第 27 卷第 5 期　　唐山师范学院学报　　2005 年 9 月
Vol. 27 No.5　　*Journal of Tangshan Teachers College*　　*Sep. 2005*

球坐标系中拉普拉斯算符表达式的推导

姚久民，石凤良

（唐山师范学院 物理系，河北 唐山 063000）

摘　要：根据直角坐标 x、y、z 与球坐标 r、θ、φ之间的关系，由直角坐标系中拉普拉斯算符的表达式推出球坐标系中拉普拉斯算符的表达式。

关键词：偏导数；球坐标系；拉普拉斯算符

中图分类号：O413.1　　文献标识码：A　　文章编号：1009-9115（2005）05-0067-05

1　引言

由直角坐标系中哈密顿算符 $\nabla=\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}$ 的表达式可推导出拉普拉斯算符

$$\Delta=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

但我们却不能根据球坐标系 $\nabla=\frac{\partial}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e_\varphi}$ 得出

$$\Delta=(\frac{\partial}{\partial r}\vec{er}+\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e_\varphi})\cdot(\frac{\partial}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e_\varphi})=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\rho^2}$$

文献[1]虽给出了正确的推导过程，但不容易理解。本文从拉普拉斯算符在直角坐标系中的表达式推导出它在球坐标系中的表达式。虽然推导过程稍嫌繁琐，但却浅显易懂，适合学生掌握。

根据直角坐标和球坐标的坐标关系

$$x=r\sin\theta\cos\varphi\qquad y=r\sin\theta\sin\varphi\qquad z=r\cos\theta\qquad r^2=x^2+y^2+z^2\qquad \cos\theta=\frac{z}{r}\qquad tg\varphi=\frac{x}{y}$$

所以 r、θ、φ 是关于 x、y、z 的函数，

即 $r=r$（x、y、z），$\theta=\theta$（x、y、z），$\varphi=\varphi$（x、y、z）。

根据微分形式的不变性可知[2]

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x},\quad \frac{\partial}{\partial y}=\frac{\partial}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial y},\quad \frac{\partial}{\partial z}=\frac{\partial}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial z}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial z}$$

$$\therefore\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial}{\partial x})=\frac{\partial}{\partial x}(\frac{\partial}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x})$$

$$=\frac{\partial}{\partial x}(\frac{\partial}{\partial r})\frac{\partial r}{\partial x}+\frac{\partial}{\partial x}(\frac{\partial r}{\partial x})\frac{\partial}{\partial r}+\frac{\partial}{\partial x}(\frac{\partial}{\partial\theta})\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial x}(\frac{\partial\theta}{\partial x})\frac{\partial}{\partial\theta}+\frac{\partial}{\partial x}(\frac{\partial}{\partial\varphi})\frac{\partial\varphi}{\partial x}+\frac{\partial}{\partial x}(\frac{\partial\varphi}{\partial x})\frac{\partial}{\partial\varphi}$$

$$=\frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r}+\frac{\partial^2\theta}{\partial x^2}\frac{\partial}{\partial\theta}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial x}(\frac{\partial}{\partial r})\frac{\partial r}{\partial x}+\frac{\partial}{\partial x}(\frac{\partial}{\partial\theta})\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial x}(\frac{\partial}{\partial\varphi})\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial r})=\frac{\partial}{\partial r}(\frac{\partial}{\partial r})\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}(\frac{\partial}{\partial r})\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial\varphi}(\frac{\partial}{\partial r})\frac{\partial\varphi}{\partial x}=\frac{\partial^2}{\partial r^2}\frac{\partial r}{\partial x}+\frac{\partial^2}{\partial\theta\partial r}\frac{\partial\theta}{\partial x}+\frac{\partial^2}{\partial\varphi\partial r}\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial\theta})=\frac{\partial}{\partial r}(\frac{\partial}{\partial\theta})\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}(\frac{\partial}{\partial\theta})\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial\varphi}(\frac{\partial}{\partial\theta})\frac{\partial\varphi}{\partial x}=\frac{\partial^2}{\partial\theta^2}\frac{\partial\theta}{\partial x}+\frac{\partial^2}{\partial r\partial\theta}\frac{\partial r}{\partial x}+\frac{\partial^2}{\partial\varphi\partial\theta}\frac{\partial\varphi}{\partial x}$$

