$$

Thus, starting from a point transformation (3.2.2)–(3.2.3) and then adding the formula (3.2.6), one obtains the transformation in the space of three independent variables, x, y, y'. This is what is known as an *extended* point transformation, or the *prolongation* of a point transformation. Specifically, the prolongation of point transformations to the first derivative is called the *first prolongation*.

Theorem 3.1. The first prolongation of the one-parameter group G is again a one-parameter group. It is called a *prolonged group*.

Proof. We deal with a canonical parameter a. Therefore the group property is given by Eq. (2.1.11), or our case

$$\begin{aligned} \bar{\bar{x}} &\equiv f(\bar{x},\bar{y},b) = f(x,y,a+b), \\ \bar{\bar{y}} &\equiv \varphi(\bar{x},\bar{y},b) = \varphi(x,y,a+b). \end{aligned} \tag{3.2.7}$$

Let us rewrite transformation (3.2.6) in the form

$$\bar{y}' = \psi(x,y,y',a) \equiv \frac{D_x\big(\varphi(x,y,a)\big)}{D_x\big(f(x,y,a)\big)}.$$

To prove the theorem, we have to show that

$$\bar{\bar{y}}' \equiv \psi(\bar{x},\bar{y},\bar{y}',b) = \psi(x,y,y',a+b). \tag{3.2.8}$$

We have

$$\bar{\bar{y}}' = \psi(\bar{x},\bar{y},\bar{y}',b) = \frac{\overline{D}_x\big(\varphi(\bar{x},\bar{y},b)\big)}{\overline{D}_x\big(f(\bar{x},\bar{y},b)\big)}.$$

Multiplying the numerator and denominator of the right-hand side of this equation by $D_x(f(x,y,a))$ and using Eqs. (3.2.4) and (3.2.7) we obtain:

$$\frac{D_x(f(x,y,a))\overline{D}_x\big(\varphi(\bar{x},\bar{y},b)\big)}{D_x(f(x,y,a))\overline{D}_x\big(f(\bar{x},\bar{y},b)\big)} = \frac{D_x\big(\varphi(x,y,a+b)\big)}{D_x\big(f(x,y,a+b)\big)} = \psi(x,y,y',a+b).$$

Hence, we have verified Eq. (3.2.8) thus completing the proof.

Applying the prolongation procedure to the transformations (3.2.2), (3.2.3) and (3.2.6), one obtains the *second prolongation* of the point transformation group G acting in the space of the variables x, y, y', y''. It is manifest that Theorem 3.1 applies also to the second prolongation. Continuing this procedure one can obtain prolongations of higher order.

3.2.2 *Prolongation with several differential variables*

Consider point transformations (3.1.4)–(3.1.5) with one independent variable x and m differential variables $u = (u^1, \ldots, u^m)$:

$$\begin{aligned} \overline{x} &= f(x,u,a), \\ \overline{u}^{\alpha} &= \varphi^{\alpha}(x,u,a), \quad \alpha = 1,\ldots,m. \end{aligned} \tag{3.2.9}$$

In this case Eq. (3.2.6) for the prolongation to the first derivatives is replaced by

$$\overline{u}_1^{\alpha} \equiv \frac{\mathrm{d}\overline{u}^{\alpha}}{\mathrm{d}\overline{x}} = \frac{D_x(\varphi^{\alpha})}{D_x(f)}, \quad \alpha = 1,\ldots,m. \tag{3.2.10}$$

Equation (3.2.10) can also be rewritten in the form (3.2.5):

$$\overline{u}_1^{\alpha} D_x(f) = D_x(\varphi^{\alpha}). \tag{3.2.11}$$

3.2.3 General case

In the case of several independent and several differential variables the chain rule (3.2.4) is written

$$D_i = D_i(f^j)\overline{D}_j. \tag{3.2.12}$$

Acting by Eq. (3.2.12) on the first equation from Eq. (3.1.5), $\overline{u}^{\alpha} = \varphi^{\alpha}(x,u,a)$, we obtain:

$$D_i(f^j)\overline{D}_j(\overline{u}^{\alpha}) = D_i(\varphi^{\alpha}(x,u,a)).$$

Invoking that

$$\overline{D}_i(\overline{u}^{\alpha}) = \overline{u}_i^{\alpha}$$

we arrive at the following *rule for the change of derivatives*:

$$\overline{u}_j^{\alpha} D_i(f^j) = D_i(\varphi^{\alpha}). \tag{3.2.13}$$

Upon solving Eq. (3.2.13) with respect to $\overline{u}_i^{\alpha}$, one obtains the transformation of the first derivatives,

$$\overline{u}_i^{\alpha} = \psi_i^{\alpha}(x,u,u_{(1)},a).$$

The prolongations to higher-order derivatives are obtained by further differentiating equation (3.2.13) by means of Eq. (3.2.12). The prolongation of a group of transformations (3.1.4)–(3.1.5) to derivatives $u_{(1)},\ldots,u_{(s)}$ of any order is again a one-parameter group (see Theorem 3.1) and is called the *prolonged group*.

3.3 Prolongation of group generators

3.3.1 One-dimensional case

The generator of the prolonged group can be obtained by applying Eqs. (2.1.16) and (2.1.17) to the extended transformation given by Eqs. (3.2.2), (3.2.3) and (3.2.6).

But the procedure can be simplified significantly by using the infinitesimal transformation (2.1.15),

$$\bar{x} \approx x + a\xi(x,y), \tag{3.3.1}$$

$$\bar{y} \approx y + a\eta(x,y), \tag{3.3.2}$$

of the group G instead of the transformation (3.2.2) and (3.2.3), or the generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{3.3.3}$$

Substituting the transformations (3.3.1) and (3.3.2) in Eq. (3.2.6), one obtains the infinitesimal transformation of y' :

$$\bar{y}' \approx y' + a\zeta_1(x,y,y'). \tag{3.3.4}$$

Namely,

$$\bar{y}' \approx \frac{D_x(y+a\eta)}{D_x(x+a\xi)} = \frac{y'+aD_x(\eta)}{1+aD_x(\xi)} \approx (y'+aD_x(\eta))(1-aD_x(\xi)),$$

or

$$\bar{y}' \approx y' + a(D_x(\eta) - y'D_x(\xi)). \tag{3.3.5}$$

Comparing Eqs. (3.3.4) and (3.3.5) we obtain:

$$\zeta_1 = D_x(\eta) - y'D_x(\xi). \tag{3.3.6}$$

Thus, given a one-parameter group with the generator (3.3.3), the *prolongation* $X_{(1)}$ of the operator to y' has the form

$$X_{(1)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_1\frac{\partial}{\partial y'}. \tag{3.3.7}$$

Equation (3.3.6) is called the *first prolongation formula.*

Example 3.1. Using the prolongation formula (3.3.6), one obtains the following first prolongation of the generator (2.1.22) of the rotation group:

$$X_{(1)} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - (1+y'^2)\frac{\partial}{\partial y'}. \tag{3.3.8}$$

The $(s+1)$-times extended infinitesimal transformations, and hence the corresponding generators $X_{(s+1)}$ are obtained recursively:

$$\bar{y}^{(s+1)} \approx (y^{(s+1)} + aD_x(\zeta_s))(1 - aD_x(\xi)) \tag{3.3.9}$$

$$\approx y^{(s+1)} + a(D_x(\zeta_s) - y^{(s+1)}D_x(\xi)), \tag{3.3.10}$$

whence, by setting

$$\bar{y}^{(s+1)} \approx y^{(s+1)} + a\zeta_{s+1},$$

we obtain the *general prolongation formula:*

$$\zeta_{s+1} = D_x(\zeta_s) - y^{(s+1)} D_x(\xi), \quad s = 1,2,\ldots. \tag{3.3.11}$$

In practical applications, one often needs the coordinates ζ_1, ζ_2 and ζ_3 of the second and third prolongations, $X_{(2)}$ and $X_{(3)}$. Namely,

$$\begin{aligned}
\zeta_1 &= D(\eta) - y' D(\xi) = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\
\zeta_2 &= D(\zeta_1) - y'' D(\xi) \\
&= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\
&\quad - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'', \\
\zeta_3 &= D(\zeta_2) - y''' D(\xi) \\
&= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 \\
&\quad + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy} y'^4 + 3[\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' \\
&\quad - 2\xi_{yy} y'^2]y'' - 3\xi_y y''^2 + (\eta_y - 3\xi_x - 4\xi_y y')y'''.
\end{aligned} \tag{3.3.12}$$

Thus, the first, second, and third prolongations of the generator X (3.3.3) are given, respectively, by (3.3.7) and by

$$\begin{aligned}
X_{(2)} &= \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial y'} + \zeta_2 \frac{\partial}{\partial y''}, \\
X_{(3)} &= \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial y'} + \zeta_2 \frac{\partial}{\partial y''} + \zeta_3 \frac{\partial}{\partial y'''}.
\end{aligned} \tag{3.3.13}$$

Remark 3.1. The notation $X_{(p)}$ for p-times prolonged infinitesimal generators is taken from Eisenhart [9]. Lie [19] used for the infinitesimal generator the term *symbol* of an infinitesimal transformation and denoted it by U; its first, second, etc. prolongations he denoted by U', U'', etc. Cohen [7] and Dickson [8] adopt Lie's notation. In examples we will employ Eisenhart's notation. In differential algebra, however, it is advantageous to consider generators X extended to all derivatives. Then X acts in the universal space $\mathscr{A}$, and there is no ambiguity if the *prolongation of a generator X to all derivatives* is denoted by the same symbol X. The operator X understood in this way truncates when it acts on differential functions $f \in \mathscr{A}$. Namely, X acts as $X_{(1)}$ if $\operatorname{ord}(f) = 1$, as $X_{(2)}$ if $\operatorname{ord}(f) = 2$, etc.

Remark 3.2. Theorem 3.1 guarantees that one can construct the first prolongation of a point transformation group by solving the Lie equations (2.1.24) for the prolonged generator (3.3.7), i.e. the equations

$$\begin{aligned} \frac{d\bar{x}}{da} &= \xi(\bar{x},\bar{y}), & \bar{x}\big|_{a=0} &= x, \\ \frac{d\bar{y}}{da} &= \eta(\bar{x},\bar{y}), & \bar{y}\big|_{a=0} &= y, \\ \frac{d\bar{y}'}{da} &= \zeta_1(\bar{x},\bar{y},\bar{y}'), & \bar{y}'\big|_{a=0} &= y'. \end{aligned} \tag{3.3.14}$$

This construction can be extended to all higher-order prolongations, e.g.

$$\begin{aligned} \frac{d\bar{x}}{da} &= \xi(\bar{x},\bar{y}), & \bar{x}\big|_{a=0} &= x, \\ \frac{d\bar{y}}{da} &= \eta(\bar{x},\bar{y}), & \bar{y}\big|_{a=0} &= y, \\ \frac{d\bar{y}'}{da} &= \zeta_1(\bar{x},\bar{y},\bar{y}'), & \bar{y}'\big|_{a=0} &= y', \\ \frac{d\bar{y}''}{da} &= \zeta_2(\bar{x},\bar{y},\bar{y}',\bar{y}''), & \bar{y}''\big|_{a=0} &= y'' \end{aligned} \tag{3.3.15}$$

for the second prolongation.

3.3.2 Several differential variables

We substitute in Eq. (3.2.10) the infinitesimal transformation (3.2.9),

$$\begin{aligned} \bar{x} &\approx x + a\xi(x,u), \\ \bar{u}^{\alpha} &\approx u^{\alpha} + a\eta^{\alpha}(x,u), \end{aligned} \tag{3.3.16}$$

and obtain the following first prolongation formula (cf. (3.3.6)):

$$\zeta_1^{\alpha} = D_x(\eta^{\alpha}) - u_1^{\alpha} D_x(\xi), \quad \alpha = 1,\ldots,m. \tag{3.3.17}$$

Furthermore, the general prolongation formula (3.3.11) is replaced by

$$\zeta_{s+1}^{\alpha} = D_x(\zeta_s^{\alpha}) - u_{s+1}^{\alpha} D_x(\xi), \quad s = 1,2,\ldots. \tag{3.3.18}$$

Hence, given a one-parameter group with the generator

$$X = \xi(x,u)\frac{\partial}{\partial x} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}},$$

we obtain, e.g., the twice-prolonged generator by the formula

$$X_{(2)} = \xi(x,u)\frac{\partial}{\partial x} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}} + \zeta_1^{\alpha}\frac{\partial}{\partial u_1^{\alpha}} + \zeta_2^{\alpha}\frac{\partial}{\partial u_2^{\alpha}} \tag{3.3.19}$$

where ζ_1^{α} is given by Eq. (3.3.17) and ζ_2^{α} has the form

$$\zeta_2^{\alpha} = D_x(\zeta_1^{\alpha}) - u_2^{\alpha}D_x(\xi). \tag{3.3.20}$$

3.3.3 General case

Taking the infinitesimal transformation (3.1.4) and (3.1.5):

$$f^i = x^i + a\xi^i, \quad \varphi^{\alpha} = u^{\alpha} + a\eta^{\alpha},$$

and setting

$$\bar{u}_i^{\alpha} = u_i^{\alpha} + a\zeta_i^{\alpha},$$

one obtains from Eq. (3.2.13) the *first prolongation formula:*

$$\zeta_i^{\alpha} = D_i(\eta^{\alpha}) - u_j^{\alpha}D_i(\xi^j). \tag{3.3.21}$$

Hence, the prolongation of the generator (3.1.6) has the form:

$$X_{(1)} = \xi^i\frac{\partial}{\partial x^i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha}\frac{\partial}{\partial u_i^{\alpha}}. \tag{3.3.22}$$

Continuing this procedure, one obtains the twice-prolonged generator

$$X_{(2)} = \xi^i\frac{\partial}{\partial x^i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha}\frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha}\frac{\partial}{\partial u_{i_1 i_2}^{\alpha}}, \tag{3.3.23}$$

where $i_2 \geq i_1$ and $\zeta_{i_1 i_2}^{\alpha}$ are defined by the *second prolongation formula*:

$$\begin{aligned}\zeta_{i_1 i_2}^{\alpha} &= D_{i_2}(\zeta_{i_1}^{\alpha}) - u_{j i_1}^{\alpha}D_{i_2}(\xi^j) \\ &\equiv D_{i_2}D_{i_1}(\eta^{\alpha}) - u_j^{\alpha}D_{i_2}D_{i_1}(\xi^j) - u_{j i_1}^{\alpha}D_{i_2}(\xi^j) - u_{j i_2}^{\alpha}D_{i_1}(\xi^j).\end{aligned} \tag{3.3.24}$$

The higher-order prolongations are defined recursively:

$$\zeta_{i_1 \ldots i_s}^{\alpha} = D_{i_s}(\zeta_{i_1 \ldots i_{s-1}}^{\alpha}) - u_{j i_1 \ldots i_{s-1}}^{\alpha}D_{i_s}(\xi^j). \tag{3.3.25}$$

Example 3.2. Let us consider the second-order prolongation in the case of two independent variables x, y and one differential variable u. In the present case, the total derivatives (3.1.2) are written

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots,$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + \cdots.$$

The generator (3.1.6) has the form

$$X = \xi^1(x,y,u) \frac{\partial}{\partial x} + \xi^2(x,y,u) \frac{\partial}{\partial y} + \eta(x,y,u) \frac{\partial}{\partial u} \tag{3.3.26}$$

and its second prolongation is

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_x} + \zeta_2 \frac{\partial}{\partial u_y} \tag{3.3.27}$$

$$+ \zeta_{11} \frac{\partial}{\partial u_{xx}} + \zeta_{12} \frac{\partial}{\partial u_{xy}} + \zeta_{22} \frac{\partial}{\partial u_{yy}}.$$

The coefficients ζ_i are given by formula (3.3.21):

$$\begin{aligned} \zeta_1 &= D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2) \\ &= \eta_x + u_x \eta_u - u_x \xi^1_x - (u_x)^2 \xi^1_u - u_y \xi^2_x - u_x u_y \xi^2_u, \end{aligned} \tag{3.3.28}$$

$$\begin{aligned} \zeta_2 &= D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2) \\ &= \eta_y + u_y \eta_u - u_x \xi^1_y - u_x u_y \xi^1_u - u_y \xi^2_y - (u_y)^2 \xi^2_u. \end{aligned} \tag{3.3.29}$$

The coefficients ζ_{ij} are given by formula (3.3.24):

$$\begin{aligned} \zeta_{11} &= D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xy} D_x(\xi^2) \\ &= D_x^2(\eta) - u_x D_x^2(\xi^1) - u_y D_x^2(\xi^2) - 2u_{xx} D_x(\xi^1) - 2u_{xy} D_x(\xi^2), \end{aligned} \tag{3.3.30}$$

$$\begin{aligned} \zeta_{12} &= D_y(\zeta_1) - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2) \\ &= D_x D_y(\eta) - u_x D_x D_y(\xi^1) - u_y D_x D_y(\xi^2) \\ &\quad - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{xy} D_x(\xi^1) - u_{yy} D_x(\xi^2), \end{aligned} \tag{3.3.31}$$

$$\begin{aligned} \zeta_{22} &= D_y(\zeta_2) - u_{xy} D_y(\xi^1) - u_{yy} D_y(\xi^2) \\ &= D_y^2(\eta) - u_x D_y^2(\xi^1) - u_y D_y^2(\xi^2) - 2u_{xy} D_y(\xi^1) - 2u_{yy} D_y(\xi^2). \end{aligned} \tag{3.3.32}$$

3.4 First definition of symmetry groups

3.4.1 Definition

Definition 3.3. A group G of transformations (3.1.4) and (3.1.5) is termed a *symmetry group* for a system of the differential equations (3.1.9) if the set of all solutions of the system of Eqs. (3.1.9) is invariant under the group G. It means that the solutions of the system are merely permuted among themselves (or are individually unaltered) by every transformation of the group G. We also say that the system of Eqs. (3.1.9) is *invariant* under the group G, or that G is *admitted* by Eqs. (3.1.9). The generator X of a symmetry group G is called an *infinitesimal symmetry*, or an *admitted operator* for Eqs. (3.1.9).

3.4.2 Examples

Example 3.3. Consider the second-order equation

$$y'' = 0. \tag{3.4.1}$$

Its general solution describes the straight lines

$$y = C_1 x + C_2. \tag{3.4.2}$$

Furthermore, equation (3.4.1) remains the same if we replace the independent and dependent variables,

$$x \leftrightarrow y. \tag{3.4.3}$$

In other words, if we let $x = x(y)$, then Eq. (3.4.1) becomes

$$x'' = 0. \tag{3.4.4}$$

Indeed, the change of variables (3.4.3) yields (see [17], Example 1.4.1):

$$x'' = -\frac{y''}{y'^3}.$$

Equation (3.4.4) describes the straight lines

$$x = C_3 y + C_4. \tag{3.4.5}$$

Equation (3.4.2) and (3.4.5) define the variety of all straight lines

$$kx + ly + m = 0 \tag{3.4.6}$$

with arbitrary constant coefficients k, l, m.

The obvious symmetries of Eq. (3.4.1) are provided by the translations

$$\begin{aligned} &\bar{x} = x + a_1, \quad \bar{y} = y, \\ &\bar{x} = x, \quad \bar{y} = y + a_2 \end{aligned} \tag{3.4.7}$$

and the dilations

$$\begin{aligned} &\bar{x} = x\mathrm{e}^{a_3}, \quad \bar{y} = y, \\ &\bar{x} = x, \quad \bar{y} = y\mathrm{e}^{a_4} \end{aligned} \tag{3.4.8}$$

with four arbitrary parameters $a_1, \ldots, a_4$. These transformations satisfy Definition 3.3. Indeed, they map the solutions (3.4.2) of Eq. (3.4.1) among themselves according to the following equations:

$$\begin{aligned} &\bar{y} = C_1\bar{x} + \bar{C}_2, \quad \bar{C}_2 = C_2 - a_1C_1, \\ &\bar{y} = C_1\bar{x} + \bar{C}_2, \quad \bar{C}_2 = C_2 - a_2 \end{aligned} \tag{3.4.9}$$

and

$$\begin{aligned} &\bar{y} = \bar{C}_1\bar{x} + C_2, \quad \bar{C}_1 = C_1\mathrm{e}^{-a_3}, \\ &\bar{y} = \bar{C}_1\bar{x} + \bar{C}_2, \quad \bar{C}_1 = C_1\mathrm{e}^{a_4}, \ \bar{C}_2 = C_2\mathrm{e}^{a_4}. \end{aligned} \tag{3.4.10}$$

The transformations (3.4.7) and (3.4.8) provide four one-parameter groups with the following infinitesimal generators:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x}, \quad X_4 = y\frac{\partial}{\partial y}. \tag{3.4.11}$$

It is also obvious that we can add to y a linear function in x, i.e. that the set of the straight lines (3.4.2) is invariant under the transformation

$$\bar{x} = x, \quad \bar{y} = y + a_5x. \tag{3.4.12}$$

Indeed, this transformation yields:

$$\bar{y} = \bar{C}_1\bar{x} + C_2, \quad \bar{C}_1 = C_1 + a_5. \tag{3.4.13}$$

The transformation (3.4.12) provides a one-parameter symmetry group with the generator

$$X_5 = x\frac{\partial}{\partial y}. \tag{3.4.14}$$

We also know from Example 1.14 that the set of the straight lines (3.4.6) is invariant under the projective group of transformation (1.3.8). Let us specify this invariance property by considering the behaviour of the solutions (3.4.2) of Eq. (3.4.1) under the special projective group (1.3.10),

$$\bar{x} = \frac{x}{1 - a_6 x}, \quad \bar{y} = \frac{y}{1 - a_6 x}. \tag{3.4.15}$$

The inverse transformation to Eqs. (3.4.15) has the form

$$x = \frac{\bar{x}}{1 + a_6 \bar{x}}, \quad y = \frac{\bar{y}}{1 + a_6 \bar{x}},$$

whence

$$\frac{1}{1 - a_6 x} = 1 + a_6 \bar{x}. \tag{3.4.16}$$

Dividing both sides of Eq. (3.4.2) by $1 - a_6 x$ and using Eqs. (3.4.15) and (3.4.16) we obtain

$$\bar{y} = C_1 \bar{x} + C_2 (1 + a_6 \bar{x})$$

or

$$\bar{y} = \overline{C}_1 \bar{x} + C_2, \quad \overline{C}_1 = C_1 + a_6 C_2. \tag{3.4.17}$$

We see that the set of the straight lines (3.4.2) is invariant under the transformation (1.3.10). Hence the generator

$$X_6 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \tag{3.4.18}$$

of the special projective one-parameter group (1.3.10) is an infinitesimal symmetry for Eq. (3.4.1).

One can obtain more symmetries using the invariance of Eq. (3.4.1) under the replacement (3.4.4) of the variables. Namely, after this replacement one obtains, instead of Eqs. (3.4.12) and (3.4.15), the transformations

$$\bar{x} = x + a_7 y, \quad \bar{y} = y \tag{3.4.19}$$

and

$$\bar{x} = \frac{y}{1 - a_8 y}, \quad \bar{y} = \frac{y}{1 - a_8 y}, \tag{3.4.20}$$

respectively. The transformations (3.4.19) and (3.4.20) map the straight lines (3.4.2) into the similar straight lines:

$$\bar{y} = \overline{C}_1 \bar{x} + \overline{C}_2, \quad \overline{C}_1 = \frac{C_1}{1 + a_7 C_1}, \ \overline{C}_2 = \frac{C_2}{1 + a_7 C_1}, \tag{3.4.21}$$

and

$$\bar{y} = \overline{C}_1 \bar{x} + \overline{C}_2, \quad \overline{C}_1 = \frac{C_1}{1 - a_8 C_2}, \ \overline{C}_2 = \frac{C_2}{1 - a_8 C_2}, \tag{3.4.22}$$

respectively. The generators of the one-parameter transformation groups (3.4.19) and (3.4.20) have the form

$$X_7 = y \frac{\partial}{\partial x}, \quad X_8 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}. \tag{3.4.23}$$

The operators X_7 and X_8 can be obtained from X_5 and X_6, respectively, by the replacement $x \leftrightarrow y$.

Thus, we have obtained the following eight linearly independent infinitesimal symmetries of Eq. (3.4.1):

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x}, \; X_2 = \frac{\partial}{\partial y}, \; X_3 = x\frac{\partial}{\partial x}, \; X_4 = y\frac{\partial}{\partial y}, \; X_5 = x\frac{\partial}{\partial y}, \\ &X_6 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_7 = y\frac{\partial}{\partial x}, \quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \end{aligned} \tag{3.4.24}$$

A general theorem due to Lie states (see, e.g. [14], Theorem 2 in Section 9.3.1 and the references therein) that second-order ordinary differential equations may admit maximum eight linearly independent infinitesimal symmetries and that this maximum is reached precisely by linear and linearizable equations. According to this theorem, we conclude that we have found all infinitesimal symmetries of Eq. (3.4.1).

Example 3.4. Consider the second-order partial differential equation

$$u_x u_{xx} + u_{yy} = 0 \tag{3.4.25}$$

describing a potential steady-state transonic planar flow of a gas. Let

$$u = h(x,y) \tag{3.4.26}$$

be any solution of Eq. (3.4.25), i.e.

$$h_x h_{xx} + h_{yy} = 0. \tag{3.4.27}$$

The obvious symmetries of Eq. (3.4.25) are provided by the translations

$$\begin{aligned} &\bar{x} = x + a_1, \quad \bar{y} = y, \quad \bar{u} = u, \\ &\bar{x} = x, \quad \bar{y} = y + a_2, \quad \bar{u} = u, \\ &\bar{x} = x, \quad \bar{y} = y, \quad \bar{u} = u + a_3. \end{aligned} \tag{3.4.28}$$

Consider, e.g. the first translation (3.4.28). It maps the solution (3.4.26) into the following function:

$$\bar{u} = h(x,y) \equiv h(\bar{x} - a_1, \bar{y}).$$

Consequently,

$$\bar{u}_{\bar{x}} = h_x, \quad \bar{u}_{\bar{x}\bar{x}} = h_{xx}, \quad \bar{u}_{\bar{y}\bar{y}} = h_{yy},$$

and hence Eq. (3.4.27) guarantees that $\bar{u}$ solves Eq. (3.4.25):

$$\bar{u}_{\bar{x}}\bar{u}_{\bar{x}\bar{x}} + \bar{u}_{\bar{y}\bar{y}} = h_x h_{xx} + h_{yy} = 0.$$

For the second and third translations from (3.4.28) the calculations are similar. For example, in the case of the third translation (3.4.28) we have:

$$\overline{u} = a_3 + h(x,y) \equiv a_3 + h(\overline{x},\overline{y}).$$

Consequently,

$$\overline{u}_{\overline{x}} = h_x, \quad \overline{u}_{\overline{x}\overline{x}} = h_{xx}, \quad \overline{u}_{\overline{y}\overline{y}} = h_{yy},$$

and hence Eq. (3.4.27) guarantees that $\overline{u}$ solves Eq. (3.4.25).

It is also obvious that we can add to u a linear function in y because Eq. (3.4.25) contains only the second-order derivative in y. Hence, equation (3.4.25) is invariant under the transformation

$$\overline{x} = x, \quad \overline{y} = y, \quad \overline{u} = u + a_4 y. \tag{3.4.29}$$

It maps the solution (3.4.26) into the following function:

$$\overline{u} = a_4 y + h(x,y) \equiv a_4 \overline{y} + h(\overline{x},\overline{y}).$$

Consequently,

$$\overline{u}_{\overline{x}} = h_x, \quad \overline{u}_{\overline{x}\overline{x}} = h_{xx}, \quad \overline{u}_{\overline{y}\overline{y}} = h_{yy},$$

and hence Eq. (3.4.27) guarantees that $\overline{u}$ solves Eq. (3.4.25).

Furthermore, the inspection shows that Eq. (3.4.25) is invariant under the following dilations:

$$\begin{aligned} &\overline{x} = x\mathrm{e}^{a_5}, \quad \overline{y} = y, \quad \overline{u} = u\mathrm{e}^{3a_5}, \\ &\overline{x} = x, \quad \overline{y} = y\mathrm{e}^{a_6}, \quad \overline{u} = u\mathrm{e}^{-2a_6}. \end{aligned} \tag{3.4.30}$$

Indeed, the first dilation from of (3.4.30) maps the solution (3.4.26) into the function

$$\overline{u} = \mathrm{e}^{3a_5} h(x,y) \equiv \mathrm{e}^{3a_5} h(\overline{x}\mathrm{e}^{-a_5},\overline{y}).$$

Consequently,

$$\overline{u}_{\overline{x}} = \mathrm{e}^{3a_5} h_x \mathrm{e}^{-a_5} = \mathrm{e}^{2a_5} h_x, \quad \overline{u}_{\overline{x}\overline{x}} = \mathrm{e}^{a_5} h_{xx}, \quad \overline{u}_{\overline{y}\overline{y}} = \mathrm{e}^{3a_5} h_{yy},$$

and hence Eq. (3.4.27) guarantees that $\overline{u}$ solves Eq. (3.4.25):

$$\overline{u}_{\overline{x}}\overline{u}_{\overline{x}\overline{x}} + \overline{u}_{\overline{y}\overline{y}} = \mathrm{e}^{3a_5}(h_x h_{xx} + h_{yy}) = 0.$$

For the second transformation from of (3.4.30) the calculations are similar. See Exercise 3.9.

The generators of the transformations (3.4.28), (3.4.29) and (3.4.30) provide the following six linearly independent infinitesimal symmetries for Eq. (3.4.25):

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = y\frac{\partial}{\partial u}, \\ X_5 = x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}, \quad X_6 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \tag{3.4.31}$$

The above calculations do not guarantee, however, that (3.4.31) contains all infinitesimal symmetries of Eq. (3.4.25). This can be done by solving the determining equations introduced in the next section.

3.5 Second definition of symmetry groups

3.5.1 Definition and determining equations

Definition 3.4. A group G of transformations (3.1.4) and (3.1.5) is termed a *symmetry group* for a system of the kth-order differential equations (3.1.9) if the frame (3.1.8) of the system of Eqs. (3.1.9) is invariant under the kth-order prolongation of the group G.

Theorem 3.2. The system of the differential equations (3.1.9) is invariant under the group with an infinitesimal generator (3.1.6),

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

if and only if

$$XF_\sigma\left(x,u,u_{(1)},\ldots,u_{(k)}\right)\Big|_{(3.1.8)} = 0, \quad \sigma = 1,\ldots,s. \tag{3.5.1}$$

In Eqs. (3.5.1), X denotes the kth-order prolongation of the generator (3.1.6), and the symbol $|_{(3.1.8)}$ means evaluated on the frame (3.1.8).

Proof. The statement is a consequence of Definition 3.4 and Theorem 2.8 from Section 2.3.1.

Definition 3.5. Equations (3.5.1) determine all infinitesimal symmetries of the differential equations (3.1.9) and therefore they are known as *determining equations.*

Remark 3.3. The determining equations (3.5.1) can also be written in the form (2.3.7),

$$XF_\sigma = \phi^\nu_\sigma F_\nu, \quad \sigma = 1,\ldots,s, \tag{3.5.2}$$

where the undeterminate coefficients $\phi^\nu_\sigma(x,u,u_{(1)},\ldots) \in \mathscr{A}$ are bounded on the frame (3.1.8).

Determining equations (3.5.1) seem, at first glance, to be more complicated than the differential equations (3.1.9) in question. However, in many practical applications, the determining equations can be integrated. This is due to the fact that the

left-hand sides of Eqs. (3.5.1) involve not only the variables x, u and the functions $\xi^i(x,u)$ and $\eta^\alpha(x,u)$, but also the derivatives $u_{(1)}, u_{(2)}, \ldots$. Accordingly, each equation of (3.5.1) splits into several equations because Eqs. (3.5.1) should be satisfied identically with respect to all variables involved in the determining equations. The resulting equations provide an overdetermined system for the unknown functions $\xi^i(x,u)$ and $\eta^\alpha(x,u)$. The examples considered below clarify the situation.

3.5.2 Determining equation for second-order ODEs

Let us restrict consideration to second-order equations written in the form

$$y'' = f(x,y,y'). \tag{3.5.3}$$

We look for the infinitesimal symmetries written in the form

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{3.5.4}$$

and obtain from Eqs. (3.5.1) the following *determining equation* for the unknown coefficients ξ and η :

$$X\Big(y'' - f(x,y,y')\Big)\Big|_{(3.5.3)} \equiv (\zeta_2 - \zeta_1 f_{y'} - \xi f_x - \eta f_y)\big|_{y''=f} = 0. \tag{3.5.5}$$

After substituting ζ_1 and ζ_2 from (3.3.12), equation (3.5.5) is written in the following form:

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} - \xi f_x - \eta f_y \\ &+(\eta_y - 2\xi_x - 3y'\xi_y)f - [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y]f_{y'} = 0. \end{aligned} \tag{3.5.6}$$

Here $f(x,y,y')$ is a known function when one deals with a given differential equation (3.5.3). Since Eq. (3.5.6) involves all three variables x, y and y', but y' does not occur in ξ and η, the determining equation (3.5.6) decomposes into several equations, thus becoming an over-determined system of differential equations for ξ and η. After solving this system, one finds all generators of point transformations admitted by Eq. (3.5.3).

3.5.3 Examples on solution of determining equations

Example 3.5. Let us find all symmetries of the second-order equation

$$y'' = 0 \tag{3.5.7}$$

by solving the determining equation (3.5.5). This equation was considered in Example 3.3 for illustrating the first definition of symmetry groups.

The determining equation (3.5.5) for Eq. (3.5.7) is written

$$\zeta_2\big|_{y''=0} = 0.$$

Substituting here the expression (3.3.12) for ζ_2 we obtain the equation

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} = 0. \tag{3.5.8}$$

Since Eq. (3.5.8) should hold identically in x, y, y', it splits into the following equations obtained by nullifying the coefficients for different degrees of y' :

$$\eta_{xx} = 0, \quad 2\eta_{xy} - \xi_{xx} = 0, \quad \eta_{yy} - 2\xi_{xy} = 0, \quad \xi_{yy} = 0. \tag{3.5.9}$$

Integrating the first and the last equations of (3.5.9) we get:

$$\xi = \phi_1(x)y + \phi_2(x), \quad \eta = \psi_1(y)x + \psi_2(y).$$

Differentiating the second equation of (3.5.9) with respect to x and the third equation with respect to y, we obtain

$$\xi_{xxx} = 0, \quad \eta_{yyy} = 0.$$

It follows that $\phi_i(x)$ and $\psi_i(y)$ are quadratic functions of x and y, respectively. Hence,

$$\xi = C_1 + C_3x + C_4y + C_7x^2 + C_8xy + Ax^2y,$$

$$\eta = C_2 + C_5x + C_6y + C_9y^2 + C_{10}xy + Bxy^2.$$

Substitution of the above expressions for ξ, η into the second and third equations of (3.5.9) yields $Ay + C_7 = 2By + C_{10}$, $2Ax + C_8 = Bx + C_9$, whence $A = B = 0$, $C_{10} = C_7$, $C_9 = C_8$. We conclude that the following general solution to Eqs. (3.5.9) is given by the following functions depending on eight arbitrary constants C_i :

$$\begin{aligned} \xi &= C_1 + C_3x + C_4y + C_7x^2 + C_8xy, \\ \eta &= C_2 + C_5x + C_6y + C_7xy + C_8y^2. \end{aligned} \tag{3.5.10}$$

Substituting (3.5.10) in (3.5.4) we obtain the general infinitesimal symmetry

$$X = C_1X_1 + C_2X_2 + C_3X_3 + C_4X_4 + C_5X_5 + C_6X_6 + C_7X_7 + C_8X_8$$

for Eq. (3.5.7), where

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = y\frac{\partial}{\partial x},\ X_5 = x\frac{\partial}{\partial y},$$
$$X_6 = y\frac{\partial}{\partial y},\quad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{3.5.11}$$

The operators (3.5.11) provide eight linearly independent infinitesimal symmetries for Eq. (3.5.7) and coincide with the symmetries (3.4.24).

Example 3.6. Let us find all symmetries of the equation

$$y'' + e^{3y}y'^4 + y'^2 = 0. \tag{3.5.12}$$

Substituting $f = -(e^{3y}y'^4 + y'^2)$ in the determining equation (3.5.6) we have

$$\begin{aligned}&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy}\\ &+3e^{3y}y'^4\,\eta - (\eta_y - 2\xi_x - 3y'\xi_y)(e^{3y}y'^4 + y'^2)\\ &+[\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y](4e^{3y}y'^3 + 2y') = 0.\end{aligned}$$

The left-hand side of this equation is a polynomial of fifth degree in y'. We proceed as in the previous example, i.e. equate to zero the coefficients of $y'^5, y'^4, \ldots$ and obtain the following four independent equations:

$$\begin{aligned}&(y')^5 : \xi_y = 0,\\ &(y')^4 : 3(\eta_y + \eta) - 2\xi_x = 0,\\ &(y')^3 : \eta_x = 0,\\ &(y')^1 : \xi_{xx} = 0.\end{aligned}$$

The coefficients for $(y')^2$ and $(y')^0$ vanish together with the coefficients of $(y')^4$ and $(y')^1$, respectively.

Let us solve the above four differential equations for $\xi(x,y)$ and $\eta(x,y)$. The first equation of this system, $\xi_y = 0$, shows that $\xi = \xi(x)$. Then the fourth equation, $\xi_{xx} = 0$, is written $\xi''(x) = 0$ and yields

$$\xi = K_1 + K_2 x, \quad K_1, K_2 = \text{const}.$$

Likewise, we obtain from the third equation, $\eta_x = 0$, that $\eta = \eta(y)$. Finally, substituting the expressions for ξ and η into the second equation,

$$3(\eta_y + \eta) - 2\xi_x = 0,$$

we obtain the first-order linear ordinary differential equation

$$\eta' + \eta = \frac{2}{3}K_2.$$

Its integration yields:

$$\eta = \frac{2}{3}K_2 + K_3 e^{-y}.$$

Denoting $K_3 = C_1$, $K_1 = C_2$ and $3C_3 = K_2$, we have:

$$\xi = C_2 + 3C_3 x, \quad \eta = 2C_3 + C_1 e^{-y},$$

where C_1, C_2, C_3 are arbitrary constants. Hence, the operator (3.5.4) admitted by Eq. (3.5.12) has the form

$$X = C_1 X_1 + C_2 X_2 + C_3 X_3,$$

where

$$X_1 = e^{-y}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 3x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}. \tag{3.5.13}$$

The operators (3.5.13) provide three linearly independent infinitesimal symmetries for Eq. (3.5.12).

Example 3.7. It is shown in [17], Section 6.5.1, that the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy} \tag{3.5.14}$$

has two independent symmetries

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.5.15}$$

Solving the Lie equations for the above generators X_1 and X_2, we obtain the admitted one-parameter groups of transformations

$$T_a: \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{y} = \frac{y}{1-ax} \tag{3.5.16}$$

and

$$T_b: \quad \bar{x} = xe^{2b}, \quad \bar{y} = ye^{b}, \tag{3.5.17}$$

respectively, where a and b are arbitrary parameters. The reckoning shows that Definition 3.4 for the invariance of Eq. (3.5.14) under the transformations (3.5.16) and (3.5.17) is satisfied in the forms (cf. Eq. (2.3.3))

$$\bar{y}'' - \frac{\bar{y}'}{\bar{y}^2} + \frac{1}{\bar{x}\bar{y}} = (1-ax)\left(y'' - \frac{y'}{y^2} + \frac{1}{xy}\right),$$

and

$$\bar{y}'' - \frac{\bar{y}'}{\bar{y}^2} + \frac{1}{\bar{x}\bar{y}} = e^{-3b}\left(y'' - \frac{y'}{y^2} + \frac{1}{xy}\right),$$

respectively. Besides, equation (3.5.14) has the discrete symmetry provided by the reflection with respect to the x axis:

$$T_*: \quad \bar{x} = x, \quad \bar{y} = -y. \tag{3.5.18}$$

For this transformation the invariance condition is written

$$\bar{y}'' - \frac{\bar{y}'}{\bar{y}^2} + \frac{1}{\bar{x}\bar{y}} = -\left(y'' - \frac{y'}{y^2} + \frac{1}{xy}\right).$$

The composition of the transformations (3.5.16), (3.5.17) and (3.5.18) provides the general *mixed* group of symmetries for Eq. (3.5.14).

Example 3.8. Consider again Eq. (3.4.25) from Example 3.4:

$$u_x u_{xx} + u_{yy} = 0. \tag{3.5.19}$$

We seek the infinitesimal symmetries in the form (3.3.26),

$$X = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u},$$

with unknown coefficients ξ^1, ξ^2 and η to be found from Eq. (3.5.1):

$$(u_{xx}\zeta_1 + u_x\zeta_{11} + \zeta_{22})\big|_{(3.5.19)} = 0, \tag{3.5.20}$$

where we substitute ζ_1, ζ_{11} and ζ_{22} given by Eqs. (3.3.28), (3.3.30) and (3.3.32), respectively. We work out the expressions (3.3.30) and (3.3.32) for ζ_{11}, ζ_{22} and obtain:

$$\begin{aligned}\zeta_{11} = {} & \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + (u_x)^2\eta_{uu} - 2u_{xx}\xi^1_x - u_x\xi^1_{xx} \\ & -2(u_x)^2\xi^1_{xu} - 3u_x u_{xx}\xi^1_u - (u_x)^3\xi^1_{uu} - 2u_{xy}\xi^2_x - u_y\xi^2_{xx} \\ & -2u_x u_y\xi^2_{xu} - (u_y u_{xx} + 2u_x u_{xy})\xi^2_u - (u_x)^2 u_y\xi^2_{uu},\end{aligned} \tag{3.5.21}$$

$$\begin{aligned}\zeta_{22} = {} & \eta_{yy} + 2u_y\eta_{yu} + u_{yy}\eta_u + (u_y)^2\eta_{uu} - 2u_{yy}\xi^2_y - u_y\xi^2_{yy} \\ & -2(u_y)^2\xi^2_{yu} - 3u_y u_{yy}\xi^2_u - (u_y)^3\xi^2_{uu} - 2u_{xy}\xi^1_y - u_x\xi^1_{yy} \\ & -2u_x u_y\xi^1_{yu} - (u_x u_{yy} + 2u_y u_{xy})\xi^1_u - u_x(u_y)^2\xi^1_{uu}.\end{aligned} \tag{3.5.22}$$

After substituting the expanded expressions (3.3.28), (3.5.21), and (3.5.22) for ζ_1, ζ_{11}, ζ_{22} and eliminating u_{yy} by setting $u_{yy} = -u_x u_{xx}$, the determining equation (3.5.20) contains the variables x, y, u, u_x, u_y, u_{xx}, u_{xy}, whereas ξ^1, ξ^2, η depend only on x, y, u. Accordingly, we isolate the terms containing u_{xy}, u_{xx}, u_x, u_y, and the terms free of these variables, and set each term equal to zero. So, the terms containing u_{xy} lead to the equation

$$\xi^1_y + u_x\xi^2_x + u_y\xi^1_u + (u_x)^2\xi^2_u = 0,$$

whence

$$\xi^1_y = 0, \quad \xi^1_u = 0, \quad \xi^2_x = 0, \quad \xi^2_u = 0. \tag{3.5.23}$$

The same argument applied to the terms containing u_{xx} yields

$$\eta_x = 0, \quad \eta_u - 3\xi^1_x + 2\xi^2_y = 0. \tag{3.5.24}$$

Then Eq. (3.5.20) reduces to

$$\eta_{yy} = 0, \quad 2\eta_{yu} - \xi^2_{yy} = 0. \tag{3.5.25}$$

Thus, we have decomposed Eq. (3.5.20) into the overdetermined system of linear partial differential equations (3.5.23)–(3.5.25). The reckoning shows that the general solution of this system is given by

$$\xi^1 = C_5 x + C_1, \quad \xi^2 = C_6 y + C_2, \quad \eta = (3C_5 - 2C_6)u + C_4 y + C_3$$

and contains six arbitrary constants C_i. Proceeding as in Example 3.5 we obtain the following six linearly independent infinitesimal symmetries:

$$\begin{gathered} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = y\frac{\partial}{\partial u}, \\ X_5 = x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}, \qquad X_6 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{gathered} \tag{3.5.26}$$

They coincide with the symmetries (3.4.31) discussed in Example 3.4.

Exercises to Chapter 3

Exercise 3.1. Verify that the transformation

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{(1 - ax)^2}$$

satisfies the group property, find the group composition law (2.1.7) and construct the group generator.

Exercise 3.2. Verify that the transformation

$$\bar{x} = x, \quad \bar{y} = y + ax^2$$

satisfies the group property and find the group generator.

Exercise 3.3. Construct the first prolongation of the rotation group (1.2.29),

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a,$$

by using the formula (3.2.6).

Exercise 3.4. Construct the first prolongation of the rotation group (1.2.29) by solving the Lie equations (3.3.14) for the prolonged generator (3.3.8).

Exercise 3.5. Give the detailed derivation of the first prolongation formula (3.3.17).

Exercise 3.6. Derive the inverse transformation to the special projective group (3.4.15).

Exercise 3.7. Construct the composition (successive application) of the transformations (3.4.7), (3.4.8), (3.4.12), (3.4.15), (3.4.19), and (3.4.20). Compare the result with the general projective group (1.3.8).

Exercise 3.8. Derive the transformations (3.4.21) and (3.4.22).

Exercise 3.9. Show that the second transformation from (3.4.30),

$$\bar{x} = x, \quad \bar{y} = y\mathrm{e}^{a_6}, \quad \bar{u} = u\mathrm{e}^{-2a_6},$$

maps the solution (3.4.26) into a solution of Eq. (3.4.25).

Exercise 3.10. Expand the expression (3.3.31) for ζ_{12}. See Eq. (3.5.21).

Chapter 4
Lie algebras of operators

Examples on calculation of infinitesimal symmetries of differential equations exhibit the specific property of the determining equations. This property states that the set of all solutions of the determining equations for a given differential equation or a system of differential equations is a *vector space closed with respect to the commutator of infinitesimal generators.* Such vector spaces are called *Lie algebras.* In this chapter we outline basic concepts and properties from the theory of Lie algebras of operators. Some of results are proved if the proofs serve practical needs of Lie group analysis of differential equations.

4.1 Basic definitions

4.1.1 Commutator

Consider two first-order linear differential operators of the form (2.1.17):

$$X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}, \quad X_2 = \xi_2^i(x)\frac{\partial}{\partial x^i}. \tag{4.1.1}$$

Definition 4.1. The *commutator* $[X_1, X_2]$ of the operators (4.1.1) is defined by the formula

$$[X_1, X_2] = X_1X_2 - X_2X_1. \tag{4.1.2}$$

Lemma 4.1. The commutator (4.1.2) is equal to the following linear partial differential operator:

$$[X_1, X_2] = \Big(X_1(\xi_2^i) - X_2(\xi_1^i)\Big)\frac{\partial}{\partial x^i}. \tag{4.1.3}$$

Proof. Let $\phi(x)$ be any differentiable function of $x = (x^1, \ldots, x^n)$. We have:

$$X_1(\phi) = \xi_1^i\frac{\partial \phi}{\partial x^i}, \quad X_2(\phi) = \xi_2^i\frac{\partial \phi}{\partial x^i}.$$

Therefore

$$X_1(X_2(\phi)) = X_1\left(\xi_2^i\right)\frac{\partial\phi}{\partial x^i} + \xi_1^j\xi_2^i\frac{\partial^2\phi}{\partial x^i\partial x^j}$$

and

$$X_2(X_1(\phi)) = X_2\left(\xi_1^i\right)\frac{\partial\phi}{\partial x^i} + \xi_2^j\xi_1^i\frac{\partial^2\phi}{\partial x^i\partial x^j}.$$

Exchanging the summation indices i and j in the last term for $X_2(X_1(\phi))$, we obtain:

$$X_1(X_2(\phi)) - X_2(X_1(\phi)) = \left(X_1\left(\xi_2^i\right) - X_2\left(\xi_1^i\right)\right)\frac{\partial\phi}{\partial x^i},$$

or

$$[X_1, X_2](\phi) = \left(X_1\left(\xi_2^i\right) - X_2\left(\xi_1^i\right)\right)\frac{\partial\phi}{\partial x^i}.$$

Since $\phi(x)$ is an arbitrary function, the latter equation means that the operator equation (4.1.3) holds. This completes the proof.

Example 4.1. Let us find the commutator of the operators (3.5.15),

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Equation (4.1.3) yields

$$[X_1, X_2] = \left(X_1(2x) - X_2(x^2)\right)\frac{\partial}{\partial x} + \left(X_1(y) - X_2(xy)\right)\frac{\partial}{\partial y}.$$

We have:

$$X_1(2x) = 2x^2, \quad X_2(x^2) = -4x^2, \quad X_1(y) = xy, \quad X_2(xy) = 3xy.$$

Consequently, we obtain

$$[X_1, X_2] = -2x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y},$$

and hence we have the following commutator of the operators (3.5.15):

$$[X_1, X_2] = -2X_1. \tag{4.1.4}$$

Example 4.2. Let us find the commutators for the operators (3.5.13),

$$X_1 = \mathrm{e}^{-y}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 3x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}.$$

It is convenient to dispose the commutators in a *commutator table* whose entry at the intersection of the X_i row with the X_j column is $[X_i, X_j]$. It is manifest that the commutator (4.1.2) is antisymmetric. Therefore the commutator table is antisymmetric

as well, with zeros on the main diagonal. Proceeding as in the previous example, one obtains the following commutator table for the operators (3.5.13).

	X_1	X_2	X_3
X_1	0	0	$2X_1$
X_2	0	0	$3X_2$
X_3	$-2X_1$	$-3X_2$	0

(4.1.5)

Example 4.3. Computing the commutators of the operators (3.5.26),

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = y\frac{\partial}{\partial u},$$

$$X_5 = x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}, \qquad X_6 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u},$$

one obtains the following commutator table:

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	X_1	0
X_2	0	0	0	X_3	0	X_2
X_3	0	0	0	0	$3X_3$	$-2X_3$
X_4	0	$-X_3$	0	0	$3X_4$	$-3X_4$
X_5	$-X_1$	0	$-3X_3$	$-3X_4$	0	0
X_6	0	$-X_2$	$2X_3$	$3X_4$	0	0

(4.1.6)

4.1.2 Properties of the commutator

It follows from the definition (4.1.2) of the commutator that it is bilinear:

$$[c_1X_1 + c_2X_2, X] = c_1[X_1, X] + c_2[X_2, X],$$
$$[X, c_1X_1 + c_2X_2] = c_1[X, X_1] + c_2[X, X_2], \tag{4.1.7}$$

where c_1, c_2 are arbitrary constants, and skew-symmetric:

$$[X_1, X_2] = -[X_2, X_1]. \tag{4.1.8}$$

Furthermore, the following equation (termed the *Jacobi identity*)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{4.1.9}$$

is satisfied for any three operators X, Y, Z of the form (2.1.17).

Lemma 4.2. The commutator (4.1.2) is invariant under changes of variables. Namely, let $\overline{X}_\alpha$ denote the operators X_α written in the variables $\tilde{x}^i$ given by Eqs. (2.1.18). Then

$$\overline{[X_\alpha, X_\beta]} = [\overline{X}_\alpha, \overline{X}_\beta]. \tag{4.1.10}$$

The proof of Lemma 4.2 in the two-dimensional case is given further, in Section 4.4.2, Lemma 4.10. For the general case, see, e.g. [14], Section 4.5.2.

Lemma 4.3. The commutator (4.1.2) is compatible with the prolongation (3.3.22). Namely, let X and Y be two operators of the form (3.1.6),

$$X = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \quad Y = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

and let $X_{(1)}$, $Y_{(1)}$ and $[X,Y]_{(1)}$ be the first prolongations (3.3.22) of the operators X, Y and of their commutator $[X,Y]$, respectively. Then

$$[X,Y]_{(1)} = [X_{(1)}, Y_{(1)}]. \tag{4.1.11}$$

In words: *The prolongation of the commutator is equal to the commutator of prolongations.* The iteration of Eq. (4.1.11) yields the similar equation for higher-order prolongations:

$$[X,Y]_{(k)} = [X_{(k)}, Y_{(k)}]. \tag{4.1.12}$$

Lemma 4.3 is an infinitesimal consequence of the Proposition given in Preface. Let us illustrate Lemma 4.3 by means of the operators (3.5.15) from Example 4.1. We take the first prolongation of the operators (3.5.15):

$$X_{1(1)} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'}, \quad X_{2(1)} = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'},$$

and calculate their commutator to obtain:

$$[X_{1(1)}, X_{2(1)}] = -2x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - 2(y - xy')\frac{\partial}{\partial y'}.$$

Hence,

$$[X_{1(1)}, X_{2(1)}] = -2X_{1(1)}.$$

On the other hand, equation (4.1.4) yields

$$[X_1, X_2]_{(1)} = -2X_{1(1)}.$$

Thus, $[X_1, X_2]_{(1)} = [X_{1(1)}, X_{2(1)}]$, i.e. equation (4.1.11) is satisfied.

4.1.3 Properties of determining equations

We will discuss now the main properties of the determining equations (3.5.1). Note that Eqs. (3.5.1) provide a system of linear homogeneous partial differential equations for unknown coefficients $\xi^i(x,u)$, $\eta^i(x,u)$ of the operator X. Therefore we have the following property of the determining equations for any system of the differential equations (3.1.9),

$$F_\sigma\left(x, u^\alpha(x), \frac{\partial u^\alpha(x)}{\partial x^i}, \ldots, \frac{\partial^k u^\alpha(x)}{\partial x^{i_1} \cdots \partial x^{i_k}}\right) = 0, \quad \sigma = 1, \ldots, s.$$

Lemma 4.4. The set of all solutions X of the determining equations (3.5.1),

$$XF_\sigma(x, u, u_{(1)}, \ldots, u_{(k)})\Big|_{(3.1.8)} = 0, \quad \sigma = 1, \ldots, s,$$

is a vector space.

This vector space has the following specific property.

Lemma 4.5. If the operators X_1 and X_2 of the form (3.1.6) satisfy the determining equations (3.5.1) then the commutator

$$X = [X_1, X_2] \tag{4.1.13}$$

also solves Eqs. (3.5.1).

Proof. We will use the determining equations written in the form (3.5.2). Namely, let

$$X_1 F_\sigma = \phi_\sigma^\nu F_\nu, \quad X_2 F_\sigma = \varphi_\sigma^\nu F_\nu, \tag{4.1.14}$$

where X_1, X_2 denote the kth-order prolongations of the operators X_1, X_2. Equations (4.1.13), (4.1.12), and (4.1.14) yield:

$$XF_\sigma = X_1(\varphi_\sigma^\nu F_\nu) - X_2(\phi_\sigma^\nu F_\nu). \tag{4.1.15}$$

Writing

$$X_1(\varphi_\sigma^\nu F_\nu) = X_1(\varphi_\sigma^\nu) F_\nu + \varphi_\sigma^\nu X_1(F_\nu),$$
$$X_2(\phi_\sigma^\nu F_\nu) = X_2(\phi_\sigma^\nu) F_\nu + \phi_\sigma^\nu X_2(F_\nu)$$

and invoking Eqs. (4.1.14) we conclude from Eq. (4.1.15) that the determining equation (3.5.2) is satisfied for X :

$$XF_\sigma = \psi_\sigma^\nu F_\nu, \tag{4.1.16}$$

where

$$\psi_\sigma^\nu = X_1(\varphi_\sigma^\nu) - X_2(\phi_\sigma^\nu) + \varphi_\sigma^\mu \phi_\mu^\nu - \phi_\sigma^\mu \varphi_\mu^\nu.$$

4.1.4 Lie algebras

Definition 4.2. A vector space L of operators of the form (2.1.17),

$$X = \xi^i(x)\frac{\partial}{\partial x^i}$$

is called a *Lie algebra* if it is closed under the commutator (4.1.2), i.e. if $X, Y \in L$ implies $[X, Y] \in L$. The Lie algebra is denoted by the same letter L. The *dimension* $\dim L$ of the Lie algebra is identified with the dimension of the vector space L. We shall use the symbol L_r to denote an r-dimensional Lie algebra.

Consider a Lie algebra L_r of a finite dimension r. Let

$$X_\alpha = \xi_\alpha^i(x)\frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, r \tag{4.1.17}$$

be a basis of the vector space L_r. Then any $X \in L_r$ is a linear combination of the operators (4.1.17),

$$X = c^\alpha X_\alpha \equiv c^1 X_1 + \cdots + c^r X_r,$$

with constant coefficients c^α. We also say that L_r is *spanned* by the operators (4.1.17) or that L_r is the *linear span* of the operators (4.1.17).

In particular, $[X_\alpha, X_\beta] \in L_r$, and hence

$$[X_\alpha, X_\beta] = c_{\alpha\beta}^\gamma X_\gamma, \quad \alpha, \beta = 1, \ldots, r. \tag{4.1.18}$$

Conversely, given r linearly independent operators (4.1.17), their linear span is an r-dimensional Lie algebra if the relations (4.1.18) hold. This is an immediate consequence of the bilinearity of the commutator.

Thus, the *commutator relations* (4.1.18) furnish a simple test for the linear span of operators X_α to be a Lie algebra. The constant coefficients $c_{\alpha\beta}^\gamma$ are called the *structure constants* of the algebra L_r.

Lemma 4.4 and Lemma 4.5 from Section 4.1.3 prove the following basic property of the determining equations.

Theorem 4.1. The set of all solutions X of the determining equations (3.5.1) is a Lie algebra. It is called a *Lie algebra admitted* by the differential equations (3.1.9), or a *symmetry Lie algebra* for Eqs. (3.1.9).

Example 4.4. According to Example 3.7, the second-order ODE (3.5.14) has the two-dimensional symmetry Lie algebra L_2 spanned by the operators (3.5.15). The commutator relations (4.1.18) are given by Eq. (4.1.4), $[X_1, X_2] = -2X_1$. Hence, the structure constants of L_2 are

$$c_{12}^1 = -2, \quad c_{21}^1 = 2, \quad c_{12}^2 = c_{21}^2 = 0.$$

4.2 Basic properties

4.2.1 Notation

The linear span of any set of operators $X_1, \ldots, X_s$ will be denoted by $\langle X_1, \ldots, X_s \rangle$. For example, a Lie algebra L_r with the basis (4.1.17) is written

$$L_r = \langle X_1, \ldots, X_r \rangle. \tag{4.2.1}$$

Let L be a vector space, and let K and N be subspaces of L. The set of all operators $X + Y$ with $X \in K$ and $Y \in N$ is denoted by

$$K + N. \tag{4.2.2}$$

Likewise, the linear span of all commutators $[X, Y]$ with $X \in K$ and $Y \in N$ will be denoted by

$$[K, N]. \tag{4.2.3}$$

For example, the condition for L to be closed under commutation, i.e. to be a Lie algebra, is written

$$[L, L] \subset L. \tag{4.2.4}$$

4.2.2 Subalgebra and ideal

Definition 4.3. Let L be a Lie algebra. A subspace $K \subset L$ of the vector space L is called a *subalgebra* of the Lie algebra L if K is closed under commutation, i.e.

$$[K, K] \subset K. \tag{4.2.5}$$

According to the notation (4.2.3), equation (4.2.5) means that

$$[X, Y] \in K \quad \text{for all} \quad X, Y \in K. \tag{4.2.6}$$

One-dimensional subalgebras exist in any Lie algebra L. Indeed, any $X \in L$ spans a one-dimensional subalgebra with elements cX, $c = \text{const}$.

Existence of two-dimensional subalgebras in Lie algebras is guaranteed, in general, only in the complex domain.

Example 4.5. (See [14], Section 7.3.3). The Lie algebra L_3 spanned by

$$X_1 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = xy\frac{\partial}{\partial x} + (1+y^2)\frac{\partial}{\partial y},$$

$$X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

has the subalgebra $L_2 = \langle X, Y\rangle$, where

$$X = X_1, \quad Y = X_2 + iX_3$$

and the commutator relation has the form

$$[X,Y] = -iY.$$

Definition 4.4. A subalgebra K is called an *ideal* of L if

$$[K,L] \subset K. \tag{4.2.7}$$

According to the notation (4.2.3), equation (4.2.7) means that

$$[X,Y] \in K \quad \text{for all} \quad X \in K,\ Y \in L. \tag{4.2.8}$$

Example 4.6. Consider the Lie algebra $L_6 = \langle X_1, \ldots, X_6\rangle$ spanned by the operators (3.5.26). The commutator table (4.1.6) shows that the subalgebra $L_3 = \langle X_1, X_2, X_3\rangle$ is an ideal of L_6.

4.2.3 Derived algebras

Lemma 4.6. Let L be a Lie algebra. The linear span of the commutators $[X,Y]$ of all operators $X, Y \in L$ is a Lie algebra.

Definition 4.5. The Lie algebra spanned by the commutators $[X,Y]$ of all operators $X, Y \in L$ is called the *derived algebra* of the Lie algebra L. The derived algebra is denoted by L' or $L^{(1)}$. Thus,

$$L' = [L,L].$$

Higher derivatives are defined by induction:

$$L^{(s+1)} = (L^{(s)})' \equiv [L^{(s)}, L^{(s)}], \quad s = 1,2,\ldots.$$

Remark 4.1. Let L_r be a Lie algebra with a basis (4.1.17). Then the derived algebra L_r' is the linear span of the commutators $[X_\alpha, X_\beta]$ of the basis. In other words, L_r' is

spanned by the linearly independent operators that appear in the table of commutators for L_r.

Example 4.7. Let L_6 be the Lie algebra spanned by the operators (3.5.26). Table (4.1.6) and Remark 4.1 show that the derived algebra is the following four-dimensional Lie algebra:

$$L_6' = \langle X_1, X_2, X_3, X_4 \rangle.$$

Remark 4.2. A Lie algebra L is said to be *Abelian* if $L^{(1)} = 0$, i.e. if $[X,Y] = 0$ for all $X, Y \in L$. It is clear that a Lie algebra with a basis (4.1.17) is Abelian if and only if the basis operators commute, $[X_\alpha, X_\beta] = 0$.

4.2.4 Solvable Lie algebras

Definition 4.6. A Lie algebra L_r, $r < \infty$, is said to be solvable if there is a sequence

$$L_r \supset L_{r-1} \supset \cdots \supset L_1 \tag{4.2.9}$$

of subalgebras of respective dimensions $r, r-1, \ldots, 1$ such that L_k is an ideal in L_{k+1}, $k = 1, \ldots, r-1$.

Theorem 4.2. A Lie algebra L_r is solvable if and only if its derived algebra of a finite order s vanishes: $L_r^{(s)} = 0$, $0 < s < \infty$.

Proof. Let L_r be a Lie algebra such that $L_r^{(s)} = 0$. Consider the sequence of successive derived algebras:

$$L_r \supset L_r^{(1)} \supset L_r^{(2)} \cdots \supset L_r^{(s)} = 0. \tag{4.2.10}$$

By the definition of derived algebras, $L_r^{(k)}$ and subspaces of $L_r^{(k-1)}$ containing $L_r^{(k)}$ are ideals of $L_r^{(k-1)}$. Hence, one can construct a link in the chain (4.2.9) between $L_r^{(k-1)}$ and $L_r^{(k)}$ merely by adding to $L_r^{(k)}$ one operator from $L_r^{(k-1)}$, then one more, etc. if $\dim L_r^{(k)} < \dim L_r^{(k-1)} - 1$. Proceeding in this manner for all parts of (4.2.10), one accomplishes the construction of (4.2.9).

Conversely, let L_r be a solvable Lie algebra. It follows from the chain (4.2.9) that the derived algebra of order r is zero. Indeed,

$$L_r^{(r)} = \left(L_r^{(1)}\right)^{(r-1)} \subset L_{r-1}^{(r-1)} \subset \cdots \subset L_1^{(1)} = 0.$$

Example 4.8. Let us examine the Lie algebra L_6 spanned by symmetries (3.5.26). One finds the derived algebras by inspecting Table (4.1.6) and arrives at the sequence (4.2.10):

$$L_6 \supset L_6^{(1)} \supset L_6^{(2)} \supset L_6^{(3)} = 0,$$

where

$$L_6 = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle,$$
$$L_6^{(1)} = \langle X_1, X_2, X_3, X_4 \rangle,$$
$$L_6^{(2)} = \langle X_3 \rangle.$$

Hence, the Lie algebra L_6 is solvable. The chain (4.2.9),

$$L_6 \supset L_5 \supset \cdots \supset L_1,$$

given by the proof of Theorem 4.2, comprises the subalgebras

$$L_5 = \langle X_1, X_2, X_3, X_4, X_5 \rangle,$$
$$L_4 = \langle X_1, X_2, X_3, X_4 \rangle,$$
$$L_3 = \langle X_1, X_2, X_3 \rangle,$$
$$L_2 = \langle X_2, X_3 \rangle,$$
$$L_1 = \langle X_3 \rangle.$$

4.3 Isomorphism and similarity

Lie's group classification of ordinary differential equations is based on the enumeration of all possible Lie algebras (infinitesimal groups in Lie's terminology) in the (x,y) plane. In this enumeration, the algebras are maximally simplified by a proper choice of bases and by means of a suitable change of variables. Associated with these two types of simplifying transformations are two distinctly different notions: *isomorphic* and *similar* (or *equivalent*) Lie algebras.