收稿日期：2004-08-26

作者简介：姚久民（1966-），男，河北丰润人，唐山师范学院物理系实验师。

© 1994-2012 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial \varphi})=\frac{\partial}{\partial r}(\frac{\partial}{\partial \varphi})\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}(\frac{\partial}{\partial \varphi})\frac{\partial \theta}{\partial x}+\frac{\partial}{\partial \varphi}(\frac{\partial}{\partial \varphi})\frac{\partial \varphi}{\partial x}=\frac{\partial^2}{\partial \varphi^2}\frac{\partial \varphi}{\partial x}+\frac{\partial^2}{\partial r\partial \varphi}\frac{\partial r}{\partial x}+\frac{\partial^2}{\partial \theta\partial \varphi}\frac{\partial \theta}{\partial x}$$

又因为 $\frac{\partial^2}{\partial \theta\partial r}=\frac{\partial^2}{\partial r\partial \theta},\frac{\partial^2}{\partial \varphi\partial r}=\frac{\partial^2}{\partial r\partial \varphi},\frac{\partial^2}{\partial \theta\partial \varphi}=\frac{\partial^2}{\partial \varphi\partial \theta}$，可得

$$\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial r}\frac{\partial^2 r}{\partial x^2}+\frac{\partial}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial}{\partial \varphi}\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2}{\partial r^2}(\frac{\partial r}{\partial x})^2+\frac{\partial^2}{\partial \theta^2}(\frac{\partial \theta}{\partial x})^2+\frac{\partial^2}{\partial \varphi^2}(\frac{\partial \varphi}{\partial x})^2$$
$$+2\frac{\partial^2}{\partial r\partial \theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}+2\frac{\partial^2}{\partial r\partial \varphi}\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x}+2\frac{\partial^2}{\partial \theta\partial \varphi}\frac{\partial \theta}{\partial x}\frac{\partial \varphi}{\partial x}\qquad(1)$$

同理可求得

$$\frac{\partial^2}{\partial y^2}=\frac{\partial}{\partial r}\frac{\partial^2 r}{\partial y^2}+\frac{\partial}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial}{\partial \varphi}\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2}{\partial r^2}(\frac{\partial r}{\partial y})^2+\frac{\partial^2}{\partial \theta^2}(\frac{\partial \theta}{\partial y})^2+\frac{\partial^2}{\partial \varphi^2}(\frac{\partial \varphi}{\partial y})^2$$
$$+2\frac{\partial^2}{\partial r\partial \theta}\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}+2\frac{\partial^2}{\partial r\partial \varphi}\frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y}+2\frac{\partial^2}{\partial \theta\partial \varphi}\frac{\partial \theta}{\partial y}\frac{\partial \varphi}{\partial y}\qquad(2)$$

$$\frac{\partial^2}{\partial z^2}=\frac{\partial}{\partial r}\frac{\partial^2 r}{\partial z^2}+\frac{\partial}{\partial \theta}\frac{\partial^2 \theta}{\partial z^2}+\frac{\partial}{\partial \varphi}\frac{\partial^2 \varphi}{\partial z^2}+\frac{\partial^2}{\partial r^2}(\frac{\partial r}{\partial z})^2+\frac{\partial^2}{\partial \theta^2}(\frac{\partial \theta}{\partial z})^2+\frac{\partial^2}{\partial \varphi^2}(\frac{\partial \varphi}{\partial z})^2$$
$$+2\frac{\partial^2}{\partial r\partial \theta}\frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z}+2\frac{\partial^2}{\partial r\partial \varphi}\frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z}+2\frac{\partial^2}{\partial \theta\partial \varphi}\frac{\partial \theta}{\partial z}\frac{\partial \varphi}{\partial z}\qquad(3)$$

∴式（1）+（2）+（3）得

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial}{\partial r}(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2})+\frac{\partial}{\partial \theta}(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2})+\frac{\partial}{\partial \varphi}(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2 \varphi}{\partial z^2})+$$
$$\frac{\partial^2}{\partial r^2}[(\frac{\partial r}{\partial x})^2+(\frac{\partial r}{\partial y})^2+(\frac{\partial r}{\partial z})^2]+\frac{\partial^2}{\partial \theta^2}[(\frac{\partial \theta}{\partial x})^2+(\frac{\partial \theta}{\partial y})^2+(\frac{\partial \theta}{\partial z})^2]+\frac{\partial^2}{\partial \varphi^2}[(\frac{\partial \varphi}{\partial x})^2+(\frac{\partial \varphi}{\partial y})^2+(\frac{\partial \varphi}{\partial z})^2]+$$
$$2\frac{\partial^2}{\partial r\partial \theta}(\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z})+2\frac{\partial^2}{\partial r\partial \varphi}(\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z})+2\frac{\partial^2}{\partial \theta\partial \varphi}(\frac{\partial \varphi}{\partial x}\frac{\partial \theta}{\partial x}+\frac{\partial \varphi}{\partial y}\times$$
$$\frac{\partial \theta}{\partial y}+\frac{\partial \varphi}{\partial z}\frac{\partial \theta}{\partial z})\qquad(4)$$