4.3.1 Isomorphic Lie algebras

An isomorphism between Lie algebras is commonly defined as a linear mapping preserving the commutators in the Lie algebras in question. We will use the following equivalent definition.

Definition 4.7. Let L and K be two finite-dimensional Lie algebras such that $\dim L = \dim K$. The algebras L and K are said to be *isomorphic* if one can find bases (real or complex) in L and K such that the algebras have, in these bases, equal structure constants, i.e. the same table of commutators.

Example 4.9. (See [19], Chapter 21, §3, Example 1). Let L and K be the three-dimensional Lie algebras spanned by the operators

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{4.3.1}$$

and

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \sin x\frac{\partial}{\partial x} + \cos x\frac{\partial}{\partial y}, \quad Y_3 = \cos x\frac{\partial}{\partial x} - \sin x\frac{\partial}{\partial y}, \tag{4.3.2}$$

respectively. The operators (4.3.1) and (4.3.2) have the following commutators:

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3, \tag{4.3.3}$$

$$[Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = -Y_2, \quad [Y_2, Y_3] = -Y_1. \tag{4.3.4}$$

If we take in K the new basis

$$Y_1' = Y_1 + Y_3, \quad Y_2' = Y_2, \quad Y_3' = Y_1 - Y_3,$$

then the commutator table (4.3.4) takes the form (4.3.3). Hence, the Lie algebras L and K are isomorphic.

Example 4.10. Let L and K be the three-dimensional Lie algebras spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = y\frac{\partial}{\partial x}, \quad X_3 = xy\frac{\partial}{\partial x} + (1+y^2)\frac{\partial}{\partial y} \tag{4.3.5}$$

and

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y}, \quad Y_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \tag{4.3.6}$$

respectively. The operators (4.3.5) and (4.3.6) have the following commutators:

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1, \tag{4.3.7}$$

$$[Y_1, Y_2] = 0, \quad [Y_1, Y_3] = Y_1, \quad [Y_2, Y_3] = -Y_2. \tag{4.3.8}$$

Now the algebras L and K are not isomorphic over the reals, but they are isomorphic in the complex domain. Namely, if we make in L the complex change of the basis (see [19], Chapter 21, §3, Example 2)

$$X_1' = X_1 - iX_2, \quad X_2' = iX_1 - X_2, \quad X_3' = -iX_3,$$

i.e. take the following new basis:

$$X_1' = (1-iy)\frac{\partial}{\partial x}, \quad X_2' = (i-y)\frac{\partial}{\partial x}, \quad X_3' = -ixy\frac{\partial}{\partial x} - i(1+y^2)\frac{\partial}{\partial y},$$

then the commutator table (4.3.7) takes the form (4.3.8). Hence, the Lie algebras L and K are complex isomorphic.

4.3.2 Similar Lie algebras

Definition 4.8. The Lie algebras L and $\overline{L}$ are said to be *similar* (or *equivalent*) if one is obtained from the other by a change of variables (real or complex). It means that the operators

$$X = \xi^i(x)\frac{\partial}{\partial x^i}$$

and

$$\overline{X} = \xi'^i(x')\frac{\partial}{\partial x'^i}$$

of L and $\overline{L}$, respectively, are related by Eqs. (2.2.6) and (2.2.7), i.e. by

$$x'^i = x'^i(x), \quad \xi'^i = \xi^k\frac{\partial x'^i}{\partial x^k}, \qquad i = 1,\ldots,n. \tag{4.3.9}$$

Lemma 4.7. If two r-dimensional Lie algebras L_r and $\overline{L}_r$ with the same number of variables are similar, they are isomorphic.

Proof. Let the structure constants $c^\gamma_{\alpha\beta}$ of the algebra L_r be defined by relations (4.1.18), $[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta}X_\gamma$. Let bases of L_r and $\overline{L}_r$ be given by the operators

$$X_\alpha = \xi^i_\alpha(x)\frac{\partial}{\partial x^i} \quad \text{and} \quad \overline{X}_\alpha = \xi'^i_\alpha(x')\frac{\partial}{\partial x'^i}, \qquad \alpha = 1,\ldots,r, \tag{4.3.10}$$

respectively, where $\overline{X}_\alpha$ are obtained from X_α by transformation (4.3.9). Then Eq. (4.1.10) from Lemma 4.2 yields:

$$[\overline{X}_\alpha, \overline{X}_\beta] = \overline{[X_\alpha, X_\beta]} = \overline{c^\gamma_{\alpha\beta}X_\gamma} = c^\gamma_{\alpha\beta}\overline{X}_\gamma.$$

Thus, L_r and $\overline{L}_r$ have the same structure constants in the bases (4.3.10), and hence they are isomorphic.

Remark 4.3. The converse is not true: two Lie algebras may be isomorphic but not similar. Therefore, there exist, e.g. *four* non-similar two-dimensional Lie algebras L_2 in the plane, but only *two* non-isomorphic L_2.

It is precisely similarity (and not simply isomorphism) that is of use in group analysis as a criterion of reducibility of one differential equation to another by a suitable change of variables. Nonetheless, establishing isomorphism is important as a first (and necessary due to Lemma 4.7) step for the determination of similarity.

Let L_r and $\overline{L}_r$ be two isomorphic r-dimensional Lie algebras with the bases (4.3.10) in which L_r and $\overline{L}_r$ have equal structure constants. We shall consider here the case when $r > n$ and assume that

$$\operatorname{rank}(\xi^i_\alpha(x)) = \operatorname{rank}(\xi'^i_\alpha(x')) = n.$$

Renumbering the indices, if necessary, one can assume that this is the value of the rank of the square matrices (ξ_h^i), $(\xi_h'^i)$ $(i,h=1,\ldots,n)$. Then the following conditions hold:

$$\xi_p^i = \varphi_p^h \xi_h^i, \quad \xi_p'^i = \varphi_p'^h \xi_h'^i, \qquad i=1,\ldots,n;\ p=n+1,\ldots,r, \tag{4.3.11}$$

where φ_p^h and $\varphi_p'^h$ are functions of x and x', respectively, and where we sum over h from 1 to n. Under the assumptions made, we have the following theorem.

Theorem 4.3. A necessary and sufficient condition that the Lie algebras L_r and $\overline{L}_r$ with the same structure constants and the same number of variables x^i and x'^i be similar is that the functions φ_p^h, $\varphi_p'^h$ satisfy the equations

$$\varphi_p'^h(x') = \varphi_p^h(x), \quad h=1,\ldots,n;\ p=n+1,\ldots,r, \tag{4.3.12}$$

and that these equations do not lead to relations between the x's alone or the x''s alone.

Sketch of the proof. (for a detailed proof, see [9], §22) Recall that the similarity of the algebras L_r and $\overline{L}_r$ means that the coordinates of the bases (4.3.10) are related by (4.3.9), i.e. there is a change of variables $x'^i = \varphi^i(x)$, such that

$$\xi_\alpha'^i = \xi_\alpha^k \frac{\partial x'^i}{\partial x^k}, \quad i=1,\ldots,n;\ \alpha=1,\ldots,r. \tag{4.3.13}$$

Necessity. Let the algebras L_r and $\overline{L}_r$ be similar. Then (4.3.13) and (4.3.11) yield $n(r-n)$ equations (4.3.12). By construction, equations (4.3.12) are compatible. Therefore it is impossible to eliminate all variables x'^i from and arrive at relations between the x's alone, and vice versa.

Sufficiency. Let Eqs. (4.3.12) be consistent and not lead to relations between the x's alone or the x''s alone. Let us consider Eqs. (4.3.13) for $\alpha=1,\ldots,n$. Under the assumptions on the ranks, these equations can be solved for derivatives and written:

$$\frac{\partial x'^i}{\partial x^k} = \xi_h'^i(x')\xi_k^h(x), \quad i,k=1,\ldots,n, \tag{4.3.14}$$

where (ξ_k^h) is the square matrix inverse to (ξ_h^i), i.e. $\xi_k^h \xi_h^i = \delta_k^i$, $\xi_i^h \xi_l^i = \delta_l^h$ with $i,k,h,l = 1,\ldots,n$. It follows from the conditions (4.3.12) that the system of Eqs. (4.3.14) (and hence the system of Eqs. (4.3.13)) is integrable. This means that there is a change of variables that establishes the similarity of the algebras L_r and $\overline{L}_r$.

Determination of the change of variables. The similarity transformation $x'^i = \varphi^i(x)$ is determined by solving, with respect to the x''s, the mixed system of equations comprising the functional equations (4.3.12) and the differential equations (4.3.14), the latter being integrable in view of Eqs. (4.3.12).

Example 4.11. Let us check if the Lie algebra L_3 spanned by (4.3.1),

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},$$

is similar to the Lie algebra $\overline{L}_3$ spanned by

$$\overline{X}_1 = \frac{\partial}{\partial x'}, \quad \overline{X}_2 = x'\frac{\partial}{\partial x'} + \frac{y'}{2}\frac{\partial}{\partial y'}, \quad \overline{X}_3 = x'^2\frac{\partial}{\partial x'} + x'y'\frac{\partial}{\partial y'}. \tag{4.3.15}$$

The commutator tables of the operators (4.3.1) and (4.3.15) are identical. They have the form (4.3.3):

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3,$$

$$[\overline{X}_1, \overline{X}_2] = \overline{X}_1, \quad [\overline{X}_1, \overline{X}_3] = 2\overline{X}_2, \quad [\overline{X}_2, \overline{X}_3] = \overline{X}_3.$$

Hence, L_3 and $\overline{L}_3$ are isomorphic and have the same structure constants. Equations (4.3.11) written in the form

$$\xi_3^k = \sum_{h=1}^{2} \varphi^h \xi_h^k, \quad \xi_3'^k = \sum_{h=1}^{2} \varphi'^h \xi_h'^k$$

yield:

$$x^2 = \varphi^1 + x\varphi^2, \quad x'^2 = \varphi'^1 + x'\varphi'^2,$$
$$y^2 = \varphi^1 + y\varphi^2, \quad 2x'y' = y'\varphi'^2.$$

Hence,

$$\varphi^1 = -xy, \quad \varphi^2 = x + y, \quad \varphi'^1 = -x'^2, \quad \varphi'^2 = 2x'.$$

With these functions φ^h and φ'^h, equations (4.3.12) assume the form

$$x'^2 = xy, \quad 2x' = x + y,$$

whence, after eliminating x', one arrives at the relation $x - y = 0$. According to Theorem 4.3, L_3 and $\overline{L}_3$ are not similar (neither over the reals, nor over the complex numbers), though they are isomorphic.

4.4 Low-dimensional Lie algebras

4.4.1 One-dimensional algebras

Any one-dimensional Lie algebra L_1 in $\mathbb{R}^n$ is spanned by one operator X_1, and hence it is Abelian. Thus, according to Theorem 2.7 from Section 2.2.2, the structure and realization of L_1 can be taken in canonical variables to be

$$[X_1, X_1] = 0, \quad X_1 = \frac{\partial}{\partial x^1}. \tag{4.4.1}$$

4.4.2 Two-dimensional algebras in the plane

Let L_2 be a two-dimensional Lie algebra with a basis

$$X_1 = \xi_1(x,y)\frac{\partial}{\partial x} + \eta_1(x,y)\frac{\partial}{\partial y},$$
$$X_2 = \xi_2(x,y)\frac{\partial}{\partial x} + \eta_2(x,y)\frac{\partial}{\partial y}. \tag{4.4.2}$$

The structure of L_2 is defined by the commutator (4.1.3) which is written:

$$[X_1,X_2] = \Big(X_1(\xi_2) - X_2(\xi_1)\Big)\frac{\partial}{\partial x} + \Big(X_1(\eta_2) - X_2(\eta_1)\Big)\frac{\partial}{\partial y}. \tag{4.4.3}$$

The operators (4.4.2) may be linearly connected (see [17], Section 4.5),

$$\lambda_1(x,y)X_1 + \lambda_2(x,y)X_2 = 0,$$

or not. Namely, the operators X_1 and X_2 (4.4.2) are linearly connected if and only if the determinant

$$\det\begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} = \xi_1\eta_2 - \eta_1\xi_2 \tag{4.4.4}$$

vanishes. We will call the determinant (4.4.4) the *pseudoscalar product* of the operators (4.4.2) and denote it by $X_1 \vee X_2$:

$$X_1 \vee X_2 = \xi_1\eta_2 - \xi_2\eta_1. \tag{4.4.5}$$

Lemma 4.8. Let us introduce in L_2 a new basis

$$X_1' = \alpha_1 X_1 + \alpha_2 X_2, \quad X_2' = \beta_1 X_1 + \beta_2 X_2 \tag{4.4.6}$$

with constant coefficients α_1, α_2 and β_1, β_2 satisfying the condition

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0 \tag{4.4.7}$$

for the linear independence of X_1' and X_2'. Then the commutator (4.4.3) of the basic operators (4.4.2) undergoes the following transformation:

$$[X_1',X_2'] = (\alpha_1\beta_2 - \alpha_2\beta_1)\,[X_1,X_2]. \tag{4.4.8}$$

Proof. Since the commutator is bilinear and anticommutative, we have:

$$[X_1',X_2'] = \alpha_1\beta_2[X_1,X_2] + \alpha_2\beta_1[X_2,X_1] = (\alpha_1\beta_2 - \alpha_2\beta_1)\,[X_1,X_2].$$

Lemma 4.9. In any two-dimensional Lie algebra one can chose a basis X_1, X_2 such that

$$[X_1,X_2] = 0 \quad \text{or} \quad [X_1,X_2] = X_1. \tag{4.4.9}$$

Proof. According to Lemma 4.8, the equation $[X_1,X_2]=0$ does not depend on a choice of a basis in L_2 : if this equation is satisfied in one basis, it holds in any other basis. Therefore, let us consider the case $[X_1,X_2]\neq 0$. Let

$$[X_1,X_2]=\alpha_1X_1+\alpha_2X_2\neq 0 \tag{4.4.10}$$

with certain constant coefficients α_1,α_2. In order to obtain a new basis X_1',X_2' in which the second equation of (4.4.9) holds, $[X_1',X_2']=X_1'$, we set

$$X_1'=\alpha_1X_1+\alpha_2X_2 \tag{4.4.11}$$

and chose a new basis (4.4.6) with any constants β_1,β_2 satisfying the condition

$$\alpha_1\beta_2-\alpha_2\beta_1=1. \tag{4.4.12}$$

Then Eqs. (4.4.8), (4.4.12), (4.4.10), and (4.4.11) yield:

$$[X_1',X_2']=[X_1,X_2]=X_1'. \tag{4.4.13}$$

This is precisely the second equation of (4.4.9) written in the basis X_1',X_2'.

Remark 4.4. If in Eq. (4.4.10) $\alpha_2=0$, i.e. if Eq. (4.4.10) has the form

$$[X_1,X_2]=\alpha_1X_1,$$

one can satisfy Eq. (4.4.12) by taking $\beta_2=1/\alpha_1$ and $\beta_1=0$. Then a new basis (4.4.6) satisfying Eq. (4.4.13) is given by

$$X_1'=\alpha_1X_1,\quad X_2'=\frac{1}{\alpha_1}X_2.$$

If $\alpha_2\neq 0$, we take $\beta_1=-1/\alpha_2$, $\beta_2=0$ and, invoking Eq. (4.4.11), obtain the following basis satisfying Eq. (4.4.13):

$$X_1'=\alpha_1X_1+\alpha_2X_2,\quad X_2'=-\frac{1}{\alpha_2}X_1.$$

Lemma 4.10. (See Lemma 4.2 in Section 4.1.2). The commutator (4.4.3) is invariant under any change of variables

$$\bar{x}=\bar{x}(x,y),\quad \bar{y}=\bar{y}(x,y). \tag{4.4.14}$$

Namely, if

$$\overline{X}_1,\quad \overline{X}_2,\quad \overline{[X_1,X_2]}$$

denote the operators X_1, X_2, $[X_1,X_2]$ written in the new variables $\bar{x},\bar{y}$, then the following equation holds:

$$[\overline{X}_1,\overline{X}_2]=\overline{[X_1,X_2]}. \tag{4.4.15}$$

Proof. We know (see Eq. (2.1.20)) that an operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{4.4.16}$$

is written in the new variables (4.4.14) as follows:

$$\overline{X} = X\big(\bar{x}(x,y)\big)\frac{\partial}{\partial \bar{x}} + X\big(\bar{y}(x,y)\big)\frac{\partial}{\partial \bar{y}}. \tag{4.4.17}$$

We will write Eq. (2.1.19) in the form

$$\overline{X}\big(F(\bar{x},\bar{y})\big) = X\big(F(\bar{x}(x,y),\bar{y}(x,y))\big), \tag{4.4.18}$$

where the operator $\overline{X}$ in the left-hand side acts on a function of the variables $\bar{x}$, $\bar{y}$, whereas the operator X in the right-hand side acts on a composite function of the variables x, y. Equation (4.4.18) means that *the result of the action of X is independent on a choice of coordinates.*

After the transformation (4.4.17) the operators (4.4.2) become

$$\overline{X}_1 = \overline{\xi}_1\frac{\partial}{\partial \bar{x}} + \overline{\eta}_1\frac{\partial}{\partial \bar{y}}, \quad \overline{X}_2 = \overline{\xi}_2\frac{\partial}{\partial \bar{x}} + \overline{\eta}_2\frac{\partial}{\partial \bar{y}},$$

where

$$\begin{aligned} \overline{\xi}_1 &= X_1\big(\bar{x}(x,y)\big), \quad \overline{\eta}_1 = X_1\big(\bar{y}(x,y)\big), \\ \overline{\xi}_2 &= X_2\big(\bar{x}(x,y)\big), \quad \overline{\eta}_2 = X_2\big(\bar{y}(x,y)\big). \end{aligned} \tag{4.4.19}$$

The commutator (4.4.3) of $\overline{X}_1$ and $\overline{X}_2$ is written

$$[\overline{X}_1,\overline{X}_2] = \Big(\overline{X}_1(\overline{\xi}_2) - \overline{X}_2(\overline{\xi}_1)\Big)\frac{\partial}{\partial \bar{x}} + \Big(\overline{X}_1(\overline{\eta}_2) - \overline{X}_2(\overline{\eta}_1)\Big)\frac{\partial}{\partial \bar{y}}.$$

Invoking Eqs. (4.4.19) we have:

$$\overline{X}_1(\overline{\xi}_2) = \overline{X}_1 X_2\big(\bar{x}(x,y)\big).$$

Applying Eq. (4.4.18) to $F = X_2\big(\bar{x}(x,y)\big)$ we obtain:

$$\overline{X}_1(\overline{\xi}_2) = \overline{X}_1 X_2\big(\bar{x}(x,y)\big) = X_1 X_2\big(\bar{x}(x,y)\big).$$

Acting likewise we get:

$$\overline{X}_2(\overline{\xi}_1) = \overline{X}_2 X_1\big(\bar{x}(x,y)\big) = X_2 X_1\big(\bar{x}(x,y)\big).$$

Hence

$$\overline{X}_1(\overline{\xi}_2) - \overline{X}_2(\overline{\xi}_1) = [X_1,X_2]\big(\bar{x}(x,y)\big),$$

and likewise

$$\overline{X}_1(\overline{\eta}_2) - \overline{X}_2(\overline{\eta}_1) = [X_1,X_2]\big(\bar{y}(x,y)\big).$$

Finally:

$$[\overline{X}_1,\overline{X}_2] = [X_1,X_2]\big(\bar{x}(x,y)\big)\frac{\partial}{\partial\bar{x}} + [X_1,X_2]\big(\bar{y}(x,y)\big)\frac{\partial}{\partial\bar{y}}\cdot \tag{4.4.20}$$

Equation (4.4.17) written for $X = [X_1,X_2]$ demonstrates that the right-hand side of Eq. (4.4.17) is identical with $\overline{[X_1,X_2]}$ thus proving Eq. (4.4.15).

Corollary 4.1. The equations (4.4.9) are invariant under the change of variables (4.4.14). Indeed, application of Eq. (4.4.15) to Eqs. (4.4.9) yields

$$[\overline{X}_1,\overline{X}_2] = 0, \quad [\overline{X}_1,\overline{X}_2] = \overline{X}_1.$$

Lemma 4.11. The pseudoscalar product of (4.4.2) transforms under the change (4.4.6) of a basis in L_2 according to

$$X_1' \vee X_2' = (\alpha_1\beta_2 - \alpha_2\beta_1)\,(X_1 \vee X_2), \tag{4.4.21}$$

and under a change of variables (4.4.14) it undergoes the transformation

$$\overline{X}_1 \vee \overline{X}_2 = \frac{\partial(\bar{x},\bar{y})}{\partial(x,y)}\,(X_1 \vee X_2), \tag{4.4.22}$$

where

$$\frac{\partial(\bar{x},\bar{y})}{\partial(x,y)} = \frac{\partial\bar{x}(x,y)}{\partial x}\frac{\partial\bar{y}(x,y)}{\partial y} - \frac{\partial\bar{x}(x,y)}{\partial y}\frac{\partial\bar{y}(x,y)}{\partial x}$$

is the Jacobian of the change of variables (4.4.14).

Proof. It is manifest that the pseudoscalar product (4.4.5) is *bilinear*:

$$(\alpha X + \beta Y) \vee Z = \alpha(X \vee Z) + \beta(Y \vee Z),$$

$$X \vee (\alpha Y + \beta Z) = \alpha(X \vee Y) + \beta(X \vee Z),$$

and anticommutative:

$$X_2 \vee X_1 = -(X_1 \vee X_2), \quad X_1 \vee X_1 = 0, \quad X_2 \vee X_2 = 0.$$

Therefore the change (4.4.6) of the basis leads to Eq. (4.4.21):

$$\begin{aligned} X_1' \vee X_2' &= \alpha_1\beta_2\,(X_1 \vee X_2) + \alpha_2\beta_1\,(X_2 \vee X_1) \\ &= (\alpha_1\beta_2 - \alpha_2\beta_1)\,(X_1 \vee X_2). \end{aligned}$$

To prove Eq. (4.4.22), we rewrite Eqs. (4.4.19) in the form

$$\overline{\xi}_1 = \xi_1 \frac{\partial \bar{x}}{\partial x} + \eta_1 \frac{\partial \bar{x}}{\partial y}, \quad \overline{\eta}_1 = \xi_1 \frac{\partial \bar{y}}{\partial x} + \eta_1 \frac{\partial \bar{y}}{\partial y},$$

$$\overline{\xi}_2 = \xi_2 \frac{\partial \bar{x}}{\partial x} + \eta_2 \frac{\partial \bar{x}}{\partial y}, \quad \overline{\eta}_2 = \xi_2 \frac{\partial \bar{y}}{\partial x} + \eta_2 \frac{\partial \bar{y}}{\partial y},$$

substitute these expression in

$$\overline{X}_1 \vee \overline{X}_2 = \overline{\xi}_1 \overline{\eta}_2 - \overline{\xi}_2 \overline{\eta}_1$$

and obtain:

$$\overline{X}_1 \vee \overline{X}_2 = \xi_1 \eta_2 \Big(\frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{y}}{\partial y} - \frac{\partial \bar{x}}{\partial y} \frac{\partial \bar{y}}{\partial x} \Big) - \xi_2 \eta_1 \Big(\frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{y}}{\partial y} - \frac{\partial \bar{x}}{\partial y} \frac{\partial \bar{y}}{\partial x} \Big),$$

and hence Eq. (4.4.22).

Since the Jacobian of the change of variables (4.4.14) does not vanish, equation (4.4.22) implies the following statement.

Corollary 4.2. The equation

$$X_1 \vee X_2 \equiv \xi_1 \eta_2 - \xi_2 \eta_1 = 0 \tag{4.4.23}$$

is invariant under the change of variables (4.4.14).

Summarizing the results of Lemma 4.9 and Corollaries 4.1 and 4.2, we arrive at the following statement.

Theorem 4.4. All two-dimensional Lie algebras L_2 in the plane fall into *four types* determined by the following *canonical structures:*

$$\begin{aligned} &\text{I. } [X_1, X_2] = 0, && \xi_1 \eta_2 - \eta_1 \xi_2 \neq 0, \\ &\text{II. } [X_1, X_2] = 0, && \xi_1 \eta_2 - \eta_1 \xi_2 = 0, \\ &\text{III. } [X_1, X_2] = X_1, && \xi_1 \eta_2 - \eta_1 \xi_2 \neq 0, \\ &\text{IV. } [X_1, X_2] = X_1, && \xi_1 \eta_2 - \eta_1 \xi_2 = 0. \end{aligned} \tag{4.4.24}$$

Theorem 4.4 provides a basis for enumeration of all non-similar realizations of two-dimensional Lie algebras in the plane ([19], Chap. 18, §1; see also [14], Section 12.2.2). The result is as follows.

Theorem 4.5. A basis of any two-dimensional Lie algebras in the plane can be transformed by a change of variables

$$t = t(x, y), \quad u = u(x, y) \tag{4.4.25}$$

into the following *canonical forms* arranged according to the types (4.4.24):

$$\begin{aligned}
&\text{I. } X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u};\\
&\text{II. } X_1 = \frac{\partial}{\partial u}, \quad X_2 = t\frac{\partial}{\partial u};\\
&\text{III. } X_1 = \frac{\partial}{\partial u}, \quad X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u};\\
&\text{IV. } X_1 = \frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u}.
\end{aligned} \tag{4.4.26}$$

The corresponding variables t, u are called *canonical variables* for L_2.

Proof. We first transform one of basis operators to a translation generator by using Eqs. (4.4.14) and (4.4.17). Then we simplify the second operator of the basis. We will consider separately each type of Eqs. (4.4.24).

Type I. Let us choose new variables (4.4.14) such that one of basis operators becomes $X_1 = \partial/\partial\bar{x}$. Let

$$X_2 = \xi(\bar{x},\bar{y})\frac{\partial}{\partial\bar{x}} + \eta(\bar{x},\bar{y})\frac{\partial}{\partial\bar{y}}$$

be the second basic operator. Denoting $\bar{x},\bar{y}$ again by x,y, we can assume that we consider a Lie algebra L_2 of type I having a basis of the form

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

Then $[X_1,X_2] = 0$ yields $\xi_x = \eta_x = 0$, while the condition $\xi_1\eta_2 - \eta_1\xi_2 \neq 0$ implies that $\eta(y) \neq 0$. Thus, in any canonical variables for X_1, a basis of L_2 of type I has the form

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \xi(y)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y}, \quad \eta(y) \neq 0. \tag{4.4.27}$$

Canonical variables for X_1 can be determined in many different ways, which fact can be used for transforming X_2 into the required form. Let us find the general form of the transformations (4.4.25) leaving unaltered the form of the first operator in Eqs. (4.4.27), i.e.

$$X_1 = \frac{\partial t}{\partial x}\frac{\partial}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} = \frac{\partial}{\partial t}.$$

It follows that

$$\frac{\partial t}{\partial x} = 1, \quad \frac{\partial u}{\partial x} = 0,$$

whence

$$t = x + f(y), \quad u = g(y). \tag{4.4.28}$$

After this transformation, the second operator (4.4.27) becomes

$$X_2 = \Big(\xi(y) + \eta(y)f'(y)\Big)\frac{\partial}{\partial t} + \eta(y)g'(y)\frac{\partial}{\partial u}\cdot$$

It takes the form

$$X_2 = \frac{\partial}{\partial u}$$

if the functions f and g solve the equations

$$\xi(y) + \eta(y)f'(y) = 0, \quad \eta(y)g'(y) = 1.$$

We have arrived at the change of variables (4.4.25) given by

$$t = x - \int \frac{\xi(y)}{\eta(y)}\mathrm{d}y, \quad u = \int \frac{\mathrm{d}y}{\eta(y)} \tag{4.4.29}$$

transforming the operators (4.4.27) to the first canonical form of (4.4.26):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u}\cdot$$

Type II. Here we start by choosing the canonical variables x, y such that the first operator has the form $X_1 = \partial/\partial y$. Then we adopt the reasoning used in the previous case. The conditions

$$[X_1, X_2] = 0, \ \xi_1\eta_2 - \eta_1\xi_2 = 0$$

yield (cf. Eqs. (4.4.27)):

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \eta(x)\frac{\partial}{\partial y}\cdot \tag{4.4.30}$$

The most general transformation preserving the X_1 in Eq. (4.4.30) is obtained from Eqs. (4.4.28) merely by interchanging x and y as well as t and u:

$$t = f(x), \ u = y + g(x). \tag{4.4.31}$$

We set here $f(x) = \eta(x)$, $g(x) = 0$ to obtain the change of variables

$$t = \eta(x), \quad u = y \tag{4.4.32}$$

transforming the operators (4.4.30) to the canonical form II:

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = t\frac{\partial}{\partial u}\cdot$$

Type III. We take again $X_1 = \partial/\partial y$ and employ transformations (4.4.31). The conditions $[X_1, X_2] = X_1, \quad \xi_1\eta_2 - \eta_1\xi_2 \neq 0$ yield:

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \xi(x)\frac{\partial}{\partial x} + \big(y + \eta(x)\big)\frac{\partial}{\partial y}, \quad \xi(x) \neq 0. \tag{4.4.33}$$

After transformation (4.4.31) the second operator of Eqs. (4.4.33) becomes

$$X_2 = \xi(x)f'(x)\frac{\partial}{\partial t} + \big(y + \eta(x) + \xi(x)g'(x)\big)\frac{\partial}{\partial u}.$$

It becomes

$$X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \equiv f(x)\frac{\partial}{\partial t} + \big(y + g(x)\big)\frac{\partial}{\partial u}$$

if

$$\xi(x)f'(x) = f(x), \quad \xi(x)g'(x) + \eta(x) = g(x). \tag{4.4.34}$$

Integrating the linear first-order ordinary differential equations (4.4.34) for unknown functions f and g and substituting into Eqs. (4.4.31), we transform the operators (4.4.33) to the canonical form III:

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

Type IV. As in the previous case we obtain using the conditions

$$[X_1, X_2] = X_1, \quad \xi_1\eta_2 - \eta_1\xi_2 = 0$$

that

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \big(y + \eta(x)\big)\frac{\partial}{\partial y}. \tag{4.4.35}$$

It is transparent that the change of variables

$$t = x, \; u = y + \eta(x) \tag{4.4.36}$$

transforms (4.4.35) to the canonical form IV:

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u}.$$

Remark 4.5. For the practical construction of *canonical variables* there is no need to follow the procedure used in proving Theorem 4.5. One has merely to determine the type of L_2 by calculating the commutator $[X_1, X_2]$ and the pseudoscalar product $X_1 \vee X_2$ of the basis X_1, X_2. If $[X_1, X_2]$ does not vanish, but $[X_1, X_2] \neq X_1$, it is necessary to find by Lemma 4.9 and Remark 4.4 a new basis satisfying the equation $[X_1, X_2] = X_1$. Denoting the new basis again by X_1, X_2 and determining the type according to the table (4.4.24), one finds *canonical variables* t, u by solving (for each type) the following *compatible overdetermined* systems of *four* first-order partial differential equations (Eqs. (6.5.15) in [17]):

$$\begin{aligned}
&\text{I.} \quad X_1(t) = 1,\ X_2(t) = 0; \quad X_1(u) = 0,\ X_2(u) = 1.\\
&\text{II.} \quad X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = t.\\
&\text{III.} \quad X_1(t) = 0,\ X_2(t) = t; \quad X_1(u) = 1,\ X_2(u) = u.\\
&\text{IV.} \quad X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = u.
\end{aligned} \tag{4.4.37}$$

Thus, we have arrive at the following table of Lie's enumeration of the non-similar realizations of two-dimensional Lie algebras in the plane.

Table 4.4.1. Structure and non-similar realizations of L_2

Type	Structure of L_2	Canonical forms of L_2
I	$[X_1, X_2] = 0,\ \xi_1\eta_2 - \eta_1\xi_2 \neq 0$	$X_1 = \dfrac{\partial}{\partial x},\ X_2 = \dfrac{\partial}{\partial y}$
II	$[X_1, X_2] = 0,\ \xi_1\eta_2 - \eta_1\xi_2 = 0$	$X_1 = \dfrac{\partial}{\partial y},\ X_2 = x\dfrac{\partial}{\partial y}$
III	$[X_1, X_2] = X_1,\ \xi_1\eta_2 - \eta_1\xi_2 \neq 0$	$X_1 = \dfrac{\partial}{\partial y},\ X_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$
IV	$[X_1, X_2] = X_1,\ \xi_1\eta_2 - \eta_1\xi_2 = 0$	$X_1 = \dfrac{\partial}{\partial y},\ X_2 = y\dfrac{\partial}{\partial y}$

4.4.3 Three-dimensional algebras in the plane

Two *non-isomorphic possibilities*, $[X_1, X_2] \neq 0$ or $[X_1, X_2] = 0$, for two-dimensional Lie algebras L_2 mean that the first derivative $L_2^{(1)}$ has the dimension 1 or 0. In the case of three-dimensional Lie algebras L_3 the derived algebra $L_3^{(1)}$ may have the dimension 3, 2, 1, or 0. Lie (see [19], Chapter 21, §2 and §3) classified the non-isomorphic structures of L_3 according to the dimension of $L_3^{(1)}$ and obtained the following result.

Theorem 4.6. The structure of any three-dimensional algebra L_3 can be transformed, by a *complex* change of its basis X_1, X_2, X_3, to one and only one of the following seven forms, where A, B, C, and D indicate that the first derived algebra $L_3^{(1)}$ has the dimension 3, 2, 1, and 0, respectively:

A. (1) $[X_1, X_2] = X_1,\ [X_1, X_3] = 2X_2,\ [X_2, X_3] = X_3,$

B. (2) $[X_1, X_2] = 0,\ [X_1, X_3] = X_1,\ [X_2, X_3] = cX_2\ (c \neq 0, \neq 1),$

(2′) $[X_1, X_2] = 0,\ [X_1, X_3] = X_1,\ [X_2, X_3] = X_2,$

(3) $[X_1, X_2] = 0,\ [X_1, X_3] = X_1,\ [X_2, X_3] = X_1 + X_2,$

C. (4) $[X_1, X_2] = 0,\ [X_1, X_3] = X_1,\ [X_2, X_3] = 0,$

(5) $[X_1, X_2] = 0,\ [X_1, X_3] = 0,\ [X_2, X_3] = X_1,$

D. (6) $[X_1, X_2] = 0,\ [X_1, X_3] = 0,\ [X_2, X_3] = 0.$

Type (2′) is distinguished from type (2) due to its autonomous significance.

For two-dimensional Lie algebras, as well as for one-dimensional ones, the classification over the reals is the same as that over the complex numbers. In the case of the three-dimensional algebras the situation is different. L. Bianchi [5] investigated the non-isomorphic structures of L_3 in *the real domain*. He showed that, in addition to Lie's seven types given in Theorem 4.6, there exist in the real domain two more types that can be reduced to Lie's table by a complex change of their bases. Bianchi's classification is well known in the literature, see, e.g. [14], Section 37.3.8.

Lie's complex classification ([19], Chapter 22, §3) of all non-similar three-dimensional algebras in the plain is given by the following theorem.

Theorem 4.7. Any three-dimensional algebra of operators in two variables can be transformed, by a complex change of variables, to one and only one of the following 13 distinctly different *standard forms* of L_3.

A. The first derived algebra has dimension three :

$$(1)\ X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},\quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},$$

$$(2)\ X_1 = \frac{\partial}{\partial x},\quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$

$$(3)\ X_1 = \frac{\partial}{\partial y},\quad X_2 = y\frac{\partial}{\partial y},\quad X_3 = y^2\frac{\partial}{\partial y}.$$

B. The first derived algebra has dimension two :

$$(4)\ X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial y},\quad X_3 = x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}\ (c \neq 0, \neq 1),$$

$$(5)\ X_1 = \frac{\partial}{\partial y},\quad X_2 = x\frac{\partial}{\partial y},$$

$$X_3 = (1-c)x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\ (c \neq 0, \neq 1),$$

$$(6)\ X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial y},\quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$(7)\ X_1 = \frac{\partial}{\partial y},\quad X_2 = x\frac{\partial}{\partial y},\quad X_3 = y\frac{\partial}{\partial y},$$

$$(8)\ X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial y},\quad X_3 = (x+y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$(9)\ X_1 = \frac{\partial}{\partial y},\quad X_2 = x\frac{\partial}{\partial y},\quad X_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

C. The first derived algebra has dimension one :

$$(10)\ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x},$$

$$(11)\ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$(12)\ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y}.$$

D. The first derived algebra has dimension zero :

$$(13)\ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = p(x)\frac{\partial}{\partial y}.$$

Remark 4.6. Classification over the reals provides ([21], see also [12], Chapter 8 written by N. H. Ibragimov and F. M. Mahomed) the additional L_3 :

$$(14)\ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$
$$X_3 = 2xy\frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial}{\partial y},$$

$$(15)\ X_1 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$X_3 = xy\frac{\partial}{\partial x} + (1+y^2)\frac{\partial}{\partial y},$$

$$(16)\ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (bx+y)\frac{\partial}{\partial x} + (by-x)\frac{\partial}{\partial y},$$

$$(17)\ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = (1+x^2)\frac{\partial}{\partial x} + (x+b)y\frac{\partial}{\partial y}.$$

4.4.4 Three-dimensional algebras in $\mathbb{R}^3$

We will consider now 1, 2, and 3-dimensional Lie algebras of operators (2.1.17) in the three-dimensional space $\mathbb{R}^3$ of variables x, y, z. For the sake of brevity, we will use the notation

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \partial_z = \frac{\partial}{\partial z}$$

and write the operator (2.1.17) in the form

$$X = \xi(x,y,z)\,\partial_x + \eta(x,y,z)\,\partial_y + \zeta(x,y,z)\,\partial_z. \tag{4.4.38}$$

First we summarize, in the notation (4.4.38), the results on Lie algebras in the (x,y) plane discussed in Sections 4.4.1, 4.4.2 and 4.4.3. According to Eq. (4.4.1), all one-dimensional Lie algebras are similar to L_1 spanned by

$$X_1 = \partial_x . \tag{4.4.39}$$

Furthermore, all two-dimensional Lie algebras in the plane are similar to one of four non-similar algebras L_2 with the following bases (see Table 4.4.1):

$$\begin{aligned} &\text{I.} && X_1 = \partial_x, \quad X_2 = \partial_y ; \\ &\text{II.} && X_1 = \partial_y, \quad X_2 = x\partial_y ; \\ &\text{III.} && X_1 = \partial_y, \quad X_2 = x\partial_x + y\partial_y ; \\ &\text{IV.} && X_1 = \partial_y, \quad X_2 = y\partial_y . \end{aligned} \tag{4.4.40}$$

Finally, the non-similar L_3 in the real domain are given by Eqs. (1)–(17) from Theorem 4.7 and Remark 4.6.

Now we return to the three dimensional space $\mathbb{R}^3$ and provide the result of classification of L_3 *in the real domain* given in [18]. The Lie algebras in $\mathbb{R}^3$ of dimensions 1 and 2 are again given by (4.4.39) and (4.4.40). The algebras of dimension 3 are enumerated in Table 4.4.2.

Table 4.4.2. Non-similar three-dimensional real Lie algebras in $\mathbb{R}^3$

1	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = \partial_y$
2	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = y\partial_x + a(y)\partial_z$
3	$X_1 = \partial_z, \quad X_2 = x\partial_z, \quad X_3 = y\partial_z$
4	$X_1 = \partial_z, \quad X_2 = x\partial_z, \quad X_3 = a(x)\partial_z, \quad a''(x) \neq 0$
5	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = \partial_y + x\partial_z$
6	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = y\partial_x + x\partial_z$
7	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = x\partial_z$
8	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = qx\partial_x + \partial_y + z\partial_z$
9	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = qx\partial_x + z\partial_z$
10	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = y\partial_x + z\partial_z,$
11	$X_1 = \partial_z, \quad X_2 = x\partial_z, \quad X_3 = (1-q)x\partial_x + z\partial_z$
12	$X_1 = \partial_z, \quad X_2 = x\partial_z, \quad X_3 = \partial_y + z\partial_z,$
13	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = x\partial_x + \partial_y + (z+x)\partial_z$
14	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = x\partial_x + (z+x)\partial_z$
15	$X_1 = \partial_z, \quad X_2 = x\partial_z, \quad X_3 = \partial_x - z\partial_z - \varepsilon\partial_y, \ \varepsilon = 0 \text{ or } 1$
16	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = (qx+z)\partial_x + \partial_y - x\partial_z$
17	$X_1 = \partial_z, \quad X_2 = \partial_x, \quad X_3 = (qx+z)\partial_x - x\partial_z$
18	$X_1 = \partial_z, \quad X_2 = x\partial_z, \quad X_3 = (x^2 - qx + 1)\partial_x + xz\partial_z$

Continued

19	$X_1 = \partial_z, \quad X_2 = \partial_x + z\partial_z, \quad X_3 = 2z\partial_x + e^x\partial_y + z^2\partial_z$
20	$X_1 = \partial_z, \quad X_2 = \partial_x + z\partial_z, \quad X_3 = 2z\partial_x + (z^2 \pm e^{2x})\partial_z$
21	$X_1 = \partial_z, \quad X_2 = z\partial_z, \quad X_3 = z^2\partial_z$
22	$X_1 = \partial_z, \quad X_2 = \sin z\partial_x + \dfrac{\cos z}{\cos x}\partial_y - \tan x \cos z\partial_z,$ $X_3 = \cos z\partial_x - \dfrac{\sin z}{\cos x}\partial_y + \tan x \sin z\partial_z$
23	$X_1 = \partial_z, \quad X_2 = \sin z\partial_x - \tan x \cos z\partial_z, \quad X_3 = \cos z\partial_x + \tan x \sin z\partial_z$

4.5 Lie algebras and multi-parameter groups

4.5.1 Definition of multi-parameter groups

We will use the notation and assumptions of Section 2.1.1, but the parameter a will be replaced now by the vector-parameter $a = (a^1, \ldots, a^r)$. We consider invertible transformations $T_a : \mathbb{R}^n \to \mathbb{R}^n$,

$$\bar{x} = f(x, a), \tag{4.5.1}$$

defined in a neighborhood of

$$a = 0 \equiv (0, \ldots, 0).$$

Here

$$f = (f^1, \ldots, f^n)$$

is a vector valued function whose components $f^i = f^i(x, a)$ are at least three times continuously differentiable with respect to the variables

$$x^1, \ldots, x^n, a^1, \ldots, a^r.$$

We impose the initial condition:

$$f\big|_{a=0} = x. \tag{4.5.2}$$

Definition 4.9. A set of transformations (4.5.1) is called an *r-parameter local group* G_r if

$$f(f(x, a), b) = f(x, c) \tag{4.5.3}$$

for all values a and b of the parameter sufficiently close to $a = 0$. Here $c = c(a, b)$ is a vector-function with components

$$c^\alpha = \phi^\alpha(a, b), \quad \alpha = 1, \ldots, r, \tag{4.5.4}$$

defined and thrice continuously differentiable for sufficiently small a and b (in the same sense as in Definition 2.1). For brevity, G_r is termed a *multi-parameter group*.

The functions (4.5.4) define a composition law in the group G_r. It is assumed that the system of equations

$$\phi^{\alpha}(a,b)=0, \quad \alpha=1,\dots,r \tag{4.5.5}$$

has a unique solution $b=(b^1,\dots,b^r)$ for any small a. Given a, the solution b of the system (4.5.5) is denoted by a^{-1}. Hence, the inverse transformation T_a^{-1} is

$$x=f(\bar{x},a^{-1}).$$

4.5.2 Construction of multi-parameter groups

The following theorem is sufficient to meet the needs of applied group analysis when the practical construction of multi-parameter groups is desired.

Theorem 4.8. Let L_r be an r-dimensional vector space spanned by the operators

$$X_{\alpha}=\xi_{\alpha}^{i}(x)\frac{\partial}{\partial x^{i}}, \quad \alpha=1,\dots,r. \tag{4.5.6}$$

The composition $T_a=T_{a^r}\cdots T_{a^1}$ of r one-parameter groups of transformations $T_{a^{\alpha}}$ generated individually by each of the base operators X_{α} via the Lie equations

$$\frac{\mathrm{d}\bar{x}^{i}}{\mathrm{d}a^{\alpha}}=\xi_{\alpha}^{i}(\bar{x}), \quad \bar{x}^{i}\big|_{a^{\alpha}=0}=x^{i}, \quad i=1,\dots,n, \tag{4.5.7}$$

is an r-parameter (local) group G_r if and only if L_r is a Lie algebra. By applying the same construction to any s-dimensional subalgebra of L_r, one generates an s-parameter subgroup of the group G_r.

Remark 4.7. The above construction of G_r depends upon the choice of a basis in L_r. Therefore, the theory of Lie groups considers more general constructions and different representations of Lie groups with a given Lie algebra. However, all these representations are equivalent (similar).

Example 4.12. Consider a three-dimensional Lie algebra spanned by

$$X_1=\frac{\partial}{\partial x}, \quad X_2=\frac{\partial}{\partial y}, \quad X_3=y\frac{\partial}{\partial x}.$$

Solution of the Lie equations (4.5.7) for these operators provides the following three one-parameter groups with the respective parameters a^1,a^2,a^3 :

$$T_{a^1}: \bar{x} = x + a^1, \qquad \bar{y} = y;$$
$$T_{a^2}: \bar{x} = x, \qquad \bar{y} = y + a^2;$$
$$T_{a^3}: \bar{x} = x + ya^3, \quad \bar{y} = y.$$

Their composition

$$T_a = T_{a^3} T_{a^2} T_{a^1},$$

where $a = (a^1, a^2, a^3)$, has the form:

$$\begin{aligned} \bar{x} &= x + ya^3 + a^1 + a^2 a^3, \\ \bar{y} &= y + a^2. \end{aligned} \tag{4.5.8}$$

The consecutive application of T_a and T_b, where $b = (b^1, b^2, b^3)$, yields the transformation $T_b T_a$:

$$\begin{aligned} \bar{\bar{x}} &= x + y(a^3 + b^3) + a^2(a^3 + b^3) + b^2 b^3 + a^1 + b^1, \\ \bar{\bar{y}} &= y + a^2 + b^2. \end{aligned}$$

The equation $T_b T_a = T_c$ leads to the equations

$$\begin{aligned} \bar{\bar{x}} &= x + yc^3 + c^1 + c^2 c^3, \\ \bar{\bar{y}} &= y + c^2, \end{aligned}$$

whence

$$\begin{aligned} c^1 &= a^1 + b^1 - b^2 a^3, \\ c^2 &= a^2 + b^2, \\ c^3 &= a^3 + b^3. \end{aligned}$$

Thus, the transformation (4.5.8) provides, in accordance with Theorem 4.8, a three-parameter group with the composition law (4.5.4) given by the following equations:

$$\begin{aligned} \phi^1(a,b) &= a^1 + b^1 - b^2 a^3, \\ \phi^2(a,b) &= a^2 + b^2, \\ \phi^3(a,b) &= a^3 + b^3. \end{aligned} \tag{4.5.9}$$

Example 4.13. Let L_2 be the two-dimensional vector space spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y}. \tag{4.5.10}$$

The operators X_1 and X_2 generate the following two one-parameter groups:

$$T_{a^1}: \bar{x} = x + a^1, \qquad \bar{y} = y;$$
$$T_{a^2}: \bar{x} = x, \qquad \bar{y} = y + a^2 x.$$

Their composition

$$T_a = T_{a^2} T_{a^1},$$

where $a = (a^1, a^2)$, has the form:

$$\begin{aligned} \bar{x} &= x + a^1, \\ \bar{y} &= y + xa^2 + a^1 a^2. \end{aligned} \tag{4.5.11}$$

The consecutive application of T_a and T_b, where $b = (b^1, b^2)$, yields the transformations $T_b T_a$:

$$\begin{aligned} \bar{\bar{x}} &= x + a^1 + b^1, \\ \bar{\bar{y}} &= y + (x + a^1)(a^2 + b^2) + b^1 b^2. \end{aligned} \tag{4.5.12}$$

The equation $T_b T_a = T_c$ implies that

$$\begin{aligned} \bar{\bar{x}} &= x + c^1, \\ \bar{\bar{y}} &= y + xc^2 + c^1 c^2. \end{aligned} \tag{4.5.13}$$

Equations (4.5.12) and (4.5.13) yield

$$\begin{aligned} c^1 &= a^1 + b^1, \\ c^2 &= a^2 + b^2, \\ c^1 c^2 &= a^1 (a^2 + b^2) + b^1 b^2. \end{aligned} \tag{4.5.14}$$

Substitution of c^1 and c^2 given by the first two equations from (4.5.14) into the third equation yields $b^1 a^2 = 0$. Since the latter equation cannot be satisfied for arbitrary a and b, the two-parameter family of transformations (4.5.11) does not satisfy the group property (4.5.3). This conclusion agrees with Theorem 4.8 because the vector space L_2 spanned by the operators (4.5.10) is not a Lie algebra.

Exercises to Chapter 4

Exercise 4.1. Make the commutator table for the operators (3.5.11).

Exercise 4.2. Check Eq. (4.1.11) for the operators from Table (4.1.5).

Exercise 4.3. Make the commutator table for the operators (3.5.11) and verify that these operators span an 8-dimensional Lie algebra.

Exercise 4.4. Consider again the operators (3.5.11). Verify the Jacobi identity (4.1.9) by taking $X = X_2$, $Y = X_4$, $Z = X_7$.

Exercise 4.5. Find the structure constants of the three-dimensional Lie algebra L_3 spanned by the operators (3.5.13).

Exercise 4.6. Prove Lemma 4.6.

Exercise 4.7. Show that the derived algebra L' is an ideal of L.

Exercise 4.8. Make the commutator table for the Lie algebra L_5 spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},$$

$$X_4 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}, \quad X_5 = 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$$

and find the structure constants. Investigate L_5 for solvability.

Exercise 4.9. Investigate for solvability the Lie algebra L_3 spanned by

$$X_1 = \left(1+x^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$

$$X_2 = xy\frac{\partial}{\partial x} + \left(1+y^2\right)\frac{\partial}{\partial y},$$

$$X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Exercise 4.10. Using the commutator table (4.1.5) and Theorem 4.2 show that the Lie algebra $L_3 = \langle X_1, X_2, X_3\rangle$ spanned by the operators (3.5.13) is solvable and construct the chain (4.2.9), $L_3 \supset L_2 \supset L_1$.

Exercise 4.11. Find the derived algebra L_3' and determine dim L_3' for the Lie algebra L_3 spanned by the operators (15) from Remark 4.6:

$$X_1 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

$$X_3 = xy\frac{\partial}{\partial x} + (1+y^2)\frac{\partial}{\partial y}.$$

Exercise 4.12. Find the derived algebra L_3' and determine dim L_3' for the Lie algebra L_3 spanned by the operators (16) from Remark 4.6:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (bx+y)\frac{\partial}{\partial x} + (by-x)\frac{\partial}{\partial y}.$$

Exercise 4.13. Construct the commutator table and find the derived algebra for the Lie algebra L_3 spanned by the operators (23) from Table 4.4.2:

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = \sin z \frac{\partial}{\partial x} - \tan x \cos z \frac{\partial}{\partial z},$$

$$X_3 = \cos z \frac{\partial}{\partial x} + \tan x \sin z \frac{\partial}{\partial z}.$$

Exercise 4.14. Determine the number of linearly unconnected operators among the operators X_1, X_2, X_3 from Exercise 4.13. (See [17], Section 4.5, Definition 4.5.1 and Lemma 4.5.1.)

Exercise 4.15. Construct the commutator table and find the derived algebra for the Lie algebra L_3 spanned by the operators (22) from Table 4.4.2:

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = \sin z \frac{\partial}{\partial x} + \frac{\cos z}{\cos x} \frac{\partial}{\partial y} - \tan x \cos z \frac{\partial}{\partial z},$$

$$X_3 = \cos z \frac{\partial}{\partial x} - \frac{\sin z}{\cos x} \frac{\partial}{\partial y} + \tan x \sin z \frac{\partial}{\partial z}.$$

Exercise 4.16. Determine the number of linearly unconnected operators among the operators X_1, X_2, X_3 from Exercise 4.15.

Chapter 5
Galois groups via symmetries

This chapter provides a bridge between Lie symmetry groups for differential equations and Galois groups for algebraic equations. Namely, the Galois groups are constructed as the restriction of Lie symmetries of algebraic equations on the roots of the equations under consideration. The approach is illustrated by several examples.

5.1 Preliminaries

We will construct the Galois groups for several simple algebraic equations by first calculating their Lie symmetries and then restricting the symmetry group to the roots of the equation in question. This procedure can be applied to differential equations as well. As a result we obtain another representation of Lie symmetries called the Galois representation of symmetries of differential equations (see [14], pp. 251–254, [16] and the references therein).

The approach will be illustrated by applying it to the following equations:

$$x^2+1=0, \tag{5.1.1}$$

$$x^4-x^2+1=0, \tag{5.1.2}$$

$$x^4+x^3+x^2+x+1=0. \tag{5.1.3}$$

It is known from the literature on Galois groups that the Galois group $\mathscr{G}$ of Eq. (5.1.1) is

$$\mathscr{G}=\{1,(x_1,x_2)\}, \tag{5.1.4}$$

where x_1, x_2 are the roots of Eq. (5.1.1), (x_1,x_2) is the permutation of the roots, and 1 is the identical permutation (unit of the group). It is also known that the Galois groups of Eqs. (5.1.2) and (5.1.3) are

$$\mathcal{G} = \{1, (x_1,x_2)(x_3,x_4), (x_1,x_3)(x_2,x_4), (x_1,x_4)(x_2,x_3)\} \tag{5.1.5}$$

and

$$\mathcal{G} = \{1, (x_1,x_2,x_4,x_3), (x_1,x_3,x_4,x_2), (x_1,x_4)(x_2,x_3)\}, \tag{5.1.6}$$

respectively. In Eq. (5.1.5), x_1, x_2, x_3, x_4 denote the roots of Eq. (5.1.2), and in Eq. (5.1.6) the roots of Eq. (5.1.3).

5.2 Symmetries of algebraic equations

5.2.1 Determining equation

Let us consider algebraic equations of the nth degree:

$$P_n(x) \equiv C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_{n-1}x + C_n = 0. \tag{5.2.1}$$

The symmetries of algebraic equations are defined as in the case of differential equations. Namely, let a transformation

$$\bar{x} = f(x) \tag{5.2.2}$$

convert an equation (5.2.1) of the nth degree into an algebraic equation

$$\overline{P}_n(\bar{x}) \equiv \bar{C}_0\bar{x}^n + \bar{C}_1\bar{x}^{n-1} + \bar{C}_2\bar{x}^{n-2} + \cdots + \bar{C}_{n-1}\bar{x} + \bar{C}_n = 0. \tag{5.2.3}$$

In general, the coefficients $\bar{C}_i$ in Eq. (5.2.3) will not coincide with the coefficients C_i in the original equation (5.2.1). If they coincide, we will say that the transformation (5.2.2) is a symmetry (a Lie symmetry) of Eq. (5.2.1). We can define the symmetry of algebraic equations as follows.

Definition 5.1. The transformation (5.2.2) is called a symmetry of Eq. (5.2.1) if Eq. (5.2.3) coincides with the equation $P_n(\bar{x}) = 0$ whenever x solves Eq. (5.2.1).

The *determining equation* for symmetries of algebraic equations can also be written, as for differential equations, either in the form (2.3.3):

$$P_n(\bar{x})\big|_{P_n(x)=0} = 0, \tag{5.2.4}$$

or in the form (2.3.8):

$$P_n(\bar{x}) = \mu(x)P_n(x), \tag{5.2.5}$$

where

$$P_n(\bar{x}) = C_0\bar{x}^n + C_1\bar{x}^{n-1} + C_2\bar{x}^{n-2} + \cdots + C_{n-1}\bar{x} + C_n.$$

The requirement that the transformation (5.2.2) maps any algebraic equation into an algebraic equation is rather restrictive. In particular, if one considers only

uniquely invertible transformations, then one can show that the general form of invertible transformations (5.2.2) converting *every* equation (5.2.1) into an algebraic equation (5.2.3) is provided by the linear fractional transformation

$$\bar{x} = \frac{ax+\varepsilon}{b+\delta x} \tag{5.2.6}$$

with complex coefficients satisfying the invertibility condition

$$ab - \varepsilon\delta \neq 0. \tag{5.2.7}$$

If we do not require existence of the uniquely determined inverse transformation, we can use transformations (5.2.2) given by the rational fractions

$$\bar{x} = \frac{A_0x^r + A_1x^{r-1} + \cdots + A_r}{B_0x^s + B_1x^{s-1} + \cdots + B_s}. \tag{5.2.8}$$

The transformation (5.2.8) maps algebraic equations into algebraic equations (in general, not of the same degree). It was considered by E. W. Tschirnhausen in 1683 and is known as *Tschirnhausen's transformation.*

In what follows, we will use the following simple result concerning Eqs. (5.2.1) that are symmetric in their coefficients.

Lemma 5.1. Equations (5.2.1) whose coefficients satisfy the conditions

$$C_n = C_0, \quad C_{n-1} = C_1, \quad \ldots, C_i = C_{n-i} \tag{5.2.9}$$

have the symmetry

$$\bar{x} = \frac{1}{x}. \tag{5.2.10}$$

Proof. Indeed,

$$\begin{aligned} P_n(\bar{x}) &= C_0\frac{1}{x^n} + C_1\frac{1}{x^{n-1}} + \cdots + C_{n-1}\frac{1}{x} + C_n \\ &= x^{-n}\left(C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n\right). \end{aligned}$$

Therefore the conditions (5.2.9) yield

$$P_n(\bar{x}) = x^{-n}P_n(\bar{x}).$$

Hence, the determining equation (5.2.5) is satisfied with $\mu(x) = x^{-n}$.

5.2.2 *First example*

Let us find the symmetries of the linear fractional form (5.2.6) for the quadratic equation (5.1.1),

$$x^2+1=0.$$

The determining equation (5.2.4) is written

$$(\bar{x}^2+1)\big|_{x^2=-1}=0. \tag{5.2.11}$$

Substituting (5.2.6) in $\bar{x}^2+1$, we have:

$$\bar{x}^2+1=(\delta x+b)^{-2}\big[(a^2+\delta^2)x^2+2(a\varepsilon+b\delta)x+b^2+\varepsilon^2\big].$$

Therefore the determining equation (5.2.11) becomes

$$2(a\varepsilon+b\delta)x+b^2+\varepsilon^2-a^2-\delta^2=0$$

and yields the following system of two equations:

$$a\varepsilon+b\delta=0,\quad b^2+\varepsilon^2-a^2-\delta^2=0. \tag{5.2.12}$$

In the case $\delta=0$ we obtain, using the first equation of (5.2.12) and the condition (5.2.7), that $\varepsilon=0$. Then the second equation of (5.2.12) becomes $b=\pm a$. Hence, we have obtained the following two symmetries of the form (5.2.6) with $\delta=0$:

$$\bar{x}=x \quad \text{and} \quad \bar{x}=-x. \tag{5.2.13}$$

If $\delta\neq 0$, equations (5.2.12) are written:

$$b=-\frac{a\varepsilon}{\delta},\quad (a^2+\delta^2)(\varepsilon^2-\delta^2)=0.$$

These equations together with the condition (5.2.7) yield $\varepsilon^2-\delta^2=0$. Hence,

$$\delta=\varepsilon,\ b=-a,\quad \text{or}\quad \delta=-\varepsilon,\ b=a,$$

where $\varepsilon\neq 0$. Thus, we have two types of transformations:

$$\bar{x}=\frac{ax+\varepsilon}{a-\varepsilon x}\quad \text{and}\quad \bar{x}=\frac{ax+\varepsilon}{\varepsilon x-a}. \tag{5.2.14}$$

If $a=0$, these transformations reduce to (see also Lemma 5.1)

$$\bar{x}=\frac{1}{x}\quad \text{and}\quad \bar{x}=-\frac{1}{x}. \tag{5.2.15}$$

If $a\neq 0$, the first transformation of (5.2.14) provides a one-parameter local group:

$$T_\alpha:\ \bar{x}=\frac{x+\alpha}{1-\alpha x}, \tag{5.2.16}$$

while the second transformation of (5.2.14) can be written in the form

$$S_\beta:\ \bar{x}=\frac{x+\beta}{\beta x-1}. \tag{5.2.17}$$

Note that (5.2.13) are obtained from (5.2.16)–(5.2.17) by letting $\alpha=\beta=0$, whereas (5.2.15) can be obtained from (5.2.16)–(5.2.17) by letting $\alpha=\beta=\infty$. The transformations (5.2.16)–(5.2.17) form a group. Indeed, the composition $S_\beta\circ T_\alpha$ acts as follows:

$$S_\beta\big(T_\alpha(x)\big)=\frac{\frac{x+\alpha}{1-\alpha x}+\beta}{\beta\frac{x+\alpha}{1-\alpha x}-1}=\frac{(1-\alpha\beta)x+\alpha+\beta}{(\alpha+\beta)x+\alpha\beta-1}=\frac{x+\frac{\alpha+\beta}{1-\alpha\beta}}{\frac{\alpha+\beta}{1-\alpha\beta}x-1}.$$

Hence,

$$S_\beta\circ T_\alpha=S_\gamma,\quad \gamma=\frac{\alpha+\beta}{1-\alpha\beta}.$$

The similar calculations show that

$$T_\alpha\circ S_\beta=S_\delta,\quad \delta=\frac{\beta-\alpha}{1+\alpha\beta}.$$

Other group properties are obviously satisfied. Thus, we have obtained the following result.

Theorem 5.1. The group of symmetries of the linear fractional form (5.2.6) for Eq. (5.1.1) is generated by the transformations (5.2.16)–(5.2.17), where the parameters α and β range over the extended complex plane.

5.2.3 Second example

Let us find the symmetries of the form (5.2.6) for Eq. (5.1.2),

$$x^4-x^2+1=0.$$

The determining equation (5.2.4) is written

$$(\bar{x}^4-\bar{x}^2+1)\big|_{x^4=x^2-1}=0.$$

Inspecting this equation as in the first example, we obtain the following four symmetries of the form (5.2.6):

$$I:\ \bar{x}=x;\quad S:\ \bar{x}=-x;\quad R:\ \bar{x}=\frac{1}{x};\quad T:\ \bar{x}=-\frac{1}{x}. \tag{5.2.18}$$

Let us verify that, e.g. the transformation R is a symmetry. We have:

$$\bar{x}^4-\bar{x}^2+1=\frac{1}{x^4}-\frac{1}{x^2}+1=\frac{1}{x^4}\Big(x^4-x^2+1\Big).$$

It follows that $\bar{x}^4 - \bar{x}^2 + 1 = 0$ whenever $x^4 - x^2 + 1 = 0$.

The transformations (5.2.18) form a group. Indeed:

$$S \circ R = R \circ S = T, \quad S \circ T = T \circ S = R, \quad T \circ R = R \circ T = S$$

and $S^{-1} = S$, $R^{-1} = R$, $T^{-1} = T$.

5.2.4 *Third example*

Consider now Eq. (5.1.3),

$$P_4(x) \equiv x^4 + x^3 + x^2 + x + 1 = 0. \tag{5.2.19}$$

This equation satisfies the conditions (5.2.9) and therefore has the symmetry (5.2.7), $\bar{x} = x^{-1}$. The reckoning shows that the transformation (5.2.7) together with the identical transformation $\bar{x} = x$ are the only symmetries of the form (5.2.6). In order to find more symmetries, we note that the transformation

$$\bar{x} = x^n \tag{5.2.20}$$

with any integer n indivisible by 5 is a symmetry for Eq. (5.2.19).