2　根据球坐标和直角坐标的关系来求式（4）中的各偏导数之值

2.1　求 $\frac{\partial r}{\partial x},\frac{\partial r}{\partial y},\frac{\partial r}{\partial z}$

由 $r^2=x\ +y^2+z^2$，得　$2r\frac{\partial r}{\partial x}=2x$，　$2r\frac{\partial r}{\partial y}=2y$，　$2r\frac{\partial r}{\partial z}=2z$，　$\therefore \frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

2.2　求 $\frac{\partial \theta}{\partial x},\frac{\partial \theta}{\partial y},\frac{\partial \theta}{\partial z}$

由 $\cos\theta=\frac{z}{r}$，得 $-\sin\theta\frac{\partial \theta}{\partial x}=\frac{-z\frac{\partial r}{\partial x}}{r^2}$　　$\therefore \frac{\partial \theta}{\partial x}=\frac{z\frac{\partial r}{\partial x}}{r^2\sin\theta}=\frac{z\frac{x}{r}}{r^2\sin\theta}=\frac{r\cos\theta\cdot r\sin\theta\cos\varphi}{r^3\sin\theta}=\frac{\cos\theta\cos\varphi}{r}$

同理　$\frac{\partial \theta}{\partial y}=\frac{z\frac{\partial r}{\partial y}}{r^2\sin\theta}=\frac{z\frac{y}{r}}{r^2\sin\theta}=\frac{r\cos\theta\cdot r\sin\theta\sin\varphi}{r^3\sin\theta}=\frac{\cos\theta\sin\varphi}{r}$

$$\frac{\partial \theta}{\partial z}=-\frac{r-z\frac{\partial \gamma}{\partial z}}{r^2\sin\theta}=-\frac{r-z\frac{z}{r}}{r^2\sin\theta}=-\frac{r^2-r^2\cos^2\theta}{r^3\sin\theta}=-\frac{1-\cos^2\theta}{r\sin\theta}=-\frac{\sin\theta}{r}$$

2.3　求 $\frac{\partial \varphi}{\partial x},\frac{\partial \varphi}{\partial y},\frac{\partial \varphi}{\partial z}$

© 1994-2012 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

由 $tg\varphi=\frac{y}{x}$，得 $\sec^2\varphi\frac{\partial\varphi}{\partial x}=\frac{-y}{x^2}=\frac{-r\sin\theta\sin\varphi}{r^2\sin^2\theta\cos^2\varphi}$ $\therefore\frac{\partial\varphi}{\partial x}=\frac{-r\sin\theta\sin\varphi\cos^2\varphi}{r^2\sin^2\theta\cos^2\varphi}=-\frac{\sin\varphi}{r\sin\theta}$

同理 $\frac{\partial\varphi}{\partial y}=\frac{\cos^2\varphi}{r\sin\theta\cos\varphi}=\frac{\cos\varphi}{r\sin\theta}$　　$\frac{\partial\varphi}{\partial z}=0$

2.4　求 $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}$

$\because\frac{\partial r}{\partial x}=\frac{x}{r}$　　$\therefore\frac{\partial^2 r}{\partial x^2}=\frac{r-x\frac{\partial r}{\partial x}}{r^2}=\frac{r-x\frac{x}{r}}{r^2}=\frac{r^2-x^2}{r^3}=\frac{r^2-r^2\sin^2\theta\cos^2\varphi}{r^3}=\frac{1-\sin^2\theta\cos^2\varphi}{r}$

$\because\frac{\partial r}{\partial y}=\frac{y}{r}$　　$\therefore\frac{\partial^2 r}{\partial y^2}=\frac{r-y\frac{\partial r}{\partial y}}{r^2}=\frac{r-y\frac{y}{r}}{r^2}=\frac{r^2-y^2}{r^3}=\frac{r^2-r^2\sin^2\theta\sin^2\varphi}{r^3}=\frac{1-\sin^2\theta\sin^2\varphi}{r}$

$\because\frac{\partial r}{\partial z}=\frac{z}{r}$　　$\therefore\frac{\partial^2 r}{\partial z^2}=\frac{r-z\frac{\partial r}{\partial z}}{r^2}=\frac{r-z\frac{z}{r}}{r^2}=\frac{r^2-z^2}{r^3}=\frac{r^2-r^2\cos^2\theta}{r^3}=\frac{1-\cos^2\theta}{r}$