Indeed, we have to substitute $\bar{x} = x^n$ in the determining equation (5.2.4):

$$P_4(\bar{x})\big|_{P_4(x)=0} \equiv (\bar{x}^4 + \bar{x}^3 + \bar{x}^2 + \bar{x} + 1)\big|_{x^4 = -(x^3+x^2+x+1)} = 0. \tag{5.2.21}$$

Here we should take into account not only the equation

$$x^4 = -(x^3 + x^2 + x + 1) \tag{5.2.22}$$

but also the equations obtained from (5.2.22) by repeated multiplication by x:

$$x^5 = x \cdot x^4 = -(x^4 + x^3 + x^2 + x) = 1, \quad x^6 = x \cdot x^5 = x, \ldots.$$

Thus, we extend (5.2.22) as follows:

$$\begin{array}{lllll} x^5 = 1, & x^6 = x, & x^7 = x^2, & x^8 = x^3, & x^9 = x^4, \\ x^{10} = 1, & x^{11} = x, & x^{12} = x^2, & x^{13} = x^3, & x^{14} = x^4, \\ x^{15} = 1, & \cdots. & \cdots. & \cdots. & \cdots. \end{array} \tag{5.2.23}$$

Let us verify that the transformation

$$\bar{x} = x^2$$

satisfies the determining equation (5.2.4). Using Eqs. (5.2.22)–(5.2.23), we have:

$$P_4(\bar{x})\big|_{P_4(x)=0} = (x^8+x^6+x^4+x^2+1)\big|_{P_4(x)=0}$$
$$= x^3+x+x^4+x^2+1=0.$$

Likewise, we verify that the transformation

$$\bar{x}=x^3$$

also satisfies the determining equation (5.2.4). In this case we have:

$$P_4(\bar{x})\big|_{P_4(x)=0} = (x^{12}+x^9+x^6+x^3+1)\big|_{P_4(x)=0}$$
$$= x^2+x^4+x+x^3+1=0.$$

The transformation

$$\bar{x}=x^4$$

is a symmetry because it is the repeated action of the symmetry transformation $\bar{x}=x^2$.

The transformation

$$\bar{x}=x^5$$

is not a symmetry because for this transformation Eqs. (5.2.23) yield

$$P_4(\bar{x})\big|_{P_4(x)=0} = x^{20}+x^{15}+x^{10}+x^5+1=5.$$

The composition of the above symmetries shows that the transformations (5.2.20) with all positive $n \neq 5m$ $(m=0,1,2,\ldots)$ are symmetries for Eq. (5.2.19). Finally, the proof for the transformation (5.2.20) with the negative values of n is obtained by taking the composition of the transformations (5.2.20) with the positive n and the symmetry (5.2.10).

Thus, we have demonstrated that the transformation (5.2.20) is a symmetry for Eq. (5.2.19), provided that $n \neq 5m$ $(m=0,\pm1,\pm2,\ldots)$.

5.3 Construction of Galois groups

5.3.1 First example

Let us construct the Galois group for Eq. (5.1.1),

$$x^2+1=0,$$

using its symmetry group G consisting of the transformations (5.2.16) and (5.2.17). Since Eq. (5.1.1) is invariant under the group G, the roots

$$x_1=i, \quad x_2=-i$$

of Eq. (5.1.1) are merely permuted among themselves (or individually unaltered) by the transformations (5.2.16)–(5.2.17). Consequently, the restriction of the group G on the set $\{x_1, x_2\}$ is well defined. It is manifest that this restriction is a group. It will be called the *induced symmetry group*, or briefly, the *induced group* and denoted by $\mathscr{G}$. The induced group comprises permutations of the roots x_1, x_2. Let us find these permutations.

The action of the transformations (5.2.16) and (5.2.17) on the roots are as follows:

$$T_\alpha(x_1) = \frac{i+\alpha}{1-\alpha i} = i = x_1, \quad T_\alpha(x_2) = \frac{-i+\alpha}{1+\alpha i} = -i = x_2,$$

$$S_\beta(x_1) = \frac{i+\beta}{\beta i - 1} = -i = x_2, \quad S_\beta(x_2) = \frac{-i+\beta}{-\beta i - 1} = i = x_1.$$

We see that the restriction of T_α on the roots is the identical transformation (the unit of the group $\mathscr{G}$) which is denoted by 1. We also see that S_β permutes the roots. This permutation is denoted by (x_1, x_2). Thus, the induced group $\mathscr{G}$ comprises two elements:

$$\mathscr{G} = \{1, (x_1, x_2)\}. \tag{5.3.1}$$

Comparison with (5.1.4) shows that the *Galois group* of Eq. (5.1.1) coincides with the induced group (5.3.1).

5.3.2 Second example

Let us construct the Galois group for Eq. (5.1.2),

$$x^4 - x^2 + 1 = 0,$$

using its symmetry group G consisting of the transformations I, S, R, T given in (5.2.18). The roots of Eq. (5.1.2) are

$$x_1 = \sqrt{\frac{1}{2}(1+i\sqrt{3})}, \quad x_2 = -x_1, \quad x_3 = \sqrt{\frac{1}{2}(1-i\sqrt{3})}, \quad x_4 = -x_3.$$

Denoting by $\widetilde{I}, \widetilde{S}, \widetilde{R}, \widetilde{T}$ the restriction of I, S, R, T on the roots, we obtain $\widetilde{I} = 1$ (the unit) and the following permutations:

$$\widetilde{S} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_1 & x_2 \end{pmatrix},$$

$$\widetilde{T} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

They are also denoted by

$$(x_1,x_2)(x_3,x_4), \quad (x_1,x_3)(x_2,x_4), \quad (x_1,x_4)(x_2,x_3).$$

Thus the induced group $\mathscr{G}$ in this case comprises four elements:

$$\mathscr{G} = \{1,\ (x_1,x_2)(x_3,x_4),\ (x_1,x_3)(x_2,x_4),\ (x_1,x_4)(x_2,x_3)\}. \tag{5.3.2}$$

Comparison with (5.1.5) shows that the *Galois group* of Eq. (5.1.2) coincides with the induced group (5.3.2).

5.3.3 *Third example*

Consider Eq. (5.1.3),

$$x^4+x^3+x^2+x+1=0.$$

We know that Eq. (5.1.3) has the infinite group G of symmetries given by the transformation (5.2.20). Let us take a finite subgroup of this infinite group, namely the subgroup comprising the transformations

$$\bar{x}=x, \quad \bar{x}=x^2, \quad \bar{x}=x^3, \quad \bar{x}=\frac{1}{x}, \tag{5.3.3}$$

i.e. the transformations (5.2.20) with $n=1,2,3,-1$. The roots of Eq. (5.1.3) are

$$x_1=\varepsilon, \quad x_2=\varepsilon^2, \quad x_3=\varepsilon^3, \quad x_4=\varepsilon^4, \tag{5.3.4}$$

where

$$\varepsilon = e^{2\pi i/5}.$$

The reckoning shows that the restriction of the transformations (5.3.3) to the roots (5.3.4) yields four permutations, namely the unit 1 and the permutations

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_4 & x_1 & x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_1 & x_4 & x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

denoted by

$$(x_1,x_2,x_4,x_3), \quad (x_1,x_3,x_4,x_2), \quad (x_1,x_4)(x_2,x_3).$$

One can verify that the restriction of the arbitrary transformations (5.2.20) to the roots (5.3.4) does not give new permutations. Hence, in this case also the induced symmetry group comprises four elements:

$$\mathscr{G} = \{1,\ (x_1,x_2,x_4,x_3),\ (x_1,x_3,x_4,x_2),\ (x_1,x_4)(x_2,x_3)\}. \tag{5.3.5}$$

Comparison with (5.1.6) shows that the *Galois group* of Eq. (5.1.3) coincides with the induced group (5.3.5).

5.3.4 Concluding remarks

I summarize the above observations in the following definition of Galois groups for algebraic equations.

Definition 5.2. Let G be the group of symmetries (5.2.2) of an algebraic equation (5.2.1). The restriction of the group G to the roots of Eq. (5.2.1) is called the Galois group of Eq. (5.2.1).

This definition provides a parallel between the role of symmetries in solvability of differential equations by quadrature and solvability of algebraic equation in radicals. Namely, we know due to S. Lie that if an nth-order ordinary differential equations admits a solvable n-parameter symmetry group, then the equation in question can be integrated by quadratures. On the other hand, it is known from the theory of Galois groups that an algebraic equation of degree n is solvable by radicals if its Galois group is an nth-order solvable group. Now we can formulate this statement in terms of symmetry groups:

An algebraic equation of degree n is solvable by radicals if it has a symmetry group G such that the induced group G *is an n*th*-order solvable group.*

Assignment to Part I

Problems

1. Find the one-parameter group with the generator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (v+x)\frac{\partial}{\partial u} - (u+y)\frac{\partial}{\partial v}.$$

2. Let x, y be two independent variables and v, u be two dependent variables. Consider the operator X from Problem 1. Find its first prolongation

$$X_{(1)} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (v+x)\frac{\partial}{\partial u} - (u+y)\frac{\partial}{\partial v}$$
$$+ \zeta_1^1\frac{\partial}{\partial u_x} + \zeta_2^1\frac{\partial}{\partial u_y} + \zeta_1^2\frac{\partial}{\partial v_x} + \zeta_2^2\frac{\partial}{\partial v_y}.$$

3. Make the commutator table for the five-dimensional Lie algebra L_5 spanned by the following operators in the three-dimensional space (t, x, y):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},$$
$$X_4 = t^2\frac{\partial}{\partial t} + ty\frac{\partial}{\partial y}, \quad X_5 = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Find the first derived algebra L_5' and determine its dimension.

4. Investigate the Lie algebra L_5 from Problem 3 for solvability.

5. Make the commutator table for the three-dimensional Lie algebra L_3 spanned by the following operators in the three-dimensional space (x, y, z):

$$X_1 = \frac{\partial}{\partial z},$$

$$X_2 = \sin z \frac{\partial}{\partial x} + \frac{\cos z}{\cos x} \frac{\partial}{\partial y} - \tan x \cos z \frac{\partial}{\partial z},$$

$$X_3 = \cos z \frac{\partial}{\partial x} - \frac{\sin z}{\cos x} \frac{\partial}{\partial y} + \tan x \sin z \frac{\partial}{\partial z}.$$

6. Investigate the Lie algebra L_3 from Problem 5 for solvability.

7. Find the one-parameter group with the generator

$$X = e^{-z}\left(\sin x \frac{\partial}{\partial x} - \cos x \frac{\partial}{\partial z}\right).$$

8. Let L_3 be the three-dimensional Lie algebra spanned by the following operators in the three-dimensional space (x, y, z) :

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Find the transformations $T_{a^1}, T_{a^2}, T_{a^3}$ generated by X_1, X_2, X_3, respectively, and construct the three-parameter group G_3 by taking the composition

$$T_a = T_{a^3} T_{a^2} T_{a^1},$$

where $a = (a^1, a^2, a^3)$. Find the composition law in G_3.

9. Find a basis of invariants for the one-parameter group with the generator

$$X = (1+t^2)\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} + (r - tv_r)\frac{\partial}{\partial v_r} - tv_\theta \frac{\partial}{\partial v_\theta} - 2t\rho\frac{\partial}{\partial \rho} - 4tp\frac{\partial}{\partial p},$$

where v_r, v_θ are the components of the velocity in the polar coordinates r, θ (see Section 2.2.3, Example 2.9).

Solutions

1. Solving the Lie equations (2.1.24) for $\bar{x}, \bar{y}$ we obtain the rotation

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a.$$

Therefore the Lie equations for $\bar{u}, \bar{v}$ are written

$$\begin{aligned} \frac{d\bar{u}}{da} &= \bar{v} + x\cos a + y\sin a, \quad \bar{u}|_{a=0} = u, \\ \frac{d\bar{v}}{da} &= -\bar{u} + x\sin a - y\cos a, \quad \bar{v}|_{a=0} = v. \end{aligned} \tag{1}$$

Differentiating the first equation of (1) and using the second one we get

$$\frac{d^2\bar{u}}{da^2} = -\bar{u},$$

whence

$$\bar{u} = C_1 \cos a + C_2 \sin a. \tag{2}$$

Substituting Eq. (2) in the first equation of (1) we find

$$\bar{v} = C_2 \cos a - C_1 \sin a - x\cos a - y\sin a. \tag{3}$$

Letting in Eqs. (2) and (3) $a = 0$ and using the initial conditions from Eqs. (1) we get $C_1 = u$, $C_2 = v + x$ and obtain

$$\bar{u} = u\cos a + v\sin a + x\sin a, \quad \bar{v} = v\cos a - u\sin a - y\sin a.$$

Answer. The group is given by the transformations

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a,$$

$$\bar{u} = u\cos a + v\sin a + x\sin a, \quad \bar{v} = v\cos a - u\sin a - y\sin a.$$

2. The prolongation formula (3.3.21) yields:

$$\begin{aligned} X_{(1)} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (v+x)\frac{\partial}{\partial u} - (u+y)\frac{\partial}{\partial v} + (1+v_x+u_y)\frac{\partial}{\partial u_x} \\ + (v_y - u_x)\frac{\partial}{\partial u_y} + (v_y - u_x)\frac{\partial}{\partial v_x} - (1+v_x+u_y)\frac{\partial}{\partial v_y}. \end{aligned}$$

3. The commutator table has the form

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_1	$2X_3 - X_5$	0
X_2	0	0	X_2	0	$2X_2$
X_3	$-X_1$	$-X_2$	0	X_4	0
X_4	$-2X_3 + X_5$	0	$-X_4$	0	0
X_5	0	$-2X_2$	0	0	0

This table shows that the derived algebra is $L_5' = < X_1, X_2, Y_3, X_4 >$, i.e. it is a four-dimensional Lie algebra spanned by the operators X_1, X_2, X_4 and

$$Y_3 = 2X_3 - X_5.$$

4. The commutator table for the first derived algebra L_5' from Problem 3 is

	X_1	X_2	Y_3	X_4
X_1	0	0	$2X_1$	Y_3
X_2	0	0	0	0
Y_3	$-2X_1$	0	0	X_4
X_4	$-Y_3$	0	$-X_4$	0

This table shows that the second derived algebra is $L_5'' =< X_1, Y_3, X_4 >$. It follows further that $L_5''' =< X_1, Y_3, X_4 >$, i.e. $L_5''' = L_5''$, and hence $L_5^{(n)} = L_5''$ for any $n \geq 2$. It means that L_5 is not solvable.

5. The reckoning shows that the commutator table has the form

	X_1	X_2	X_3
X_1	0	X_3	$-X_2$
X_2	$-X_3$	0	X_1
X_3	X_2	$-X_1$	0

6. The commutator table from Problem 5 yields $L_3' = L_3$. Hence, L_3 is not solvable.

7. The Lie equations (2.1.23) are written as follows:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}a} = \mathrm{e}^{-\bar{z}} \sin\bar{x}, \quad \frac{\mathrm{d}\bar{z}}{\mathrm{d}a} = -\mathrm{e}^{-\bar{z}} \cos\bar{x}.$$

Dividing the second equation by the first one, we obtain

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{x}} = -\frac{\cos\bar{x}}{\sin\bar{x}},$$

or

$$\mathrm{d}\bar{z} = -\frac{\cos\bar{x}}{\sin\bar{x}}\mathrm{d}\bar{x} = -\frac{\mathrm{d}\sin\bar{x}}{\sin\bar{x}}.$$

Thus, we have the first integral

$$\mathrm{e}^{\bar{z}} \sin\bar{x} = C. \tag{4}$$

Now we write the first Lie equation in the form

$$\mathrm{d}a = \frac{\mathrm{e}^{\bar{z}}}{\sin\bar{x}}\mathrm{d}\bar{x} = \frac{C}{\sin^2\bar{x}}\mathrm{d}\bar{x},$$

integrate it and obtain:

$$-C\frac{\cos\bar{x}}{\sin\bar{x}} = a + K,$$

or, substituting the expression (4) for C :

$$-e^{\bar{z}}\cos\bar{x} = a + K.$$

Using the initial conditions, $\bar{x} = x, \bar{z} = z$ when $a = 0$, we find the constant of integration K and obtain:

$$e^{\bar{z}}\cos\bar{x} = e^{z}\cos x - a. \tag{5}$$

Determining C in Eq. (4) by using the initial conditions we conclude that the group transformation is defined implicitly by the equations

$$e^{\bar{z}}\sin\bar{x} = e^{z}\sin x, \quad e^{\bar{z}}\cos\bar{x} = e^{z}\cos x - a. \tag{6}$$

Checking. The group composition law is verified as follows:

$$e^{\bar{\bar{z}}}\sin\bar{\bar{x}} = e^{\bar{z}}\sin\bar{x} = e^{z}\sin x,$$
$$e^{\bar{\bar{z}}}\cos\bar{\bar{x}} - b = e^{\bar{z}}\cos\bar{x} - b = e^{z}\cos x - (a+b).$$

In addition, we can obtain from Eqs. (6) the coordinates of the generator X. To this end, we write the first equation of Eqs. (6) for the infinitesimal transformation

$$\bar{x} = x + a\xi, \quad \bar{z} = z + a\eta$$

and obtain:

$$\begin{aligned} e^{z}\sin x = e^{z+a\eta}\sin(x+a\xi) &= e^{z}e^{a\eta}(\sin x\cos a\xi + \cos x\sin a\xi) \\ &\approx e^{z}(1+a\eta)(\sin x + a\xi\cos x) \\ &\approx e^{z}\sin x + ae^{z}(\xi\cos x + \eta\sin x), \end{aligned}$$

whence

$$\xi\cos x + \eta\sin x = 0. \tag{7}$$

Dealing likewise with the second equation of Eqs. (6) we obtain

$$\begin{aligned} e^{z}\cos x - a = e^{z+a\eta}\cos(x+a\xi) \\ \approx e^{z}\cos x + ae^{z}(\eta\cos x - \xi\sin x), \end{aligned}$$

whence

$$e^{z}(\eta\cos x - \xi\sin x) = -1. \tag{8}$$

Solving Eqs. (7)–(8) for ξ, η we get the coordinates of our operator X.

Answer. The group transformation is defined implicitly by Eqs. (6).

8. Following the construction described in Section 4.5.2 we obtain:

$$T_{a^1}: \bar{x} = x + a^1, \quad \bar{y} = y, \quad \bar{z} = z;$$
$$T_{a^2}: \bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = z + a^2;$$
$$T_{a^3}: \bar{x} = x, \quad \bar{y} = y + a^3, \quad \bar{z} = z + xa^3.$$

Their composition $T_a = T_{a^3} T_{a^2} T_{a^1}$ has the form:

$$\bar{x} = x + a^1, \quad \bar{y} = y + a^3,$$
$$\bar{z} = z + a^2 + xa^3 + a^1 a^3.$$

The consecutive application of T_a and T_b, where $b = (b^1, b^2, b^3)$, yields the transformation $T_b T_a$:

$$\bar{\bar{x}} = x + a^1 + b^1, \quad \bar{\bar{y}} = y + a^3 + b^3,$$
$$\bar{\bar{z}} = z + a^2 + b^2 + x(a^3 + b^3) + a^1(a^3 + b^3) + b^1 b^3.$$

The equation $T_b T_a = T_c$ leads to the equations

$$\bar{x} = x + c^1, \quad \bar{y} = y + c^3,$$
$$\bar{z} = z + c^2 + xc^3 + c^1 c^3,$$

whence

$$c^1 = a^1 + b^1,$$
$$c^3 = a^3 + b^3,$$
$$c^2 = a^2 + b^2 - a^3 b^1.$$

Answer. The transformation T_a has the form

$$\bar{x} = x + a^1, \quad \bar{y} = y + a^3, \quad \bar{z} = z + a^2 + xa^3 + a^1 a^3.$$

The composition law is

$$\phi^1(a, b) = a^1 + b^1,$$
$$\phi^2(a, b) = a^2 + b^2 - a^3 b^1,$$
$$\phi^3(a, b) = a^3 + b^3.$$

9. The equation (2.2.1) for the invariant $J(t, r, v_r, v_\theta, \rho, p)$ is written

$$X(J) \equiv 2t(1 + t^2)\frac{\partial J}{\partial t} + tr\frac{\partial J}{\partial r} + (r - tv_r)\frac{\partial J}{\partial v_r}$$
$$- tv_\theta \frac{\partial J}{\partial v_\theta} - 2tp\frac{\partial J}{\partial \rho} - 4tp\frac{\partial J}{\partial p} = 0.$$

We will use its characteristic system:

$$\frac{dt}{1+t^2} = \frac{dr}{tr} = \frac{dv_r}{r-tv_r} = -\frac{dv_\theta}{tv_\theta} = -\frac{d\rho}{2t\rho} = -\frac{dp}{4tp}\cdot$$

Integration of the first equation of this system,

$$\frac{dt}{1+t^2} = \frac{dr}{tr},$$

gives $\ln(1+t^2)+\ln C_1 = \ln r$. It yields $r = C_1\sqrt{1+t^2}$, whence solving with respect to the constant of integration C_1 we obtain the first integral, i.e. the invariant

$$\lambda = \frac{r}{\sqrt{1+t^2}}\cdot$$

Let us integrate now the second equation, namely the equation

$$\frac{dt}{1+t^2} = \frac{dv_r}{r-tv_r}\cdot$$

We write it in the form

$$\frac{dv_r}{dt} = -\frac{tv_r}{1+t^2} + \frac{r}{1+t^2},$$

substitute here $r = \lambda\sqrt{1+t^2}$ and obtain the following non-homogeneous linear first-order equation:

$$\frac{dv_r}{dt} = -\frac{t}{1+t^2}v_r + \frac{\lambda}{\sqrt{1+t^2}}, \quad \lambda = \text{const.}$$

It can be readily solved, e.g. by the method of variation of parameter (see, e.g. [17], Section 3.2.7). The homogeneous equation

$$\frac{dv_r}{dt} = -\frac{t}{1+t^2}v_r$$

yields $\ln|v_r| = -\frac{1}{2}\ln(1+t^2)+\ln C$, C =const., whence

$$v_r = \frac{C}{\sqrt{1+t^2}}\cdot$$

Thus, we set

$$v_r = \frac{C(t)}{\sqrt{1+t^2}}$$

substitute this expression and its derivative

$$\frac{dv_r}{dt} = -\frac{tC(t)}{(1+t^2)\sqrt{1+t^2}} + \frac{C'(t)}{\sqrt{1+t^2}}$$

in the non-homogeneous equation and obtain:

$$-\frac{tC(t)}{(1+t^2)\sqrt{1+t^2}}+\frac{C'(t)}{\sqrt{1+t^2}}=-\frac{t}{1+t^2}\,\frac{C(t)}{\sqrt{1+t^2}}+\frac{\lambda}{\sqrt{1+t^2}},$$

whence $C'(t)=\lambda$, and hence $C(t)=\lambda t+K$, $K=\text{const}$. Thus,

$$v_r=\frac{\lambda t+K}{\sqrt{1+t^2}}.$$

Let us solve this equation for K. Using the equation $r=\lambda\sqrt{1+t^2}$, we get

$$K=v_r\sqrt{1+t^2}-\lambda t=\frac{1}{\lambda}\left(rv_r-\lambda^2 t\right)=\frac{1}{\lambda}\left(rv_r-\frac{tr^2}{1+t^2}\right).$$

Since λ is a constant on the solutions of the characteristic system, it follows that $K\lambda=\text{const}$. Hence, setting $U=K\lambda$ we arrive at the invariant

$$U=rv_r-\frac{tr^2}{1+t^2}.$$

The equation

$$\frac{\mathrm{d}r}{tr}=-\frac{\mathrm{d}v_\theta}{tv_\theta}$$

yields $\ln|t|=\ln C_2-\ln|v_\theta|$, $C_2=\text{const}$., whence the invariant is $V=rv_\theta$.

Other invariants are obtained likewise. The equation

$$\frac{\mathrm{d}t}{1+t^2}=-\frac{\mathrm{d}\rho}{2t\rho}$$

yields the invariant $R=(1+t^2)\rho$ and the last equation

$$\frac{\mathrm{d}t}{1+t^2}-\frac{\mathrm{d}p}{4tp},$$

the invariant $P=(1+t^2)^2p$.

Answer. The following five invariants provide a basis:

$$\lambda=\frac{r}{\sqrt{1+t^2}},\quad U=rv_r-\frac{tr^2}{1+t^2},$$

$$V=rv_\theta,\quad R=(1+t^2)\rho,\quad P=(1+t^2)^2p.$$

Part II
Approximate Transformation Groups

Theory of approximate transformation groups is a new direction in Lie group analysis of differential equations. It has been developed in [2], [3] for tackling differential equations with a small parameter.

The challenge has been as follows. The classical Lie group analysis allows one to single out equations with remarkable symmetry properties among differential equations used in mathematical modelling. However, small perturbations of model equations may break the admissible group thus reducing the value of group methods in general applied sciences. For example, it is well known that the Burgers equation

$$u_t = uu_x + u_{xx}$$

and the Korteweg-de Vries equation

$$u_t = uu_x + u_{xxx},$$

considered separately, have infinite sets of Lie-Bäcklund symmetries. On the other hand, the combined Burgers–Korteweg-de Vries equation

$$u_t = uu_x + \varepsilon(au_{xxx} + bu_{xx})$$

with non-vanishing parameters a, b, ε does not have this remarkable property. Therefore, *development of methods of group analysis that are stable with respect to small perturbations of equations has become vital.* An attempt to develop such methods has been undertaken in [2].

Our approach is based on the concept of approximate transformations and a representation of local Lie groups by approximate transformations. The extension of Lie's infinitesimal characterization of local groups to approximate transformation groups leads to approximate Lie algebras and approximate symmetries of differential equations with small parameters. This extension augments possibilities of the classical Lie group analysis by adding stability with respect to small perturbations of mathematical models. For instance, the infinite series of symmetries of the Burgers equation and the Korteweg-de Vries equation do not disappear when we combine them together as above, but pass from the category of "exact" symmetries to the new category of "approximate" symmetries [3]. Numerous examples show that symmetries of many other physically interesting differential equations are stable with respect to small perturbations (see [12], Chapter 9).

The new theory maintains the basic attributes of Lie group analysis. In particular, in the classical Lie group theory, one-parameter transformation groups with given infinitesimal generators are obtained either by solving the Lie equations or applying the exponential map. The same constructions apply to approximate transformation groups by adding the adjective *approximate* to the *generators, Lie equations* and *exponential map.*

Chapter 6
Preliminaries

6.1 Motivation

The initiation and subsequent development of the theory of approximate transformation groups were inspired by the following circumstances.

A variety of differential equations recognized as mathematical models in engineering and physical sciences, involve empirical parameters or constitutive laws. Therefore coefficients of model equations are defined approximately with an inevitable error. Consequently, differential equations depending on a small parameter occur frequently in applications. Unfortunately, any small perturbation of coefficients of a differential equation disturbs its symmetry properties. This *instability of Lie symmetries* was the first circumstance that led us [2] to the idea of modifying Lie's theory by introducing the concept of approximate transformation groups.

The second circumstance is that utilization of *Lie group techniques may lead to unjustified complexities.* This happens, e.g. in developing the theory of relativity in the de Sitter space-time (see [15]) with the metric

$$ds^2 = -\left(1+\varepsilon\sigma^2\right)^{-2}\sum_{\mu=1}^{4}(dx^\mu)^2, \quad \sigma^2 = \sum_{\mu=1}^{4}(x^\mu)^2.$$

Here the following notation is used:

$$(x^1,x^2,x^3,x^4) = (x,y,z,ict), \quad i=\sqrt{-1},$$

where $c = 2.99793\times 10^{10}$ cm/s is the velocity of light in empty space, $\varepsilon = K/4$ with K denoting the curvature of the de Sitter universe. According to cosmological data, the curvature K is a small constant ($\sim 10^{-54}$ cm^{-2}), and hence ε can be treated as a small parameter. We assume here that $\varepsilon \geq 0$. The *de Sitter group* (i.e., the group of isometric motions in the de Sitter space-time) differs from the *Poincaré group* (i.e., the group of isometries in the Minkowski space-time) in that the usual translations of space-time coordinates x^μ are replaced by more complicated transformations,

the so-called "generalized translations" in the de Sitter space-time. The generalized translation, e.g. along the x^1 axis has the generator

$$X = \Big(1+\varepsilon[(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2]\Big)\frac{\partial}{\partial x^1}$$
$$+2\varepsilon x^1\Big(x^2\frac{\partial}{\partial x^2}+x^3\frac{\partial}{\partial x^3}+x^4\frac{\partial}{\partial x^4}\Big).$$

The corresponding group transformations (with the group parameter a) have the form

$$\bar{x}^1 = 2\frac{x^1\cos(2a\sqrt{\varepsilon})+\dfrac{1-\varepsilon\sigma^2}{2\sqrt{\varepsilon}}\sin(2a\sqrt{\varepsilon})}{1+\varepsilon\sigma^2+(1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon})-2x^1\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})}, \tag{6.1.1}$$
$$\bar{x}^j = 2\frac{x^j}{1+\varepsilon\sigma^2+(1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon})-2x^1\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},$$
$$j=2,3,4.$$

However, since ε is small, it is sufficient to use an approximate expression of these transformations, e.g. by expanding them in powers of ε and considering only the leading terms of the first order. That is, to consider the de Sitter group as a perturbation of the Poincaré group by the curvature K. Then the result is given by the following simple approximate transformation:

$$\bar{x}^1 \approx x^1+a+\varepsilon\Big([(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2]a+x^1a^2+\frac{1}{3}a^3\Big),$$
$$\bar{x}^j \approx x^j+\varepsilon(2ax^1+a^2)x^j, \quad j=2,3,4. \tag{6.1.2}$$

The question naturally arises of how to calculate this perturbation directly, without using the complicated group transformation (6.1.1). The theory of approximate groups gives a method of calculation. Namely, one can use either the *approximate Lie equations* or the *approximate exponential map*. We calculate the one-parameter approximate transformation group (6.1.2) in Example 7.6 by applying Theorem 7.2 on the approximate exponential map to the above generator X.

Thus, the approximate transformation group techniques often simplify calculations significantly. Moreover, approximate symmetries can help to solve with the desired accuracy differential equations that have no Lie point symmetries and hence cannot be solved by Lie group methods. An example of this kind, namely the equation

$$y''-x-\varepsilon y^2=0$$

will be discussed in Section 9.1.1.

6.2 A sketch on Lie transformation groups

A brief sketch of the Lie equations and the exponential map is given here to draw a parallel between the classical and approximate group theories.

6.2.1 One-parameter transformation groups

Let $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. Consider a one-parameter family of invertible transformation T_a:

$$\bar{x} = f(x, a), \tag{6.2.1}$$

or in coordinates,

$$\bar{x}^i = f^i(x, a), \quad i = 1, \ldots, n.$$

The transformation T_a carries the point $x \in \mathbb{R}^n$ to the point $\bar{x} \in \mathbb{R}^n$.

Here the parameter a ranges over all real numbers from a neighborhood $U \subset \mathbb{R}$ of $a = 0$, and we impose the condition that Eq. (6.2.1) is the identity transformation if and only if $a = 0$, i.e.

$$f(x, 0) = x, \tag{6.2.2}$$

and, conversely, the equation $f(x, a) = x$ with $a \in U$ implies $a = 0$.

Definition 6.1. A set G of transformations (6.2.1) is called a continuous one-parameter transformation group in $\mathbb{R}^n$ if the functions $f^i(x, a)$ satisfy the condition (6.2.2) and the *group property*

$$f^i(f(x, a), b) = f^i(x, c), \quad i = 1, \ldots, n, \tag{6.2.3}$$

for all $a, b \in U$, where $c \in U$ is a certain (smooth) function of a and b:

$$c = \phi(a, b), \tag{6.2.4}$$

such that the equation

$$\phi(a, b) = 0 \tag{6.2.5}$$

has a unique solution $b \in U$ for any $a \in U$. Given a, the solution b of Eq. (6.2.5) is denoted by a^{-1}. The function $\phi(a, b)$ is termed a group composition law.

According to Definition 6.1, a continuous group G contains the (unique) identity transformation $I = T_0$. Further, the group property (6.2.3) means that any two transformations $T_a, T_b \in G$ carried out one after the other result in a transformation which also belongs to G:

$$T_b T_a = T_c, \quad c = \phi(a, b) \in U,$$

for any $a, b \in U$. The solvability of Eq. (6.2.5), together with the group property (6.2.3), provides the inverse transformation $T_a^{-1} = T_{a^{-1}} \in G$ to $T_a \in G$:

$$T_{a^{-1}}T_a = T_aT_{a^{-1}} = I$$

for any $a \in U$.

6.2.2 Canonical parameter

Definition 6.2. The group parameter a is said to be *canonical* if the composition law (6.2.4) has the form

$$\phi(a,b) = a+b, \tag{6.2.6}$$

i.e., if the group property is written

$$f^i(f(x,a),b) = f^i(x,a+b), \quad i = 1,\ldots,n. \tag{6.2.7}$$

Theorem 6.1. Given an arbitrary group composition law (6.2.4), there exists the canonical parameter $\tilde{a}$. Namely, it is defined by the formula

$$\tilde{a} = \int_0^a \frac{\mathrm{d}a}{A(a)}, \tag{6.2.8}$$

where

$$A(a) = \left.\frac{\partial\phi(a,b)}{\partial b}\right|_{b=0}. \tag{6.2.9}$$

Example 6.1. We let $n = 1$ and consider the transformation

$$\bar{x} = x + ax.$$

Using Eqs. (6.2.3) one can verify that this transformation provides a one-parameter group where the composition law (6.2.4) has the form

$$\phi(a,b) = a+b+ab.$$

The parameter a is not canonical. Therefore we apply Theorem 6.1, Equation (6.2.9) yields:

$$A(a) = 1+a,$$

and hence the canonical parameter (6.2.8) is written

$$\tilde{a} = \int_0^a \frac{\mathrm{d}a}{1+a} = \ln(1+a).$$

In what follows, we will adopt the canonical parameter when referring to one-parameter Lie groups as well as approximate transformation groups.

6.2.3 Group generator and Lie equations

Let G be a group of transformations (6.2.1) with $f(x,a)$ satisfying the condition (6.2.2) and the group property (6.2.7). Its infinitesimal transformation

$$\bar{x}^i \approx x^i + a\xi^i(x), \tag{6.2.10}$$

where

$$\xi^i(x) = \left.\frac{\partial f^i(x,a)}{\partial a}\right|_{a=0}, \tag{6.2.11}$$

is associated with the following *generator* of the group G:

$$X = \xi^i(x)\frac{\partial}{\partial x^i}. \tag{6.2.12}$$

One-parameter groups are determined by their generators according to the following theorem due to Lie.

Theorem 6.2. The transformation (6.2.1), $\bar{x} = f(x,a)$, satisfies the group property (6.2.7) and has the operator (6.2.12) as its generator if and only if it solves the ordinary differential equations (called the *Lie equations*)

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}a} = \xi^i(\bar{x}), \quad i = 1,\ldots,n, \tag{6.2.13}$$

with the initial conditions

$$\bar{x}^i\big|_{a=0} = x^i, \quad i = 1,\ldots,n. \tag{6.2.14}$$

Proof. Let us assume that $f(x,a)$ satisfies the group property (6.2.7):

$$f^i(x,a+b) = f^i(f(x,a),b).$$

Let us consider b as an increment of a, i.e. set $b = \Delta a$, and expand both sides of Eq. (6.2.7) in powers of Δa keeping only the principal parts. We have:

$$f^i(x,a+b) \approx f^i(x,a) + \frac{\partial f^i(x,a)}{\partial a}\Delta a$$

and

$$f^i(f(x,a),b) \approx f^i(x,a) + \left.\frac{\partial f^i\big(f(x,a),b\big)}{\partial b}\right|_{b=0}\Delta a.$$

The substitution in Eq. (6.2.7) yields:

$$\frac{\partial f^i(x,a)}{\partial a} = \left.\frac{\partial f^i\big(f(x,a),b\big)}{\partial b}\right|_{b=0}. \tag{6.2.15}$$

According to Eq. (6.2.11), the right-hand side of Eq. (6.2.15) is equal to $\xi^i(f(x,a))$. Invoking that $f(x,a)=\bar{x}$ and considering Eq. (6.2.15) for any fixed x, we arrive at Eqs. (6.2.13).

Let us prove now that the solution $\bar{x}=f(x,a)$ of Eqs. (6.2.13)–(6.2.14) satisfies the group property (6.2.7). Consider, for any fixed x and a, the functions $u=(u^1,\ldots,u^n)$ and $v=(v^1,\ldots,v^n)$ of the variable b defined as follows:

$$u(b)=f(\bar{x},b)\equiv f(f(x,a),b),\quad v(b)=f(x,a+b).$$

To prove the group property (6.2.7), it suffices to show that $u(b)=v(b)$ in a neighborhood of $b=0$. Since $f(x,a)$ solves Eqs. (6.2.13)–(6.2.14), one has:

$$\frac{\mathrm{d}u}{\mathrm{d}b}\equiv\frac{\mathrm{d}f(\bar{x},b)}{\mathrm{d}b}=\xi(u),\quad u\big|_{b=0}=f(x,a);$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}b}\equiv\frac{\mathrm{d}f(x,a+b)}{\mathrm{d}b}=\xi(v),\quad v\big|_{b=0}=f(x,a).$$

Hence, both $z=u(b)$ and $z=v(b)$ solve one and the same Cauchy problem:

$$\frac{\mathrm{d}z}{\mathrm{d}b}=\xi(z),\quad z\big|_{b=0}=f(x,a).$$

The uniqueness of the solution of the Cauchy problem guarantees that, in a neighborhood of $b=0$, we have $u(b)=v(b)$, i.e.

$$f(f(x,a),b)=f(x,a+b).$$

This completes the proof.

Example 6.2. Theorem 6.2 allows one to obtain a one-parameter group of transformations (6.2.1) by solving the Lie equations (6.2.13) with the initial conditions (6.2.14) for any given operator (6.2.12) or the corresponding infinitesimal transformation (6.2.10).

Consider, e.g. the infinitesimal transformation

$$\bar{x}\approx x+ax^2,\quad \bar{y}\approx y+axy$$

in the (x,y) plane. The corresponding differential operator (6.2.12) is

$$X=x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}$$

and the Lie equations (6.2.13) have the form

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}a}=\bar{x}^2,\quad \frac{\mathrm{d}\bar{y}}{\mathrm{d}a}=\bar{x}\bar{y}.$$

Integrating these equations, one obtains

$$\bar{x} = \frac{1}{C_1 - a}, \quad \bar{y} = \frac{C_2}{C_1 - a},$$

where C_1 and C_2 are arbitrary constants. The initial conditions (6.2.14) provide

$$x = \frac{1}{C_1}, \quad y = \frac{C_2}{C_1},$$

whence

$$C_1 = \frac{1}{x}, \quad C_2 = \frac{y}{x}.$$

Thus the one-parameter group of transformations has the form

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ax}. \tag{6.2.16}$$

6.2.4 Exponential map

The solution of the Lie equations (6.2.13)–(6.2.14) can be represented explicitly by the exponential map,

$$\bar{x}^i = \mathrm{e}^{aX}(x^i), \quad i = 1, \ldots, n, \tag{6.2.17}$$

where the exponent is given by the infinite sum:

$$\mathrm{e}^{aX} = 1 + aX + \frac{a^2}{2!}X^2 + \frac{a^3}{3!}X^3 + \cdots. \tag{6.2.18}$$

Example 6.3. The exponential map (6.2.17) is written in the (x, y) plane in the form:

$$\bar{x} = \mathrm{e}^{aX}(x), \quad \bar{y} = \mathrm{e}^{aX}(y), \tag{6.2.19}$$

where e^{aX} is given by the series (6.2.18). Let us apply the exponential map to the generator

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

considered in Example 6.2. We have:

$$X(x) = x^2, \quad X^2(x) = X(X(x)) = X(x^2) = 2!x^3,$$
$$X^3(x) = 3!x^4, \ldots.$$

These equations hint the general formula

$$X^n(x) = n!x^{n+1}, \quad n = 1, 2, \ldots.$$

The proof is given by induction:

$$X^{n+1}(x) = X(n!x^{n+1}) = (n+1)!x^2x^n = (n+1)!x^{n+2}.$$

It follows:

$$e^{aX}(x) = x + ax^2 + \cdots + a^n x^{n+1} + \cdots.$$

One can rewrite the right-hand side of this equation in the form

$$x(1 + ax + \cdots + a^n x^n + \cdots) = \frac{x}{1-ax},$$

where the well known Taylor expansion of the function $(1-ax)^{-1}$ is used provided that $|ax| < 1$. Hence,

$$e^{aX}(x) = \frac{x}{1-ax}.$$

Similarly,

$$X(y) = xy,$$

$$X^2(y) = X(xy) = yX(x) + xX(y) = y(x^2) + y(xy) = 2!yx^2,$$

$$X^3(y) = 2![yX(x^2) + x^2X(y)] = 2![y(2x^3) + x^2(xy)] = 3!yx^3.$$

This hints the general formula

$$X^n(y) = n!yx^n, \quad n = 1,2,\ldots$$

that can be readily verified by induction:

$$X^{n+1}(y) = n!X(yx^n) = n![nyx^{n+1} + x^n(xy)] = (n+1)!yx^{n+1}.$$

It follows:

$$e^{aX}(y) = y + ayx + a^2yx^2 + \cdots + a^nyx^n + \cdots$$

$$= y(1 + ax + \cdots + a^nx^n + \cdots) = \frac{y}{1-ax}.$$

Thus, we arrive at the transformation (6.2.16):

$$\bar{x} = \frac{x}{1-ax}, \quad \bar{y} = \frac{y}{1-ax}.$$

6.3 Approximate Cauchy problem

6.3.1 Notation

Consider functions $f(x,\varepsilon)$ depending on variables $x = (x^1,\ldots,x^n)$ and a parameter ε. These functions are defined in a neighborhood of $\varepsilon = 0$ and are assumed to be

continuous in x, ε together with derivatives of an order required in the subsequent discussion.

Definition 6.3. Let $p \geq 1$ be an integer. A function $f(x, \varepsilon)$ is said to be *of order less than* ε^p and is written

$$f(x, \varepsilon) = o(\varepsilon^p) \tag{6.3.1}$$

if it satisfies the condition

$$\lim_{\varepsilon \to 0} \frac{f(x, \varepsilon)}{\varepsilon^p} = 0. \tag{6.3.2}$$

Remark 6.1. Equation (6.3.2) is satisfied if there exists a constant $C > 0$ such that

$$|f(x, \varepsilon)| \leq C|\varepsilon|^{p+1}. \tag{6.3.3}$$

It is also satisfied if there exists a function $\chi(x, \varepsilon)$ that is analytic near $\varepsilon = 0$ and such that

$$f(x, \varepsilon) = \varepsilon^{p+1} \chi(x, \varepsilon). \tag{6.3.4}$$

Definition 6.4. If

$$f(x, \varepsilon) - g(x, \varepsilon) = o(\varepsilon^p), \tag{6.3.5}$$

the functions f and g are said to be *approximately equal* (with an error $o(\varepsilon^p)$) and written

$$f(x, \varepsilon) = g(x, \varepsilon) + o(\varepsilon^p), \tag{6.3.6}$$

or briefly

$$f \approx g \tag{6.3.7}$$

when there is no ambiguity.

The approximate equality defines an equivalence relation, and we join functions into equivalence classes by letting $f(x, \varepsilon)$ and $g(x, \varepsilon)$ be members of the same class if and only if $f \approx g$.

Given a function $f(x, \varepsilon)$, let

$$f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x) \tag{6.3.8}$$

be the approximating polynomial of degree p in ε obtained via the Taylor series expansion of $f(x, \varepsilon)$ in powers of ε about $\varepsilon = 0$. Then any function $g \approx f$ (in particular, the function f itself) has the form

$$g(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x) + o(\varepsilon^p). \tag{6.3.9}$$

Consequently the function (6.3.8) is called a *canonical representative* of the equivalence class of functions containing f.

Thus, the equivalence class of functions $g(x, \varepsilon) \approx f(x, \varepsilon)$ is determined by the ordered set of $p + 1$ functions

$$f_0(x),\ f_1(x),\ \ldots,\ f_p(x).$$

6.3.2 Definition of the approximate Cauchy problem

Theorem 6.3. Let functions $f(x,\varepsilon)$, $g(x,\varepsilon)$ be analytic near the point (x_0,ε) and satisfy the condition

$$g(x,\varepsilon) = f(x,\varepsilon) + o(\varepsilon^p). \tag{6.3.10}$$

Let $x = x(t,\varepsilon)$ and $\tilde{x} = \tilde{x}(t,\varepsilon)$ solve the problems

$$\frac{dx}{dt} = f(x,\varepsilon), \quad x|_{t=0} = \alpha(\varepsilon)$$

and

$$\frac{d\tilde{x}}{dt} = g(\tilde{x},\varepsilon), \quad \tilde{x}|_{t=0} = \beta(\varepsilon),$$

respectively, where

$$\alpha(0) = \beta(0) = x_0, \quad \beta(\varepsilon) = \alpha(\varepsilon) + o(\varepsilon^p).$$

Then,

$$\tilde{x}(t,\varepsilon) = x(t,\varepsilon) + o(\varepsilon^p). \tag{6.3.11}$$

In other words, the solutions of two Cauchy problems with approximately equal right-hand sides and initial conditions, are equal in the same order of approximation.

Proof. The function $u(t,\varepsilon) = x(t,\varepsilon) - \tilde{x}(t,\varepsilon)$ satisfies the conditions

$$u(0,\varepsilon) = o(\varepsilon^p), \tag{6.3.12}$$

$$\left|\frac{du}{dt}\right| \le |f(x,\varepsilon) - g(\tilde{x},\varepsilon)| \le |f(x,\varepsilon) - g(x,\varepsilon)| + |g(x,\varepsilon) - g(\tilde{x},\varepsilon)|. \tag{6.3.13}$$

Using Eq. (6.3.10) written in the form of the inequality (6.3.3)

$$|g(x,\varepsilon) - f(x,\varepsilon)| \le C\varepsilon^{p+1}, \; C = \text{const.},$$

and the Lipschitz condition

$$|g(x,\varepsilon) - g(\tilde{x},\varepsilon)| \le K|x - \tilde{x}|, \quad K = \text{const.},$$

one obtains from Eq. (6.3.13) the following inequality:

$$\left|\frac{du}{dt}\right| \le K|u| + C\varepsilon^{p+1}. \tag{6.3.14}$$

For every fixed ε there exists such t_ε, that the function $u(t,\varepsilon)$ has a constant sign on the interval form 0 to t_ε. Then inequality (6.3.14) yields that on this interval one has

$$\frac{d}{dt}|u| \le K|u| + C\varepsilon^{p+1}.$$

Dividing this inequality by

$$|u| + \frac{C}{K}\varepsilon^{p+1}$$

and integrating from 0 to t with $|t| \le |t_\varepsilon|$, one obtains

$$|u(t,\varepsilon)| \le \frac{C}{K}\left(e^{Kt_\varepsilon} - 1\right)\varepsilon^{p+1} + u(0,\varepsilon)e^{Kt_\varepsilon}.$$

Whence, using Eq. (6.3.12), one obtains the required Eq. (6.3.11).

Definition 6.5. An approximate Cauchy problem

$$\frac{dz}{dt} \approx f(z,\varepsilon), \tag{6.3.15}$$

$$z|_{t=0} \approx \alpha(\varepsilon), \tag{6.3.16}$$

is determined as follows. The approximate differential equation (6.3.15) is interpreted as a family of differential equations

$$\frac{dz}{dt} = g(z,\varepsilon) \quad \text{with} \quad g(z,\varepsilon) \approx f(z,\varepsilon). \tag{6.3.17}$$

The approximate initial condition (6.3.16) is interpreted likewise:

$$z|_{t=0} = \beta(\varepsilon) \quad \text{with} \quad \beta(\varepsilon) \approx \alpha(\varepsilon). \tag{6.3.18}$$

The approximate equations in (6.3.17) and (6.3.18) have the accuracy p as in Eqs. (6.3.15) and (6.3.16).

Definition 6.6. The solution of the approximate initial value problem (6.3.15) and (6.3.16) is the equivalence class (see Section 6.3.1) of the solutions of the exact Cauchy problems (6.3.17) and (6.3.18). In other words, the solution of the approximate Cauchy problem (6.3.15) and (6.3.16) is the solution of any problem (6.3.17) and (6.3.18) considered with the accuracy up to $o(\varepsilon^p)$.

Theorem 6.4. The solution to the approximate Cauchy problem (6.3.15) and (6.3.16) is unique.

Proof. The uniqueness of the solution follows from Theorem 6.3. Indeed, according to this theorem, the solutions of all problems of the form (6.3.17) and (6.3.18) coincide with the mentioned accuracy.

Chapter 7
Approximate transformations

In the rest of the book we will consider the approximation in the first order of precision in ε. In other words, equation (6.3.5) for approximate equality will be considered for $p = 1$.

7.1 Approximate transformations defined

Definition 7.1. An approximate transformation

$$\bar{x} \approx f(x, a, \varepsilon) \tag{7.1.1}$$

in $\mathbb{R}^n$ is written in the first-order of precision in the form

$$\bar{x}^i \approx f_0^i(x, a) + \varepsilon f_1^i(x, a), \quad i = 1, \ldots, n, \tag{7.1.2}$$

and is defined as the set of all invertible transformations

$$\bar{x}^i = g^i(x, a, \varepsilon) \tag{7.1.3}$$

such that

$$g^i(x, a, \varepsilon) \approx f_0^i(x, a) + \varepsilon f_1^i(x, a).$$

We will deal in what follows with approximate transformations (7.1.1) obeying the initial condition $\bar{x}|_{a=0} \approx x$, or in the coordinate form

$$\bar{x}^i\big|_{a=0} \approx x^i, \quad i = 1, \ldots, n. \tag{7.1.4}$$

Accordingly, we will assume that the functions $g^i(x, a, \varepsilon)$ in the representations (7.1.3) of the approximate transformation (7.1.2) are defined in a neighborhood of $a = 0$ and satisfy the conditions

$$g^i(x, a, \varepsilon) \approx x^i, \quad i = 1, \ldots, n,$$

if and only if $a = 0$.

7.2 Approximate one-parameter groups

7.2.1 *Introductory remark*

Consider transformations

$$\bar{x} = f(x, a, \varepsilon)$$

depending on a small parameter ε and defining a one-parameter group with respect to a, so that the vector-function $f = (f^1, \ldots, f^n)$ satisfies the following equations:

$$\begin{gathered} f(f(x,a,\varepsilon),b,\varepsilon) = f(x,a+b,\varepsilon), \\ f(x,0,\varepsilon) = x. \end{gathered} \tag{7.2.1}$$

Let a vector-function $g = (g^1, \ldots, g^n)$ be approximately equal to f, i.e.

$$g(x,a,\varepsilon) = f(x,a,\varepsilon) + o(\varepsilon). \tag{7.2.2}$$

Consider now, together with $\bar{x}$, the "neighboring points" $\tilde{x}$ determined by

$$\tilde{x} = g(x,a,\varepsilon). \tag{7.2.3}$$

Substituting Eq. (7.2.2) into Eq. (7.2.1), we obtain the following approximate equations in the first-order of precision in ε :

$$\begin{gathered} g(g(x,a,\varepsilon),b,\varepsilon) \approx g(x,a+b,\varepsilon), \\ g(x,0,\varepsilon) \approx x. \end{gathered} \tag{7.2.4}$$

7.2.2 *Definition of one-parameter approximate transformation groups*

Definition 7.2. An *approximate transformation* (7.1.1),

$$\bar{x} \approx f(x,a,\varepsilon)$$

satisfying the initial condition (7.1.4),

$$\bar{x}\big|_{a=0} \approx x, \tag{7.2.5}$$

is called a one-parameter approximate transformation group in $\mathbb{R}^n$ if the group property is satisfied with the accuracy $o(\varepsilon)$:

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon). \tag{7.2.6}$$

Remark 7.1. For approximate transformations written in the form (7.1.2),

$$\bar{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a), \quad i = 1,\dots,n,$$

the definition means that the vector-functions $g = (g^1,\dots,g^n)$ in all *exact representations* (7.1.3) of form (7.1.2):

$$\bar{x}^i = g^i(x,a,\varepsilon), \quad i = 1,\dots,n, \tag{7.2.7}$$

satisfy the approximate equation (7.2.6):

$$g(g(x,a,\varepsilon),b,\varepsilon) \approx g(x,a+b,\varepsilon). \tag{7.2.8}$$

Remark 7.2. In Eqs. (7.2.6) and (7.2.8), unlike the exact group property (6.2.7) in the classical Lie group theory, f and g, respectively, do not necessarily denote the same functions at each occurrence. They can be replaced by any equivalent functions (see the next example).

Example 7.1. Let us take $n = 1$ and consider the functions

$$f(x,a,\varepsilon) = x + a\Big(1 + \varepsilon x + \frac{1}{2}\varepsilon a\Big)$$

and

$$g(x,a,\varepsilon) = x + a(1+\varepsilon x)\Big(1 + \frac{1}{2}\varepsilon a\Big).$$

They are equal in the first order of precision, namely Eq. (6.3.4) is satisfied:

$$g(x,a,\varepsilon) = f(x,a,\varepsilon) + \varepsilon^2\chi(x,a), \quad \chi(x,a) = \frac{1}{2}a^2x.$$

They also obey the approximate group property (7.2.6). Indeed,

$$f(g(x,a,\varepsilon),b,\varepsilon) = f(x,a+b,\varepsilon) + \varepsilon^2\widetilde{\chi}(x,a,b,\varepsilon), \tag{7.2.9}$$

where

$$\widetilde{\chi}(x,a,b,\varepsilon) = \frac{1}{2}a(ax + ab + 2bx + \varepsilon abx).$$

7.2.3 Generator of approximate transformation group

Definition 7.3. The *generator of an approximate transformation group* (7.1.2) is the set of all first-order linear differential operators

$$X = \xi^i(x,\varepsilon)\frac{\partial}{\partial x^i} \tag{7.2.10}$$

with the coefficients

$$\xi^i(x,\varepsilon) \approx \xi_0^i(x) + \varepsilon\xi_1^i(x), \quad i = 1,\dots,n,$$

where

$$\xi_0^i(x) = \left.\frac{\partial f_0^i(x,a)}{\partial a}\right|_{a=0}, \quad \xi_1^i(x) = \left.\frac{\partial f_1^i(x,a)}{\partial a}\right|_{a=0}. \tag{7.2.11}$$

According to this definition, the generator of an approximate transformation group (7.1.1) is an *approximate differential operator*

$$X \approx \xi^i(x,\varepsilon)\frac{\partial}{\partial x^i}, \tag{7.2.12}$$

where $\xi^i(x,\varepsilon)$ are the components of the vector

$$\xi(x,\varepsilon) \approx \left.\frac{\partial f(x,a,\varepsilon)}{\partial a}\right|_{a=0}. \tag{7.2.13}$$

In practice, it is convenient to identify X with its *canonical representative:*

$$X = \left(\xi_0^i(x) + \varepsilon\xi_1^i(x)\right)\frac{\partial}{\partial x^i}. \tag{7.2.14}$$

7.3 Infinitesimal description

7.3.1 Approximate Lie equations

Lemma 7.1. Let an approximate transformation (7.1.1) in $\mathbb{R}^n$,

$$\bar{x} \approx f(x,a,\varepsilon),$$

satisfy the approximate group property (7.2.6) as well as the initial condition

$$\bar{x}|_{a=0} \approx x$$

and have the approximate generator (7.2.12) with ξ defined by Eq. (7.2.13):

$$\xi(x,\varepsilon) \approx \left.\frac{\partial f(x,a,\varepsilon)}{\partial a}\right|_{a=0}.$$

Then the vector-function $f(x,a,\varepsilon)$ solves the approximate equation

$$\frac{\partial f(x,a,\varepsilon)}{\partial a} \approx \xi\left(f(x,a,\varepsilon),\varepsilon\right). \tag{7.3.1}$$

Conversely, given any smooth vector-function $\xi(x,\varepsilon)$, an approximate transformation (7.1.1) obtained by solving the approximate Cauchy problem

$$\frac{d\bar{x}}{da} \approx \xi(\bar{x},\varepsilon), \quad \bar{x}\big|_{a=0} \approx x \tag{7.3.2}$$

satisfies the approximate group property (7.2.6).

Proof. Let us assume that $f(x,a,\varepsilon)$ satisfies the approximate equation (7.2.6) written in the form (6.3.4):

$$f(x,a+b,\varepsilon) = f(f(x,a,\varepsilon),b,\varepsilon) + \varepsilon^2 \chi(x,a,b,\varepsilon). \tag{7.3.3}$$

Let us set $b = \Delta a$ and expand both sides of Eq. (7.3.3) in powers of Δa keeping only the principal parts. We have:

$$f(x,a+b,\varepsilon) \approx f(x,a,\varepsilon) + \frac{\partial f(x,a,\varepsilon)}{\partial a}\Delta a$$

and

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a,\varepsilon) + \left.\frac{\partial f^i(f(x,a,\varepsilon),b,\varepsilon)}{\partial b}\right|_{b=0} \Delta a.$$

The substitution in Eq. (7.3.3) yields:

$$\frac{\partial f(x,a,\varepsilon)}{\partial a} = \left.\frac{\partial f(f(x,a,\varepsilon),b,\varepsilon)}{\partial b}\right|_{b=0} + \varepsilon^2 \chi(x,a,b,\varepsilon). \tag{7.3.4}$$

Invoking Eq. (7.2.13), we replace the first term in the right-hand side of Eq. (7.3.4) by $\xi(f(x,a,\varepsilon),\varepsilon)$ and arrive at Eq. (7.3.1).

Let us prove now that the solution $\bar{x} \approx f(x,a,\varepsilon)$ of the Cauchy problem (7.3.2) satisfies the group property (7.2.6). Consider, for any fixed x and a, the functions $u(b,\varepsilon)$ and $v(b,\varepsilon)$ defined as follows:

$$u(b,\varepsilon) = f(\bar{x},b,\varepsilon) \equiv f(f(x,a,\varepsilon),b,\varepsilon), \quad v(b,\varepsilon) = f(x,a+b,\varepsilon).$$

To prove the group property (7.2.6), it suffices to show that $u(b,\varepsilon) \approx v(b,\varepsilon)$ in a neighborhood of $b = 0$. Since $f(x,a,\varepsilon)$ solves Eqs. (7.3.2), one has:

$$\frac{du}{db} \equiv \frac{df(\bar{x},b,\varepsilon)}{db} \approx \xi(u,\varepsilon), \quad u\big|_{b=0} \approx f(x,a,\varepsilon);$$

and

$$\frac{dv}{db} \equiv \frac{df(x,a+b,\varepsilon)}{db} \approx \xi(v,\varepsilon), \quad v\big|_{b=0} \approx f(x,a,\varepsilon).$$

Hence, both $z = u(b,\varepsilon)$ and $z = v(b,\varepsilon)$ solve one and the same Cauchy problem:

$$\frac{dz}{db} \approx \xi(z,\varepsilon), \quad z\big|_{b=0} \approx f(x,a,\varepsilon).$$

Theorem 6.4 on uniqueness of the solution of the approximate Cauchy problem guarantees that in a neighborhood of $b = 0$ we have $u(b,\varepsilon) \approx v(b,\varepsilon)$, i.e. equation (7.2.6):

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon).$$

This completes the proof.

In computations, it is convenient to use Eqs. (7.3.2) in a modified form given below. Let us write the approximate transformation (7.1.2),

$$\bar{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a), \quad i = 1,\ldots,n,$$

in the following form:

$$\bar{x} \approx \bar{x}_0 + \varepsilon \bar{x}_1, \tag{7.3.5}$$

where

$$\bar{x}_0 = f_0(x,a), \quad \bar{x}_1 = f_1(x,a).$$

Accordingly, the vector (7.2.13) is written

$$\xi(x,\varepsilon) \approx \xi_0(x) + \varepsilon \xi_1(x), \tag{7.3.6}$$

where $\xi_0(x)$ and $\xi_1(x)$ are defined by Eqs. (7.2.11).

We will also write the canonical representation (7.2.14) of the approximate generator in the form

$$X = X_0 + \varepsilon X_1 \tag{7.3.7}$$

with

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}.$$

Theorem 7.1. The approximate transformation (7.3.5), $\bar{x} \approx \bar{x}_0 + \varepsilon \bar{x}_1$, satisfies the approximate group property (7.2.6) and has the operator (7.3.7) as its approximate generator if and only if the following ordinary differential equations with initial conditions are satisfied:

$$\frac{d\bar{x}_0^i}{da} = \xi_0^i(\bar{x}_0), \quad \bar{x}_0^i\big|_{a=0} = x^i, \quad i = 1,\ldots,n, \tag{7.3.8}$$

$$\frac{d\bar{x}_1^i}{da} = \sum_{k=1}^{n}\left(\left.\frac{\partial \xi_0^i(x)}{\partial x^k}\right|_{x=\bar{x}_0}\bar{x}_1^k\right) + \xi_1^i(\bar{x}_0), \quad \bar{x}_1^i\big|_{a=0} = 0. \tag{7.3.9}$$

Equations (7.3.8)–(7.3.9) are called the *approximate Lie equations*.

Proof. Substituting the representations (7.3.5) and (7.3.6) for $\bar{x}$ and $\xi(x,\varepsilon)$, respectively, in Eqs. (7.3.2), we have:

$$\begin{aligned} &\frac{d\bar{x}_0}{da} + \varepsilon\frac{d\bar{x}_1}{da} \approx \xi_0(\bar{x}_0 + \varepsilon\bar{x}_1) + \varepsilon\xi_0(\bar{x}_0 + \varepsilon\bar{x}_1), \\ &(\bar{x}_0 + \varepsilon\bar{x}_1)\big|_{a=0} \approx x. \end{aligned} \tag{7.3.10}$$

Equations (7.3.8) and (7.3.9) are obtained from Eqs. (7.3.10) by singling out the terms not containing ε and the terms that are linear in ε.

Example 7.2. Let $n = 1$ and let

$$X = (1 + \varepsilon x)\frac{\partial}{\partial x}.$$

Here $\xi_0(x) = 1$, $\xi_1(x) = x$, and Eqs. (7.3.8)–(7.3.9) are written:

$$\frac{d\bar{x}_0}{da} = 1, \quad \bar{x}_0\big|_{a=0} = x, \tag{7.3.11}$$

$$\frac{d\bar{x}_1}{da} = \bar{x}_0, \quad \bar{x}_1\big|_{a=0} = 0. \tag{7.3.12}$$

The solution of Eq. (7.3.11) is $\bar{x}_0 = x + a$. Substituting it in the differential equation in (7.3.12) we have:

$$\frac{d\bar{x}_1}{da} = x + a.$$

Integrating and using the initial condition we obtain the following solution of the system (7.3.11)–(7.3.12).

$$\bar{x}_0 = x + a, \quad \bar{x}_1 = ax + \frac{a^2}{2}.$$

Hence, the approximate transformation group is given by

$$\bar{x} \approx x + a + \varepsilon\left(ax + \frac{a^2}{2}\right).$$

Example 7.3. Let $n = 2$ and let

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}. \tag{7.3.13}$$

Here $\xi_0(x,y) = (1,0)$, $\xi_1(x,y) = (x^2, xy)$, and Eqs. (7.3.8)–(7.3.9) are written:

$$\frac{d\bar{x}_0}{da} = 1, \qquad \frac{d\bar{y}_0}{da} = 0, \qquad \bar{x}_0\big|_{a=0} = x,\ \bar{y}_0\big|_{a=0} = y,$$

$$\frac{d\bar{x}_1}{da} = (\bar{x}_0)^2,\ \frac{d\bar{y}_1}{da} = \bar{x}_0\bar{y}_0,\ \bar{x}_1\big|_{a=0} = 0,\ \bar{y}_1\big|_{a=0} = 0.$$

The integration gives the following approximate transformation group:

$$\bar{x} \approx x + a + \varepsilon\left(ax^2 + a^2x + \frac{a^3}{3}\right), \quad \bar{y} \approx y + \varepsilon\left(axy + \frac{a^2}{2}y\right).$$

7.3.2 *Approximate exponential map*

Theorem 7.2. Given an operator

$$X = X_0 + \varepsilon X_1 \tag{7.3.14}$$

with a small parameter ε, where

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}, \tag{7.3.15}$$

the corresponding approximate group transformation

$$\bar{x}^i = \bar{x}_0^i + \varepsilon \bar{x}_1^i, \quad i = 1,\dots,n, \tag{7.3.16}$$

is determined by the following equations:

$$\bar{x}_0^i = \mathrm{e}^{aX_0}(x^i), \quad \bar{x}_1^i = \langle\langle aX_0, aX_1\rangle\rangle(\bar{x}_0^i), \quad i = 1,\dots,n, \tag{7.3.17}$$

where

$$\mathrm{e}^{aX_0} = 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \tag{7.3.18}$$

and

$$\langle\langle aX_0, aX_1\rangle\rangle = aX_1 + \frac{a^2}{2!}[X_0, X_1] + \frac{a^3}{3!}[X_0,[X_0,X_1]] + \cdots. \tag{7.3.19}$$

In other words, the approximate operator $X = X_0 + \varepsilon X_1$ generates the one-parameter approximate transformation group given by the following *approximate exponential map*:

$$\bar{x}^i = \big(1 + \varepsilon\langle\langle aX_0, aX_1\rangle\rangle\big)\mathrm{e}^{aX_0}(x^i), \quad i = 1,\dots,n. \tag{7.3.20}$$

Proof. (see [13]) The substitution of the operator (7.3.14),

$$X_0 + \varepsilon X_1$$

in the definition (6.2.18) of the exponent yields:

$$\mathrm{e}^{a(X_0+\varepsilon X_1)} = 1 + a\big(X_0 + \varepsilon X_1\big) + \frac{a^2}{2!}\big(X_0 + \varepsilon X_1\big)^2 + \frac{a^3}{3!}\big(X_0 + \varepsilon X_1\big)^3 + \cdots.$$

Now we single out the sum of terms up to the first degree in ε and obtain:

$$\begin{aligned}
&\mathrm{e}^{a(X_0+\varepsilon X_1)}\\
\approx&1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\\
&+ \varepsilon\Big[aX_1 + \frac{a^2}{2!}\big(X_0X_1 + X_1X_0\big) + \frac{a^3}{3!}\big(X_0^2X_1 + X_0X_1X_0 + X_1X_0^2\big)\\
&+ \frac{a^4}{4!}\big(X_0^3X_1 + X_0^2X_1X_0 + X_0X_1X_0^2 + X_1X_0^3\big) + \cdots\Big].
\end{aligned} \tag{7.3.21}$$

By using the identities

$$X_0X_1 = X_1X_0 + [X_0, X_1],$$

$$X_0^2X_1 + X_0X_1X_0 = 2X_1X_0^2 + 3[X_0, X_1]X_0 + [X_0, [X_0, X_1]], \ldots$$

we rewrite Eq. (7.3.21) in the form:

$$\begin{aligned}
&e^{a(X_0+\varepsilon X_1)} \\
&\approx 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \\
&+ \varepsilon\left[aX_1\left(1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\right)\right. \\
&+ \frac{a^2}{2!}[X_0, X_1]\left(1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\right) \\
&\left.+ \frac{a^3}{3!}[X_0, [X_0, X_1]]\left(1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\right) + \cdots\right].
\end{aligned}$$

Whence, using the exponent (6.2.18) we have:

$$e^{a(X_0+\varepsilon X_1)} \approx \left(1 + \varepsilon\langle\langle aX_0, aX_1\rangle\rangle\right)e^{aX_0}. \tag{7.3.22}$$

In other words, the exponential map

$$\bar{x}^i = e^{aX}(x^i)$$

written for the operator (7.3.14) and evaluated in the first order of precision with respect to ε has the form (7.3.20). Taking into account Eq. (7.3.16), one arrives at Eqs. (7.3.17). This completes the proof of the theorem.

Example 7.4. Let us apply Theorem 7.2 to the operator

$$X = (1 + \varepsilon x)\frac{\partial}{\partial x}$$

considered in Example 7.2. Here

$$X_0 = \frac{\partial}{\partial x}, \quad X_1 = x\frac{\partial}{\partial x}.$$

Therefore

$$X_0(x) = 1, \; X_0^2(x) = X_0^3(x) = \cdots = 0,$$

and

$$[X_0, X_1] = \frac{\partial}{\partial x} = X_0,$$

$$[X_0, [X_0, X_1]] = [X_0, X_0] = 0, \ldots .$$

Consequently,

$$\bar{x}_0 = e^{aX_0}(x) = x + a,$$

and

$$\langle\langle aX_0, aX_1 \rangle\rangle = \left(ax + \frac{a^2}{2!}\right)\frac{\partial}{\partial x},$$

whence

$$\bar{x}_1 = \langle\langle aX_0, aX_1 \rangle\rangle(\bar{x}_0) = \left(ax + \frac{a^2}{2!}\right)\frac{\partial}{\partial x}(x + a) = ax + \frac{a^2}{2!}.$$

Hence,

$$\bar{x} \approx x + a + \varepsilon\left(ax + \frac{a^2}{2}\right).$$

Example 7.5. Let us apply Theorem 7.2 to the operator (7.3.13) from Example 7.3. In this case we have

$$X_0 = \frac{\partial}{\partial x}, \quad X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

Therefore,

$$\bar{x}_0 = e^{aX_0}(x) = x + a,$$

and

$$[X_0, X_1] = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$[X_0, [X_0, X_1]] = 2\frac{\partial}{\partial x},$$

$$[X_0, [X_0, [X_0, X_1]]] = 0, \ldots .$$

Consequently,

$$\langle\langle aX_0, aX_1 \rangle\rangle = aX_1 + \frac{a^2}{2!}\left(2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + 2\frac{a^3}{3!}\frac{\partial}{\partial x}$$

$$= \left(ax^2 + a^2x + \frac{a^3}{3}\right)\frac{\partial}{\partial x} + \left(axy + \frac{a^2}{2}y\right)\frac{\partial}{\partial y}.$$

Whence

$$\bar{x}_1 = \langle\langle aX_0, aX_1 \rangle\rangle(\bar{x}_0) = \left(ax^2 + a^2x + \frac{a^3}{3}\right)\frac{\partial}{\partial x}(x + a),$$

$$\bar{y}_1 = \langle\langle aX_0, aX_1 \rangle\rangle(\bar{y}_0) = \left(axy + \frac{a^2}{2}y\right)\frac{\partial}{\partial y}(y).$$

Hence,

$$\bar{x}_1 = ax^2 + a^2x + \frac{a^3}{3}, \quad \bar{y}_1 = axy + \frac{a^2}{2}y.$$

We thus arrive at the result of Example 7.3:

$$\bar{x} \approx x + a + \varepsilon\left(ax^2 + a^2x + \frac{a^3}{3}\right),$$

$$\bar{y} \approx y + \varepsilon\left(axy + \frac{a^2}{2}y\right).$$

Example 7.6. Let us apply the approximate exponential map to the following operator:

$$X = \left(1 + \varepsilon[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]\right)\frac{\partial}{\partial x^1}$$

$$+2\varepsilon x^1\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right).$$

Here $X = X_0 + \varepsilon X_1$ with

$$X_0 = \frac{\partial}{\partial x^1},$$

$$X_1 = \left((x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\right)\frac{\partial}{\partial x^1}$$

$$+\ 2x^1\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right).$$

The operator X_0 generates the translation group:

$$\bar{x}_0^1 = x^1 + a, \quad \bar{x}_0^j = x^j,\ j = 2,3,4.$$

We have:

$$[X_0, X_1] = 2\left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right),$$

$$[X_0,[X_0,X_1]] = 2\frac{\partial}{\partial x^1}, \quad [X_0,[X_0,[X_0,X_1]]] = 0, \ldots.$$

Consequently, equation (7.3.19) takes the form:

$$\langle\langle aX_0, aX_1\rangle\rangle = \left([(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\right)\frac{\partial}{\partial x^1}$$

$$+(2ax^1 + a^2)\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right).$$

Therefore Eq. (7.3.17) yields

$$\bar{x}_1^1 = \langle\langle aX_0, aX_1\rangle\rangle(\bar{x}_0^1) = [(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3,$$

$$\bar{x}_1^j = (2ax^1 + a^2)x^j, \quad j = 2,3,4.$$

We have arrived at the following approximate transformation group:

$$\bar{x}^1 \approx \bar{x}_0^1 + \varepsilon\bar{x}_1^1 = x^1 + a + \varepsilon\left([(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\right),$$

$$\bar{x}^j \approx \bar{x}_0^j + \varepsilon\bar{x}_1^j = x^j + \varepsilon(2ax^1 + a^2)x^j, \quad j = 2,3,4.$$

Exercises to Chapter 7

Exercise 7.1. Derive Eq. (7.2.9) in Example 7.1.

Exercise 7.2. Provide details of the proof of Theorem 7.1.

Exercise 7.3. Rewrite the approximate Lie equations (7.3.8)–(7.3.9) in the case of one variable x.

Exercise 7.4. Rewrite the approximate Lie equations (7.3.8)–(7.3.9) for approximate transformation groups on the (x,y) plane.

Exercise 7.5. Solve the approximate Lie equations and find the approximate transformation group for the generator

$$X = \left(t + \frac{\varepsilon}{6}t^2\right)\frac{\partial}{\partial t} - \left(u + \frac{\varepsilon}{3}tu\right)\frac{\partial}{\partial u}. \tag{1}$$

Exercise 7.6. Solve the approximate Lie equations (7.3.8)–(7.3.9) for the operator

$$X = \left(1 + \varepsilon[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]\right)\frac{\partial}{\partial x^1} + 2\varepsilon x^1\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right)$$

from Example 7.6.

Chapter 8
Approximate symmetries

8.1 Definition of approximate symmetries

Definition 8.1. Let G be a one-parameter approximate transformation group considered in the first order of precision:

$$\bar{z}^i \approx f(z,a,\varepsilon) \equiv f_0^i(z,a) + \varepsilon f_1^i(z,a), \quad i = 1,\ldots,N. \tag{8.1.1}$$

Let

$$F(z,\varepsilon) \approx F_0(z) + \varepsilon F_1(z)$$

and

$$F(\bar{z},\varepsilon) \approx F(f(z,a,\varepsilon),\varepsilon).$$

An approximate equation

$$F(z,\varepsilon) \approx 0 \tag{8.1.2}$$

is said to be *approximately invariant* with respect to G (or *admits* G) if

$$F(\bar{z},\varepsilon) \approx 0$$

whenever $z = (z^1,\ldots,z^N)$ satisfies Eq. (8.1.2).

If $z = (x,u,u_{(1)},\ldots,u_{(k)})$, then equation (8.1.2) becomes an approximate differential equation of order k, and G is an *approximate symmetry group* of the differential equation. Recall that in our approximation $F(z,\varepsilon) \approx 0$ means $F(z,\varepsilon) = o(\varepsilon)$.

A different approach to approximate symmetries, based on decomposing differential equations with a small parameter, has been proposed in [10]. For a comparison of two approaches I refer the reader to [11].

8.2 Calculation of approximate symmetries

8.2.1 Determining equations

Theorem 8.1. Equation (8.1.2) is approximately invariant under the approximate transformation group (8.1.1) with the generator

$$X = X^0 + \varepsilon X^1 \equiv \xi_0^i(z)\frac{\partial}{\partial z^i} + \varepsilon\xi_1^i(z)\frac{\partial}{\partial z^i}, \tag{8.2.1}$$

if and only if

$$\big[XF(z,\varepsilon)\big]_{F\approx 0} = o(\varepsilon),$$

or

$$\Big[X^0F_0(z) + \varepsilon\Big(X^1F_0(z) + X^0F_1(z)\Big)\Big]_{(8.1.2)} = o(\varepsilon). \tag{8.2.2}$$

Proof. Equation (8.2.2) is obtained by substituting Eqs. (8.1.2) and (8.2.1) in the determining equation

$$\big[XF(z,\varepsilon)\big]_{F=0} = 0$$

and considering the result in the first-order of precision with respect to ε.

The operator (8.2.1) satisfying Eq. (8.2.2) is called an infinitesimal *approximate symmetry* of, or an *approximate operator admitted* by Eq. (8.1.2). Accordingly, equation (8.2.2) is termed the *determining equation* for approximate symmetries.

Remark 8.1. The determining equation (8.2.2) can be written as follows:

$$X^0F_0(z) = \lambda(z)F_0(z), \tag{8.2.3}$$

$$X^1F_0(z) + X^0F_1(z) = \lambda(z)F_1(z). \tag{8.2.4}$$

The factor $\lambda(z)$ is determined by Eq. (8.2.3) and then substituted in Eq. (8.2.4). The latter equation must hold for all solutions of $F_0(z) = 0$.

8.2.2 Stable symmetries

Comparing Eq. (8.2.3) with the determining equation of exact symmetries, we obtain the following statement.

Theorem 8.2. If equation (8.1.2) admits an approximate transformation group with the generator $X = X^0 + \varepsilon X^1$, where $X^0 \neq 0$, then the operator

$$X^0 = \xi_0^i(z)\frac{\partial}{\partial z^i} \tag{8.2.5}$$

is an exact symmetry of the equation

$$F_0(z) = 0. \tag{8.2.6}$$

Remark 8.2. It is manifest from Eqs. (8.2.3), (8.2.4) that if X^0 is an *exact* symmetry of Eq. (8.2.6) then $X = \varepsilon X^0$ is an *approximate* symmetry of Eq. (8.1.2). Even though such approximate are "trivial", they are significant for the *approximate Lie algebra structure* (see Section 8.3.2).

Definition 8.2. Equations (8.2.6) and (8.1.2) are termed an *unperturbed equation* and a *perturbed equation,* respectively. Under the conditions of Theorem 8.2, the operator X^0 is called a *stable symmetry* of the unperturbed equation (8.2.6). The corresponding approximate symmetry generator $X = X^0 + \varepsilon X^1$ for the perturbed equation (8.2.5) is called a *deformation of the infinitesimal symmetry* X^0 of Eq. (8.2.6) caused by the perturbation $\varepsilon F_1(z)$. In particular, if the most general symmetry Lie algebra of Eq. (8.2.6) is stable, we say that the perturbed equation (8.1.2) *inherits the symmetries of the unperturbed equation.*

8.2.3 Algorithm for calculation

Remark 8.1 and Theorem 8.2 provide an infinitesimal method for calculating approximate symmetries (8.2.1) for differential equations with a small parameter. Implementation of the method requires the following three steps.

1st step. Calculation of the exact symmetries X^0 of the unperturbed equation (8.2.6), e.g. by solving the determining equation

$$X^0 F_0(z)\Big|_{F_0(z)=0} = 0. \tag{8.2.7}$$

2nd step. Determination of the *auxiliary function* H by virtue of Eqs. (8.2.3), (8.2.4) and (8.1.2), i.e. by the equation

$$H = \frac{1}{\varepsilon}\left[X^0\Big(F_0(z) + \varepsilon F_1(z)\Big)\Big|_{F_0(z)+\varepsilon F_1(z)=0}\right] \tag{8.2.8}$$

with known X^0 and $F_1(z)$.

3rd step. Calculation of the operators X^1 by solving the *determining equation for deformations:*

$$X^1 F_0(z)\Big|_{F_0(z)=0} + H = 0. \tag{8.2.9}$$

Note that Eq. (8.2.9), unlike the determining equation (8.2.7) for exact symmetries, is *inhomogeneous.*

8.3 Examples

8.3.1 First example

Example 8.1. Let us find the approximate symmetries of the following *perturbed* nonlinear wave equation:

$$u_{tt} - \left(u^2 u_x\right)_x + \varepsilon u_t = 0. \tag{8.3.1}$$

We write the approximate operators admitted by Eq. (8.3.1) in the form

$$X = X^0 + \varepsilon X^1 \equiv (\tau_0 + \varepsilon\tau_1)\frac{\partial}{\partial t} + (\xi_0 + \varepsilon\xi_1)\frac{\partial}{\partial x} + (\eta_0 + \varepsilon\eta_1)\frac{\partial}{\partial u}, \tag{8.3.2}$$

where τ_ν, ξ_ν, and η_ν $(\nu = 0, 1)$ are unknown functions of t, x and u.

1st step. Solving the determining equation (8.2.7) for the exact symmetries X^0 of the unperturbed equation

$$u_{tt} - \left(u^2 u_x\right)_x = 0 \tag{8.3.3}$$

one obtains (it follows from the group classification given in [1])

$$\tau_0 = C_1 + C_3 t, \quad \xi_0 = C_2 + (C_3 + C_4)x, \quad \eta_0 = C_4 u, \tag{8.3.4}$$

where $C_1, \ldots, C_4$ are arbitrary constants. Hence,

$$X^0 = (C_1 + C_3 t)\frac{\partial}{\partial t} + (C_2 + C_3 x + C_4 x)\frac{\partial}{\partial x} + C_4 u\frac{\partial}{\partial u}\cdot \tag{8.3.5}$$

In other words, equation (8.3.3) admits the four-dimensional Lie algebra L_4 with the basis

$$X_1^0 = \frac{\partial}{\partial t}, \quad X_2^0 = \frac{\partial}{\partial x}, \quad X_3^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4^0 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\cdot \tag{8.3.6}$$

2nd step. Substituting the expression (8.3.5) of the generator X^0 in Eq. (8.2.8) we obtain the auxiliary function

$$H = C_3 u_t.$$

3rd step. Now the determining equation (8.2.9) for deformations is written

$$X^1\left(u_{tt} - u^2 u_{xx} - 2u u_x^2\right)\Big|_{(8.3.3)} + C_3 u_t = 0, \tag{8.3.7}$$

where X^1 denotes the prolongation of the operator

$$X^1 = \tau_1\frac{\partial}{\partial t} + \xi_1\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial u}$$

to the derivatives of u involved in Eq. (8.3.1). Upon setting $u_{tt} = \left(u^2 u_x\right)_x$ the left-hand side of Eq. (8.3.7) becomes a polynomial in the variables u_{tx}, u_{xx}, u_t, u_x. Equating to zero its coefficients we obtain:

$$\tau_1 = \tau_1(t), \quad \xi_1 = \xi_1(x), \quad 3\tau_1'' = C_3,$$
$$\xi_1'' = 0, \quad \eta_1 = (\xi_1'(x) - \tau_1'(t))u.$$

Hence,

$$\tau_1 = A_1 + A_3 t + \frac{1}{6} C_3 t^2,$$
$$\xi_1 = A_2 + (A_3 + A_4)x, \tag{8.3.8}$$
$$\eta_1 = \left(A_4 - \frac{1}{3} C_3 t\right)u.$$

Substituting Eqs. (8.3.4) and (8.3.8) in Eq. (8.3.2), we obtain the following approximate symmetries for Eq. (8.3.1):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x},$$
$$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{\varepsilon}{6}\left(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right), \tag{8.3.9}$$
$$X_4 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_5 = \varepsilon X_1, \quad X_6 = \varepsilon X_2,$$
$$X_7 = \varepsilon X_4, \quad X_8 = \varepsilon X_3.$$

Remark 8.3. Equations (8.3.8) upon letting $A_1 = C_3 = A_2 = A_4 = 0$ yield

$$X_8 = \varepsilon\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right).$$

However, in the first-order of precision, the operator X_8 can be written in the form given in Eqs. (8.3.9).

Remark 8.4. Equations (8.3.9) show that all symmetries (8.3.6) of Eq. (8.3.3) are stable. It means that the Lie algebra L_4 spanned by the operators (8.3.6) is stable. Hence, the *perturbed equation* (8.3.1) *inherits the symmetries of the unperturbed equation* (8.3.3).

8.3.2 Approximate commutator and Lie algebras

The following table of commutators, evaluated in the first-order of precision, shows that the operators (8.3.9) span an eight-dimensional *approximate Lie algebra* L_8, and

hence generate an eight-parameter approximate transformations group. For the sake of brevity, we will write only the off-diagonal elements of the table of commutators.

Table 8.3.1 shows that although the approximate symmetries X_5, X_6, X_7 and X_8 are "trivial" according to Remark 8.2, they are necessary for the Lie algebra structure, and hence for constructing multi-parameter approximate transformation groups.

Table 8.3.1. Approximate commutators

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$X_1+\frac{1}{3}(X_8-X_7)$	0	0	0	0	X_5
X_2		0	X_2	X_2	0	0	X_6	X_6
X_3			0	0	$-X_5$	$-X_6$	0	0
X_4				0	0	$-X_6$	0	0
X_5					0	0	0	0
X_6						0	0	0
X_7							0	0
X_8								0

8.3.3 Second example

Example 8.2. Consider the equation

$$u_{tt} - \left(u^{-4}u_x\right)_x + \varepsilon u_t = 0. \tag{8.3.10}$$

1st step. The unperturbed equation

$$u_{tt} - \left(u^{-4}u_x\right)_x = 0 \tag{8.3.11}$$

admits the five-dimensional Lie algebra L_5 consisting of the operators

$$X^0 = \left(C_1 + C_3 t + C_5 t^2\right)\frac{\partial}{\partial t} + \left(C_2 + C_3 x + C_4 x\right)\frac{\partial}{\partial x}$$
$$+\left(-\frac{1}{2}C_4 + C_5 t\right)u\frac{\partial}{\partial u}.$$

A basis of this algebra is

$$X_1^0 = \frac{\partial}{\partial t}, \quad X_2^0 = \frac{\partial}{\partial x}, \quad X_3^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},$$
$$X_4^0 = x\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u}, \quad X_5^0 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}. \tag{8.3.12}$$

2nd step. Equation (8.2.8) provides the following auxiliary function:

$$H = C_3 u_t + 2C_5 t u_t + C_5 u.$$

3rd step. The determining equation (8.2.9) for deformations yields that $C_3 = C_5 = 0$. Proceeding further as in the previous example, we obtain the following approximate symmetries of the perturbed equation (8.3.10):

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \\
&X_3 = \varepsilon\Big(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\Big), \quad X_4 = x\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u}, \\
&X_5 = \varepsilon X_1, \quad X_6 = \varepsilon X_2, \\
&X_7 = \varepsilon X_4, \quad X_8 = \varepsilon\Big(t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}\Big).
\end{aligned} \tag{8.3.13}$$

Equations (8.3.13) show that not all symmetries (8.3.12) of Eq. (8.3.11) are stable. Namely, the operators

$$X_3^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \quad \text{and} \quad X_5^0 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$$

from Eqs. (8.3.12) are unstable. Hence, the perturbed equation (8.3.10) does not inherit the symmetries of the unperturbed equation (8.3.11).

8.3.4 Third example

Example 8.3. Consider the equation

$$u_{tt} - \big(u^{-4/3}u_x\big)_x + \varepsilon u_t = 0. \tag{8.3.14}$$

The unperturbed equation

$$u_{tt} - \Big(u^{-4/3}u_x\Big)_x = 0 \tag{8.3.15}$$

admits the five-dimensional Lie algebra L_5 spanned by the operators

$$\begin{aligned}
&X_1^0 = \frac{\partial}{\partial t}, \quad X_2^0 = \frac{\partial}{\partial x}, \quad X_3^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\
&X_4^0 = x\frac{\partial}{\partial x} - \frac{3}{2}u\frac{\partial}{\partial u}, \quad X_5^0 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.
\end{aligned} \tag{8.3.16}$$

Proceeding as before, one arrives at the following approximate symmetries of the perturbed equation (8.3.14):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \left(t - \frac{\varepsilon}{4}t^2\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{3}{4}\varepsilon t u\frac{\partial}{\partial u},$$

$$X_4 = x\frac{\partial}{\partial x} - \frac{3}{2}u\frac{\partial}{\partial u}, \quad X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}, \quad X_6 = \varepsilon X_1, \tag{8.3.17}$$

$$X_7 = \varepsilon X_2, \quad X_8 = \varepsilon X_3^0, \quad X_9 = \varepsilon X_4, \quad X_{10} = \varepsilon X_5.$$

They span a ten-dimensional approximate Lie algebra L_{10}. We see that in this example the algebra L_5 is stable, i.e. the perturbed equation (8.3.14) inherits the symmetries of the unperturbed equation.

Remark 8.5. Equations (8.3.1) and (8.3.10) with an arbitrary parameter $\varepsilon \neq 0$ have only *three exact symmetries* (see [12], page 227, Section 9.2.1.1.2). Namely, the exact symmetries of Eq. (8.3.1) are the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_4 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$$

from (8.3.9), and the exact symmetries of Eq. (8.3.10) are the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_4 = x\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u}$$

from (8.3.13).

More examples can be found in [12], Chapters 2 and 9.

Exercises to Chapter 8

Exercise 8.1. Verify that the operator X_3 from (8.3.9),

$$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{\varepsilon}{6}\left(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right),$$

is an approximate symmetry for the perturbed equation (8.3.1).

Exercise 8.2. Make the table of approximate commutators for the operators (8.3.13).

Exercise 8.3. Verify that the operator X_3 from (8.3.17),

$$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{\varepsilon}{4}\left(t^2\frac{\partial}{\partial t} + 3tu\frac{\partial}{\partial u}\right),$$

is an approximate symmetry for the perturbed equation (8.3.14).

Exercise 8.4. Make the table of approximate commutators for the operators (8.3.17).

Exercise 8.5. Construct the one-parameter approximate transformation group with the generator X_3 from Exercise 8.3.

Chapter 9
Applications

9.1 Integration of equations with a small parameter using approximate symmetries

Let us discuss, by way of examples, group methods of integration of differential equations with a small parameter with known approximate symmetries.

9.1.1 Equation having no exact point symmetries

Example 9.1. The second-order equation

$$y'' - x - \varepsilon y^2 = 0 \tag{9.1.1}$$

has no exact point symmetries if $\varepsilon \neq 0$ is regarded as a constant coefficient, and hence cannot be integrated by the Lie method. Moreover, this equation cannot be integrated by quadrature. However, it possesses approximate symmetries if ε is treated as a small parameter, e.g.

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y} + \frac{\varepsilon}{3}\left[2x^3\frac{\partial}{\partial x} + \left(3yx^2 + \frac{11}{20}x^5\right)\frac{\partial}{\partial y}\right], \\ X_2 &= x\frac{\partial}{\partial y} + \frac{\varepsilon}{6}\left[x^4\frac{\partial}{\partial x} + \left(2yx^3 + \frac{7}{30}x^6\right)\frac{\partial}{\partial y}\right]. \end{aligned} \tag{9.1.2}$$

The commutator of the operators (9.1.2) is $[X_1, X_2] \approx 0$. Hence they span a two-dimensional approximate Abelian Lie algebra and can be used for consecutive integration of Eq. (9.1.1) as follows ([14], Section 12.4).

Equations $X_1(t) \approx 1$, $X_1(u) \approx 0$ yield the canonical variables

$$t = y - \varepsilon\left(\frac{1}{2}x^2y^2 + \frac{11}{60}yx^5\right), \quad u = x - \frac{2}{3}\varepsilon yx^3 \tag{9.1.3}$$

for X_1 from (9.1.2). Thus we have $X_1 \approx \partial/\partial t$, while Eq. (9.1.1) reads

$$u'' + uu'^3 + \varepsilon\left[3u^2u' + \frac{1}{6}(u^2u')^2 - \frac{11}{60}(u^2u')^3\right] = 0.$$

Integration by the standard substitution $u' = p(u)$ yields:

$$p' + up^2 + \varepsilon\left(3u^2 + \frac{1}{6}u^4p - \frac{11}{60}u^6p^2\right) = 0. \tag{9.1.4}$$

The second operator of (9.1.2) is written

$$X_2 = p^2\frac{\partial}{\partial p} + \varepsilon\left[\frac{1}{2}u^4\frac{\partial}{\partial u} + \left(2u^3p - \frac{13}{15}u^5p^2\right)\right]\frac{\partial}{\partial p}$$

and becomes the infinitesimal translation

$$X_2 \approx \frac{\partial}{\partial v}$$

upon introducing the new independent variable z and dependent variable v defined by the equations

$$z = u + \varepsilon\frac{u^4}{2p}, \quad v = -\frac{1}{p} + \varepsilon\left(\frac{u^3}{p^2} - \frac{13u^5}{15p}\right). \tag{9.1.5}$$

In the variables of (9.1.5), equation (9.1.4) is written

$$v' + z + \frac{11}{60}\varepsilon z^6 = 0$$

and yields

$$v = -\frac{11}{60}z^2 - \frac{11}{420}\varepsilon z^7 + C.$$

Substituting this expression for v in Eqs. (9.1.5) and eliminating z, one obtains the function $p(u)$. The quadrature

$$\int \frac{du}{p(u)} = t + C$$

followed by the substitution in Eqs. (9.1.3), completes the approximate integration of Eq. (9.1.1).

9.1.2 Utilization of stable symmetries

Let us consider the van der Pol equation

$$y''+y=\varepsilon(y'-y'^3),\quad \varepsilon=\text{const.} \tag{9.1.6}$$

If ε is treated as an arbitrary constant, equation (9.1.6) has only one *exact* point symmetry, namely the translation of the independent variable x with the generator

$$X=\frac{\partial}{\partial x}.$$

On the other hand, it can be shown that Eq. (9.1.6) with a small parameter ε inherits all 8 point symmetries of the unperturbed equation

$$y''+y=0. \tag{9.1.7}$$

The stability (see Definition 8.2.1) of all symmetries is a necessary condition for existence of an approximate transformation connecting perturbed and unperturbed equations. Let us use this circumstance for the integration of the van der Pol equation in the first order of precision.

The unperturbed equation (9.1.7) is homogeneous, and hence admits the dilation generator

$$X^0=y\frac{\partial}{\partial y}. \tag{9.1.8}$$

The reckoning shows that although the perturbed equation (9.1.6) is not homogeneous, it admits, as an approximate symmetry, the following deformation of the operator (9.1.8):

$$X=\left(y-\frac{\varepsilon}{4}\left[y^2y'+3xy(y^2+y'^2)\right]\right)\frac{\partial}{\partial y}. \tag{9.1.9}$$

We will write Eq. (9.1.7) and its symmetry (9.1.8) in the form

$$z''+z=0 \tag{9.1.10}$$

and

$$X^0=z\frac{\partial}{\partial z}, \tag{9.1.11}$$

respectively, and look for an approximate transformation

$$y=z+\varepsilon f$$

connecting the van der Pol equation (9.1.6) with Eq. (9.1.10).

We can simplify the calculations by using the stability of the dilation generator (9.1.8). Namely, we will begin with constructing the approximate transformations mapping the operator (9.1.11) into the operator (9.1.9). Then we will subject the resulting transformations to the condition that they connect Eqs. (9.1.10) and (9.1.6).

Noting that the approximate symmetry (9.1.9) depends on the first-order derivative y', we will search the approximate transformation of (9.1.11) into (9.1.9) in the form

$$y=z+\varepsilon f(x,z,z'). \tag{9.1.12}$$

The operator (9.1.11) is written in the variable y given by (9.1.12) as follows:

$$\overline{X}^0 = X^0(y)\frac{\partial}{\partial y},$$

where the prolongation of X^0 to the derivative y' is understood. So we have:

$$X^0(y) = \left(z\frac{\partial}{\partial z} + z'\frac{\partial}{\partial z'}\right)(z + \varepsilon f(x,z,z')) = z + \varepsilon\left(z\frac{\partial f}{\partial z} + z'\frac{\partial f}{\partial z'}\right).$$

Since $\overline{X}^0$ should be identical with the operator (9.1.9), we have:

$$z + \varepsilon\left(z\frac{\partial f}{\partial z} + z'\frac{\partial f}{\partial z'}\right) = y - \frac{\varepsilon}{4}[y^2y' + 3xy(y^2 + y'^2)].$$

Substituting in the right-hand side of this equation the expression (9.1.12) for y and noting that $y = z$ up to terms of order ε according to Eq. (9.1.12), we get

$$z + \varepsilon\left(z\frac{\partial f}{\partial z} + z'\frac{\partial f}{\partial z'}\right) = z + \varepsilon\left(f - \frac{1}{4}[z^2z' + 3xz(z^2 + z'^2)]\right),$$

whence

$$z\frac{\partial f}{\partial z} + z'\frac{\partial f}{\partial z'} = f - \frac{1}{4}[z^2z' + 3xz(z^2 + z'^2)]. \tag{9.1.13}$$

Let us solve the non-homogeneous linear first-order partial differential equation (9.1.13). The characteristic system of Eq. (9.1.13) is

$$\frac{\mathrm{d}z}{z} = \frac{\mathrm{d}z'}{z'} = \frac{\mathrm{d}f}{f - \frac{1}{4}[z^2z' + 3xz(z^2 + z'^2)]}.$$

The first equation of this system yields the first integral $z'/z = \lambda = \text{const}$. Writing the second equation of the characteristic system in the form

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{1}{z}\left(f - \frac{1}{4}[z^2z' + 3xz(z^2 + z'^2)]\right),$$

and substituting $z' = \lambda z$, we obtain the following linear first-order ordinary differential equation for f:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{1}{z}f - \frac{1}{4}[\lambda z^2 + 3x(1 + \lambda^2)z^2].$$

The integration yields

$$f = -\frac{1}{8}[K(x)z + \lambda z^3 + 3xz(1 + \lambda^2)z^2],$$

where the "constant of integration" $K(x)$ is an arbitrary function of x. Substituting $\lambda z = z'$, we obtain

$$f = -\frac{1}{8}\left[K(x)z + z^2z' + 3xz(z^2 + z'^2)\right]$$

and finally arrive at the following transformation (9.1.12) mapping the operator (9.1.11) into (9.1.9):

$$y = z - \frac{\varepsilon}{8}\left[K(x)z + z^2z' + 3xz(z^2 + z'^2)\right]. \tag{9.1.14}$$

Since $y = z$ up to terms of order ε, the inverse to (9.1.11) in the first order of approximation is given by

$$z = y + \frac{\varepsilon}{8}\left[K(x)y + y^2y' + 3xy(y^2 + y'^2)\right]. \tag{9.1.15}$$

Now we will determine $K(x)$ using the requirement that (9.1.14) maps Eq. (9.1.10) into Eq. (9.1.6). Let us substitute (9.1.15) in Eq. (9.1.10). Since $y = z$ up to terms of order ε, the equation $z'' = -z$ yields that

$$y'' = -y$$

up to terms of order ε., Therefore, differentiating (9.1.15), we have:

$$z' = y' + \frac{\varepsilon}{8}\left[K(x)y' + K'(x)y + 2y^3 + 5yy'^2 + 3x(y^2y' + y'^3)\right]$$

and

$$z'' = y'' + \frac{\varepsilon}{8}\left[2K'(x)y' + K''(x)y - K(x)y - y^2y' + 8y'^3 - 3xy(y^2 + y'^2)\right].$$

Consequently,

$$z'' + z = y'' + y + \frac{\varepsilon}{8}\left[8y'^3 + 2K'(x)y' + K''(x)y\right].$$

Hence, we obtain Eq. (9.1.6) if we let $K(x) = -4x$. Thus, we have arrived at the following transformation mapping Eq. (9.1.10) into Eq. (9.1.6):

$$y = z + \frac{\varepsilon}{8}\left[4xz - z^2z' - 3xz\left(z^2 + z'^2\right)\right]. \tag{9.1.16}$$

Substituting in (9.1.16) the general solution $z = A\cos t + B\sin t$ of Eq. (9.1.10) one obtains the following *general approximate solution* of the van der Pol equation (9.1.6):

$$y = A\cos x + B\sin x + \frac{\varepsilon}{8}\Big[\big(4 - 3(A^2+B^2)\big)x(A\cos x + B\sin x) + (A\sin x - B\cos x)(A\cos x + B\sin x)^2\Big]. \tag{9.1.17}$$

Letting here $A = 1, B = 0$ and then $A = 0, B = 1$, i.e. taking a fundamental set of solutions $z_1 = \cos x$ and $z_2 = \sin x$ of the linear equation $z'' + z = 0$, one obtains the following *particular approximate solutions* of Eq. (9.1.6):

$$y_1 = \cos x + \frac{\varepsilon}{8}\left[x\cos x + \sin x\cos^2 x\right],$$

$$y_2 = \sin x + \frac{\varepsilon}{8}\left[x\sin x - \cos x\sin^2 x\right].$$

9.2 Approximately invariant solutions

9.2.1 Nonlinear wave equation

Example 9.2. Consider Eq. (8.3.1),

$$u_{tt} - \left(u^2 u_x\right)_x + \varepsilon u_t = 0, \tag{9.2.1}$$

with known approximate symmetries (8.3.9) and find an approximately invariant solution based on the approximate symmetry $X = X_3 - X_4$ with X_3, X_4 from (8.3.9). We have:

$$X = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + \frac{\varepsilon}{6}\left(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right). \tag{9.2.2}$$

The approximate invariants for the operator (9.2.2) are written in the form

$$J(t,x,u,\varepsilon) = J^0(t,x,u) + \varepsilon J^1(t,x,u) + o(\varepsilon).$$

They are determined by the equation

$$X(J) = o(\varepsilon).$$

Using for the operator (9.2.2) the notation

$$X = X^0 + \varepsilon X^1,$$

where

$$X^0 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad X^1 = \frac{1}{6}\left(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right),$$

we write the determining equation $X(J) = o(\varepsilon)$ for the approximate invariants in the form

$$X^0(J^0)+\varepsilon\left[X^0(J^1)+X^1(J^0)\right]=0,$$

whence

$$X^0(J^0)=0,\ X^0(J^1)+X^1(J^0)=0,$$

or

$$\begin{aligned} &t\frac{\partial J^0}{\partial t}-u\frac{\partial J^0}{\partial u}=0,\\ &t\frac{\partial J^1}{\partial t}-u\frac{\partial J^1}{\partial u}=-\frac{1}{6}\left(t^2\frac{\partial J^0}{\partial t}-2tu\frac{\partial J^0}{\partial u}\right). \end{aligned} \tag{9.2.3}$$

Solving Eqs. (9.2.3), we will find two functionally independent invariants

$$\begin{aligned} J_1&=J_1^0(t,x,u)+\varepsilon J_1^1(t,x,u),\\ J_2&=J_2^0(t,x,u)+\varepsilon J_2^1(t,x,u) \end{aligned} \tag{9.2.4}$$

for the operator (9.2.2).

Remark 9.1. Recall that functions (9.2.4) are said to be functionally dependent if $J_2=\Psi(J_1)$, in other words if the equation

$$J_2^0(t,x,u)+\varepsilon J_2^1(t,x,u)=\Psi\left(J_1^0(t,x,u)+\varepsilon J_1^1(t,x,u)\right)+o(\varepsilon)$$

with a certain function Ψ holds identically in t,x,u. If such a function Ψ *does not exist,* the functions (9.2.4) are said to be *functionally independent.* It is manifest that if $J_1^0(t,x,u)$ and $J_2^0(t,x,u)$ are functionally independent, then so are the functions (9.2.4) as well.

Let us proceed to the integration of the system (9.2.3). The first equation from (9.2.3) has precisely two functionally independent solutions, namely

$$J_1^0=x,\quad J_2^0=tu.$$

Substituting $J_1^0=x$ in the second equation of (9.2.3) and taking its simplest solution $J_1^1=0$, we obtain one invariant of (9.2.4),

$$J_1=x. \tag{9.2.5}$$

Note that it does not involve the dependent variable u. Now we substitute the solution $J_2^0=tu$ of the first equation of (9.2.3) into the second equation of (9.2.3) and obtain the following non-homogeneous linear equation:

$$t\frac{\partial J_2^1}{\partial t}-u\frac{\partial J_2^1}{\partial u}=\frac{1}{6}t^2u.$$

The first equation of the characteristic system

$$\frac{dt}{t} = -\frac{du}{u} = 6\frac{dJ_2^1}{t^2 u}$$

yields the first integral $tu = \lambda = \text{const}$. Therefore the second equation

$$\frac{dt}{t} = 6\frac{dJ_2^1}{t^2 u}$$

of the characteristic system, upon replacing tu by λ, is written $\lambda\, dt = 6\, dJ_2^1$ and yields

$$J_2^1 = \frac{1}{6} t\lambda + C = \frac{1}{6} t^2 u + C.$$

Letting here $C = 0$, we obtain the second invariant (9.2.4),

$$J_2 = tu + \frac{\varepsilon}{6} t^2 u. \tag{9.2.6}$$

By Remark 9.1 the invariants (9.2.5), (9.2.6) are functionally independent.

Letting $J_2 = \varphi(J_1)$, i.e.

$$\left(1 + \frac{\varepsilon}{6} t\right) tu = \varphi(x)$$

and solving for tu in the first order of precision,

$$tu = \left(1 + \frac{\varepsilon t}{6}\right)^{-1} \varphi(x) = \left(1 - \frac{\varepsilon t}{6}\right) \varphi(x) + o(\varepsilon),$$

we obtain the following form for the approximately invariant solutions:

$$u = \left(\frac{1}{t} - \frac{\varepsilon}{6}\right) \varphi(x). \tag{9.2.7}$$

Now we substitute (9.2.7) in Eq. (9.2.1). Differentiation of (9.2.7) yields:

$$u_t = -\frac{1}{t^2} \varphi, \quad u_{tt} = \frac{2}{t^3} \varphi, \quad u_x = \left(\frac{1}{t} - \frac{\varepsilon}{6}\right) \varphi'.$$

Furthermore,

$$u^2 u_x = \left(\frac{1}{t} - \frac{\varepsilon}{6}\right)^3 \varphi^2 \varphi' = \left(\frac{1}{t^3} - \frac{\varepsilon}{2t^2}\right) \varphi^2 \varphi' + o(\varepsilon),$$

and hence we have in our approximation:

$$(u^2 u_x)_x = \left(\frac{1}{t^3} - \frac{\varepsilon}{2t^2}\right) (\varphi^2 \varphi')'.$$

Therefore

$$u_{tt} - \left(u^2 u_x\right)_x + \varepsilon u_t = \frac{1}{t^3}\left(1 - \frac{\varepsilon t}{2}\right)\left[2\varphi - (\varphi^2\varphi')'\right].$$

Thus, equation (9.2.1) yields

$$(\varphi^2\varphi')' = 2\varphi. \tag{9.2.8}$$

Let us integrate Eq. (9.2.8). We have

$$\varphi^2\varphi'' + 2\varphi\varphi'^2 = 2\varphi,$$

or

$$\varphi\varphi'' + 2\varphi'^2 = 2.$$

The latter equation is reduced to a first-order equation by the standard substitution $\varphi' = p(\varphi)$. Since $\varphi'' = pp'$, we have

$$\varphi pp' + 2p^2 = 2,$$

or

$$\varphi\frac{\mathrm{d}p^2}{\mathrm{d}\varphi} + 4p^2 = 4.$$

Now we denote $p^2 = v$, integrate the resulting linear equation

$$\varphi\frac{\mathrm{d}v}{\mathrm{d}\varphi} + 4v = 4$$

and obtain $v = 1 + C\varphi^{-4}$. The equation $p \pm \sqrt{v}$ yields

$$p = \pm\sqrt{1 + C\varphi^{-4}}.$$

Thus,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \pm\sqrt{1 + C\varphi^{-4}}$$

and the integration provides the general solution to Eq. (9.2.8) in the following implicit form involving one quadrature:

$$\int \frac{\varphi^2 \mathrm{d}\varphi}{\sqrt{\varphi^4 + C}} = C \pm x. \tag{9.2.9}$$

Substituting $\varphi(x)$ given by (9.2.9) in (9.2.7) we obtain the invariant solution to Eq. (9.2.1). For example, setting $C = 0$ we have from (9.2.9) $\varphi(x) = \pm x$, and the invariant solution (9.2.7) is

$$u = \pm\left(\frac{x}{t} - \varepsilon\frac{x}{6}\right).$$

9.2.2 Approximate travelling waves of KdV equation

Example 9.3. Let us find the *approximate travelling waves* for the following perturbed Korteweg-de Vries equation:

$$u_t = uu_x + u_{xxx} + \varepsilon u. \tag{9.2.10}$$

We will begin with the well-known solitary wave solution $u = U_0$,

$$U_0 = 3\kappa \operatorname{sech}^2 z, \tag{9.2.11}$$

of the Korteweg-de Vries equation

$$u_t = uu_x + u_{xxx}. \tag{9.2.12}$$

Here

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}},$$

and

$$z = \frac{\sqrt{\kappa}}{2}(x + \kappa t)$$

is an invariant of the group with the generator

$$X^0 = \frac{\partial}{\partial t} - \kappa \frac{\partial}{\partial x}, \quad \kappa = \text{const}. \tag{9.2.13}$$

The function U_0 given by Eq. (9.2.11) is invariant under the operator (9.2.13). Let us check that $u = U_0$ is an invariant solution, i.e. that it solves Eq. (9.2.12). Using the equations

$$(\operatorname{sech} z)' = -\operatorname{sech}^2 z \cdot \sinh z, \quad (\operatorname{sech}^2 z)' = -2\operatorname{sech}^3 z \cdot \sinh z,$$

where

$$\sinh z = \frac{e^z - e^{-z}}{2},$$

we have:

$$\begin{aligned}
(U_0)_t &= -3\kappa^{5/2} \operatorname{sech}^3 z \cdot \sinh z, \\
(U_0)_x &= -3\kappa^{3/2} \operatorname{sech}^3 z \cdot \sinh z, \\
(U_0)_{xx} &= \frac{9}{2}\kappa^2 \operatorname{sech}^4 z \cdot \sinh^2 z - \frac{3}{2}\kappa^2 \operatorname{sech}^2 z, \\
(U_0)_{xxx} &= 3\kappa^{5/2}\left(3\operatorname{sech}^2 z - 1\right)\operatorname{sech}^3 z \cdot \sinh z.
\end{aligned} \tag{9.2.14}$$

In the last equation of (9.2.14) the following identity has been used:

$$\sinh^3 z = \left(\cosh^2 z - 1\right)\sinh z.$$

It is manifest from Eqs. (9.2.14) that (9.2.11) solves Eq. (9.2.12):

$$(U_0)_t = U_0(U_0)_x + (U_0)_{xxx}.$$

Using the generators of the Galilean and scaling transformations,

$$X_1 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \quad X_2 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$$

admitted by the Korteweg-de Vries equation (9.2.12), and invoking Remark 8.2, we consider the approximate symmetry $X_\varepsilon = X^0 + \varepsilon(\kappa X_1 - X_2)$ of the perturbed equation (9.2.10). Thus we will take the approximate symmetry

$$X_\varepsilon = \frac{\partial}{\partial t} - \kappa\frac{\partial}{\partial x} - \varepsilon\Big[3t\frac{\partial}{\partial t} + (x-\kappa t)\frac{\partial}{\partial x} + (\kappa - 2u)\frac{\partial}{\partial u}\Big] \tag{9.2.15}$$

and use it for finding an approximately invariant solution by looking for the invariant perturbation of the solitary wave (9.2.11):

$$u = U_0 + \varepsilon v(t,x). \tag{9.2.16}$$

Let us write the operator (9.2.15) in the form

$$X_\varepsilon = X^0 + \varepsilon X^1,$$

where X^0 is the operator (9.2.13) and X^1 is given by

$$X^1 = -\Big[3t\frac{\partial}{\partial t} + (x-\kappa t)\frac{\partial}{\partial x} + (\kappa - 2u)\frac{\partial}{\partial u}\Big]. \tag{9.2.17}$$

Then the invariant equation test for Eq. (9.2.16),

$$X_\varepsilon(u - U_0 - \varepsilon v)\Big|_{(9.2.16)} = o(\varepsilon),$$

is written

$$\Big(X^0(u-U_0) + \varepsilon\big[X^1(u-U_0) - X^0(v)\big]\Big)_{u=U_0} = 0. \tag{9.2.18}$$

Note that $X^0(u-U_0)$ vanishes identically since the operator X^0 does not contain the differentiation in u and since the function U_0 is an invariant because it depends only on the invariant z. Therefore Eq. (9.2.18) becomes

$$\big[X^1(u-U_0) - X^0(v)\big]_{u=U_0} = 0,$$

whence, invoking definitions (9.2.13) and (9.2.17) of X^0 and X^1, respectively, we obtain the following differential equation for $v(t,x)$:

$$v_t - \kappa v_x = 3t(U_0)_t + (x-\kappa t)(U_0)_x + 2U_0 - \kappa. \tag{9.2.19}$$

Since $U_0 = U_0(z)$, it is easy to integrate Eq. (9.2.19) in the "natural" variables

$$z = \frac{\sqrt{\kappa}}{2}(x + \kappa t), \quad y = t.$$

Then, denoting by U_0' the derivative of U_0 with respect to z, we have:

$$(U_0)_t = \frac{1}{2}\kappa^{3/2} U_0', \quad (U_0)_x = \frac{1}{2}\kappa^{1/2} U_0',$$

and Eq. (9.2.19) becomes

$$v_y = \left(z + \frac{1}{2}\kappa^{3/2} y\right) U_0' + 2U_0 - \kappa.$$

The integration yields:

$$v = \left(zy + \frac{1}{4}\kappa^{3/2} y^2\right) U_0' + (2U_0 - \kappa)y + g(z).$$

Returning to the variables t, x, we have:

$$v = \left(xt + \frac{3}{2}t^2\right)(U_0)_x + (2U_0 - \kappa)t + h(x + \kappa t).$$

Inserting this v in (9.2.16) and substituting in the perturbed KdV equation (9.2.10) we obtain $h = x + \kappa t$.

Thus, the approximate symmetry (9.2.15) provides the following approximately invariant solution (*approximate travelling wave*):

$$u = U_0 + \varepsilon\left[x + 2tU_0 + \left(xt + \frac{3}{2}t^2\right)(U_0)_x\right], \tag{9.2.20}$$

or invoking Eqs. (9.2.11) and (9.2.14):

$$u = 3\kappa \operatorname{sech}^2 z + \varepsilon\left[x + 6\kappa t \operatorname{sech}^2 z - 3\kappa^{\frac{3}{2}}\left(tx + \frac{3}{2}\kappa t^2\right)\operatorname{sech}^3 z \cdot \sinh z\right].$$

9.3 Approximate conservation laws

Noether's theorem can be generalized to equations with a small parameter admitting approximate transformation groups. For the sake of brevity, we restrict the discussion to first-order Lagrangians $L(x, u, u_{(1)}, \varepsilon)$ and to approximate point transformation groups. Then one can prove the following statement on approximate conservation laws [4] (see also [12], Section 2.6).

Theorem 9.1. Let approximate Euler-Lagrange equations

$$\frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u_i^\alpha}\right) = o(\varepsilon) \tag{9.3.1}$$

possess an approximate symmetry

$$X = \left[\xi_0^i(x,u) + \varepsilon\xi_1^i(x,u)\right]\frac{\partial}{\partial x^i} + \left[\eta_0^\alpha(x,u) + \varepsilon\eta_1^\alpha(x,u)\right]\frac{\partial}{\partial u^\alpha}$$

such that the following holds:

$$XL + LD_i(\xi^i) = D_i(B^i) + o(\varepsilon). \tag{9.3.2}$$

Then the functions $C^i(x,u,u_{(1)},\varepsilon)$ defined by

$$C^i = L\xi^i + (\eta^\alpha - \xi^j u_j^\alpha)\frac{\partial L}{\partial u_i^\alpha} - B^i + o(\varepsilon) \tag{9.3.3}$$

satisfy the *approximate conservation law* for Eq. (9.3.1), i.e.

$$D_i(C^i)\Big|_{(9.3.1)} = o(\varepsilon). \tag{9.3.4}$$

The following notation is used in Eqs. (9.3.2)–(9.3.3):

$$\xi^i = \xi_0^i(x,u) + \varepsilon\xi_1^i(x,u),$$
$$\eta^\alpha = \eta_0^\alpha(x,u) + \varepsilon\eta_1^\alpha(x,u).$$

Theorem 9.1 on approximate conservation laws can be extended to Euler-Lagrange equations with higher-order Lagrangians.

Example 9.4. Equation (9.1.1),

$$y'' - x - \varepsilon y^2 = 0,$$

has the Lagrangian

$$L = \frac{1}{2}\left(y'^2 - x^2 y'\right) + \frac{\varepsilon}{3}y^3.$$

Consider an approximate symmetry of Eq. (9.1.1), e.g. the operator X_1 from (9.1.2). Let us verify that the condition (9.3.2) is satisfied. Taking the first prolongation of the operator X_1 :

$$X_1 = \frac{2}{3}\varepsilon x^3\frac{\partial}{\partial x} + \left[1 + \varepsilon\left(yx^2 + \frac{11}{60}x^5\right)\right]\frac{\partial}{\partial y} + \varepsilon\left[2xy - x^2y' + \frac{11}{12}x^4\right]\frac{\partial}{\partial y'}$$

we have, ignoring the higher-order terms in ε :

$$X_1 L + L D_x(\xi) = \varepsilon\Big(2xyy' + y^2 - x^3 y - \frac{1}{4}x^4 y' - \frac{11}{24}x^6\Big)$$

$$= \varepsilon D_x\Big(xy^2 - \frac{1}{4}x^4 y - \frac{11}{24\cdot 7}x^7\Big).$$

Hence, the condition (9.3.2) holds with

$$B = \varepsilon\Big(xy^2 - \frac{1}{4}x^4 y - \frac{11}{24\cdot 7}x^7\Big).$$

The reckoning shows that

$$L\xi + (\eta - \xi y')\frac{\partial L}{\partial y'}$$

$$= y' - \frac{x^2}{2} + \varepsilon\Big(x^2 yy' - \frac{1}{3}x^3 y'^2 + \frac{11}{60}x^5 y' - \frac{1}{2}x^4 y - \frac{11}{60\cdot 2}x^7\Big).$$

Therefore, the formula (9.3.3) and the conservation equation (9.3.4) provide the following *approximate first integral* for Eq. (9.1.1):

$$y' - \frac{x^2}{2} + \varepsilon\Big(x^2 yy' - \frac{1}{3}x^3 y'^2 - xy^2 - \frac{1}{4}x^4 y + \frac{11}{60}x^5 y' - \frac{11}{35\cdot 12}x^7\Big) = C. \qquad (9.3.5)$$

Thus, we have reduced the second-order equation $y'' - x - \varepsilon y^2 = 0$ to the first-order equation (9.3.5) with an arbitrary constant C.

Exercises to Chapter 9

Exercise 9.1. The unperturbed equation (9.1.1),

$$y'' - x = 0,$$

has, as any linear second-order ordinary differential equation, eight symmetries. Find them.

Exercise 9.2. Verify that the operators (9.1.2) are approximate symmetries for Eq. (9.1.1).

Exercise 9.3. Find the one-parameter approximate transformation groups for each operator of (9.1.2):

$$X_1 = \frac{\partial}{\partial y} + \frac{\varepsilon}{3}\left[2x^3\frac{\partial}{\partial x} + \Big(3yx^2 + \frac{11}{20}x^5\Big)\frac{\partial}{\partial y}\right],$$

$$X_2 = x\frac{\partial}{\partial y} + \frac{\varepsilon}{6}\left[x^4\frac{\partial}{\partial x} + \Big(2yx^3 + \frac{7}{30}x^6\Big)\frac{\partial}{\partial y}\right].$$

Assignment to Part II

Problems

1. Find the approximate transformation group for the generator

$$X = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{\varepsilon}{6}\left(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right). \tag{1}$$

2. Verify that the operator (1) is an approximate symmetry for the following equation with the small parameter ε :

$$u_{tt} - \left(u^2 u_x\right)_x + \varepsilon u_t = 0. \tag{2}$$

3. Make the table of approximate commutators for the operators

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \varepsilon\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right),\ X_4 = x\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u},$$

$$X_5 = \varepsilon X_1,\quad X_6 = \varepsilon X_2,\quad X_7 = \varepsilon X_4,\quad X_8 = \varepsilon\left(t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}\right). \tag{3}$$

4. Show that the approximate transformation

$$y = z + \frac{\varepsilon}{8}\left[4xz - z^2 z' - 3xz\left(z^2 + z'^2\right)\right] + o(\varepsilon) \tag{4}$$

with a small parameter ε maps the equation

$$z'' + z = 0 \tag{5}$$

into the van der Pol equation

$$y'' + y = \varepsilon(y' - y'^3). \tag{6}$$

5. The approximate transformation (4) maps the symmetries

$$X_1^0 = \frac{\partial}{\partial x}, \quad X_2^0 = z\frac{\partial}{\partial z} \tag{7}$$

of Eq. (5) into approximate symmetries

$$X_1 = X_1^0 + \varepsilon X_1^1, \quad X_2 = X_2^0 + \varepsilon X_2^1$$

of the perturbed equation (6). Find these approximate symmetries.

Solutions

1. We have to solve the approximate Lie equations (7.3.8)–(7.3.9) for the operator (1). Here $n = 3$ and

$$\xi_0(t,x,u) = (t,x,0), \quad \xi_1(t,x,u) = \left(\frac{1}{6}t^2,\ 0,\ -\frac{1}{3}tu\right).$$

Accordingly, equations (7.3.8) have the form

$$\frac{\mathrm{d}\bar{t}_0}{\mathrm{d}a} = \bar{t}_0, \quad \frac{\mathrm{d}\bar{x}_0}{\mathrm{d}a} = \bar{x}_0, \quad \frac{\mathrm{d}\bar{u}_0}{\mathrm{d}a} = 0;$$

$$a = 0: \quad \bar{t}_0 = t, \quad \bar{x}_0 = x, \quad \bar{u}_0 = u$$

and yield

$$\bar{t}_0 = t\,\mathrm{e}^a, \quad \bar{x}_0 = x\mathrm{e}^a, \quad \bar{u}_0 = u. \tag{8}$$

Therefore Eqs. (7.3.9) are written:

$$\begin{gathered}\frac{\mathrm{d}\bar{t}_1}{\mathrm{d}a} = \bar{t}_1 + \frac{1}{6}t^2\mathrm{e}^{2a}, \quad \frac{\mathrm{d}\bar{x}_1}{\mathrm{d}a} = \bar{x}_1, \quad \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}a} = -\frac{1}{3}tu\,\mathrm{e}^a; \\ a = 0: \quad \bar{t}_1 = \bar{x}_1 = u_1 = 0.\end{gathered} \tag{9}$$

The first differential equation in Eqs. (9) is a non-homogeneous linear first-order equation for $\bar{t}_1$. Its integration yields

$$\bar{t}_1 = \frac{1}{6}t^2\mathrm{e}^{2a} + K\mathrm{e}^a.$$

Using the initial condition $\bar{t}_1 = 0$ at $a = 0$, we find the constant K:

$$K = -\frac{1}{6}t^2.$$

Hence,

$$\bar{t}_1 = \frac{1}{6}t^2\left(\mathrm{e}^a - 1\right)\mathrm{e}^a. \tag{10}$$

The solution to the second differential equation in Eqs. (9) is given by $\bar{x}_1 = Ce^a$. The initial condition, $\bar{x}_1 = 0$ at $a = 0$, yields $C = 0$. Hence,

$$\bar{x}_1 = 0. \tag{11}$$

The solution to the third differential equation in Eqs. (9) is given by

$$\bar{u}_1 = -\frac{1}{3}tue^a + C.$$

The initial condition, $\bar{u}_1 = 0$ at $a = 0$, yields $C = tu/3$. Hence,

$$\bar{u}_1 = \frac{1}{3}tu\left(1 - e^a\right). \tag{12}$$

Collecting Eqs. (8), (10), (11) and (12), we conclude that the approximate transformation group with the generator (1) has the form

$$\bar{t} = \left[t + \frac{\varepsilon}{6}t^2\left(e^a - 1\right)\right]e^a, \quad \bar{x} = xe^a, \quad \bar{u} = u + \frac{\varepsilon}{3}tu\left(1 - e^a\right).$$

2. First we prolong the operator (1) to the derivatives involved in Eq. (2), i.e. we write it in the form

$$X = \left(t + \frac{\varepsilon}{6}t^2\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{\varepsilon}{3}tu\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{11}\frac{\partial}{\partial u_{tt}} + \zeta_{22}\frac{\partial}{\partial u_{xx}},$$

where

$$\zeta_1 = D_t\left(-\frac{\varepsilon}{3}tu\right) - u_tD_t\left(t + \frac{\varepsilon}{6}t^2\right) = -u_t - \frac{\varepsilon}{3}(u + 2tu_t),$$

$$\zeta_2 = D_x\left(-\frac{\varepsilon}{3}tu\right) - u_xD_x(x) = -\left(1 + \frac{\varepsilon}{3}t\right)u_x,$$

$$\zeta_{11} = D_t(\zeta_1) - u_{tt}D_t\left(t + \frac{\varepsilon}{6}t^2\right) = -2u_{tt} - \varepsilon(u_t + tu_{tt}),$$

$$\zeta_{22} = D_x(\zeta_2) - u_{xx}D_x(x) = -\left(2 + \frac{\varepsilon}{3}t\right)u_{xx}.$$

Thus, the prolonged operator (1) has the form

$$X = \left(t + \frac{\varepsilon}{6}t^2\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{\varepsilon}{3}tu\frac{\partial}{\partial u} - \left[u_t + \frac{\varepsilon}{3}(u + 2tu_t)\right]\frac{\partial}{\partial u_t} - \left(1 + \frac{\varepsilon}{3}t\right)u_x\frac{\partial}{\partial u_x} - \left[2u_{tt} + \varepsilon(u_t + tu_{tt})\right]\frac{\partial}{\partial u_{tt}} - \left(2 + \frac{\varepsilon}{3}t\right)u_{xx}\frac{\partial}{\partial u_{xx}}.$$

The calculation shows that the approximate invariance condition (8.2.2) for Eq. (2) is satisfied:

$$\begin{aligned} &X\left(u_{tt}-u^2u_{xx}-2uu_x^2+\varepsilon u_t\right)\\ &=-2\left(u_{tt}-u^2u_{xx}-2uu_x^2+\varepsilon u_t\right)\\ &\quad-\varepsilon t\left(u_{tt}-u^2u_{xx}-2uu_x^2\right)=\varepsilon^2tu_t=o(\varepsilon). \end{aligned}$$

Hence the operator (1) is an approximate symmetry for Eq. (2).

3. The calculation leads to the following table of approximate commutators for the operators (3):

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	X_5	0	0	0	0	$2(X_3-X_7)$
X_2		0	X_6	X_2	0	0	X_6	0
X_3			0	0	0	0	0	0
X_4				0	0	$-X_6$	0	0
X_5					0	0	0	0
X_6						0	0	0
X_7							0	0
X_8								0

4. The inverse to the transformation (4) is given, in the first order of precision, by

$$z=y-\frac{\varepsilon}{8}\left[4xy-y^2y'-3xy\left(y^2+y'^2\right)\right]. \tag{13}$$

Equation (6) yields:

$$y''=-y+O(\varepsilon).$$

Therefore, we have:

$$z'=y'-\frac{\varepsilon}{8}\left[4y-2y^3-5yy'^2+4xy'-3xy^2y'-3xy'^3\right]+o(\varepsilon)$$

and

$$z''=y''-\frac{\varepsilon}{8}\left[8y'+y^2y'-8y'^3-4xy+3xy^3+3xyy'^2\right]+o(\varepsilon). \tag{14}$$

Substituting Eqs. (13) and (14) in the left-hand side of Eq. (5) we obtain

$$z''+z=y''+y-\varepsilon(y'-y'^3)+o(\varepsilon).$$

Hence Eq. (5) leads to Eq. (6).

5. After the approximate transformation (4) the operator X_1^0 from Eqs. (7) becomes

$$X_1 = \frac{\partial}{\partial x} + \varepsilon \left[\frac{1}{2} y - \frac{3}{8} y \left(y^2 + y'^2 \right) \right] \frac{\partial}{\partial y} .$$

Turning to the operator X_2^0 from Eqs. (7), we write it in the prolonged form

$$X_2^0 = z \frac{\partial}{\partial z} + z' \frac{\partial}{\partial z'}$$

and act on Eq. (4) to obtain

$$\begin{aligned} X_2^0 (y) &= z + \frac{\varepsilon}{8} \left[4xz - z^2 z' - 3xz \left(z^2 + z'^2 \right) \right] \\ &\quad - \frac{\varepsilon}{8} \left[2z^2 z' + 6xz \left(z^2 + z'^2 \right) \right] + o(\varepsilon) \\ &= y - \frac{\varepsilon}{4} \left[y^2 y' + 3xy \left(y^2 + y'^2 \right) \right] + o(\varepsilon). \end{aligned}$$

Hence, the operator X_2^0 becomes

$$X_2 = \left(y - \frac{\varepsilon}{4} \left[y^2 y' + 3xy (y^2 + y'^2) \right] \right) \frac{\partial}{\partial y} .$$

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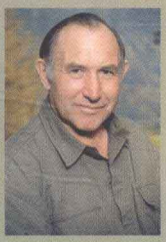

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