$\therefore\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}=\frac{1-\sin^2\theta\cos^2\varphi+1-\sin^2\theta\sin^2\varphi+1-\cos^2\theta}{r}=\frac{2}{r}$

2.5　求 $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}$

$\because\frac{\partial\theta}{\partial x}=\frac{\cos\theta\cos\varphi}{r}$

$\therefore\frac{\partial^2\theta}{\partial x^2}=\frac{(\cos\theta\cos\varphi)'r-\cos\theta\cos\varphi\frac{\partial r}{\partial x}}{r^2}=\frac{-\sin\theta\cos\varphi r\frac{\partial\theta}{\partial x}-r\cos\theta\sin\varphi\frac{\partial\varphi}{\partial x}-\cos\theta\cos\varphi\frac{\partial r}{\partial x}}{r^2}$

$=\frac{-r\sin\theta\cos\varphi\frac{\cos\theta\cos\varphi}{r}-r\cos\theta\sin\varphi\frac{-\sin\varphi}{r\sin\theta}-\cos\theta\cos\varphi\frac{r\sin\theta\cos\varphi}{r}}{r^2}$

$=\frac{-\sin\theta\cos\theta\cos^2\varphi+\frac{\cos\theta\sin^2\varphi}{\sin\theta}-\sin\theta\cos\theta\cos^2\varphi}{r^2}$

同理：$\frac{\partial^2\theta}{\partial y^2}=\frac{-\sin\theta\cos\theta\sin^2\varphi+\frac{\cos\theta\cos^2\varphi}{\sin\theta}-\sin\theta\cos\theta\sin^2\varphi}{r^2}$

$\frac{\partial^2\theta}{\partial z^2}=\frac{2\sin\theta\cos\theta}{r^2}\therefore\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=\frac{\cos\theta}{r^2\sin\theta}$

2.6　求 $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$

$\because\frac{\partial\varphi}{\partial x}=\frac{-\sin\varphi}{r\sin\theta}$

$\therefore\frac{\partial^2\varphi}{\partial x^2}=\frac{-r\sin\theta\cos\varphi\frac{\partial\varphi}{\partial x}+\sin\varphi(r\sin\theta)'}{r^2\sin^2\theta}=\frac{-r\sin\theta\cos\varphi\frac{-\sin\varphi}{r\sin\theta}+\sin\varphi(\sin\theta\frac{\partial r}{\partial x}+r\cos\theta\frac{\partial\theta}{\partial x})'}{r^2\sin^2\theta}$

$=\frac{\sin\varphi\cos\varphi+\sin\varphi(\sin\theta\sin\theta\cos\varphi+r\cos\theta\frac{\cos\theta\cos\varphi}{r}}{r^2\sin^2\theta}=\frac{\sin\varphi\cos\varphi+\sin\varphi\cos\varphi}{r^2\sin^2\theta}=\frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}$

© 1994-2012 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

同理：$\frac{\partial^2\varphi}{\partial y^2}=-\frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}$

$\because \frac{\partial\varphi}{\partial z}=0 \qquad \therefore \frac{\partial^2\varphi}{\partial z^2}=0 \therefore \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$

2.7　求$(\frac{\partial r}{\partial x})^2+(\frac{\partial r}{\partial y})^2+(\frac{\partial r}{\partial z})^2$

$\because \frac{\partial r}{\partial x}=\frac{x}{r} \qquad \therefore (\frac{\partial r}{\partial x})^2=\frac{x^2}{r^2}=\frac{r^2\sin^2\theta\cos^2\varphi}{r^2}=\sin^2\theta\cos^2\varphi$

$\because \frac{\partial r}{\partial y}=\frac{y}{r} \qquad \therefore (\frac{\partial r}{\partial y})^2=\frac{y^2}{r^2}=\frac{r^2\sin^2\theta\sin^2\varphi}{r^2}=\sin^2\theta\sin^2\varphi$

$\because \frac{\partial r}{\partial z}=\frac{z}{r} \qquad \therefore (\frac{\partial r}{\partial z})^2=\frac{z^2}{r^2}=\frac{r^2\cos^2\theta}{r^2}=\cos^2\theta$

$\therefore (\frac{\partial r}{\partial x})^2+(\frac{\partial r}{\partial y})^2+(\frac{\partial r}{\partial z})^2=1$

2.8　求$\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2+\left(\frac{\partial\theta}{\partial z}\right)^2$

容易求得$\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2+\left(\frac{\partial\theta}{\partial z}\right)^2=\frac{1}{r^2}$

2.9　求$(\frac{\partial\varphi}{\partial x})^2+(\frac{\partial\varphi}{\partial y})^2+(\frac{\partial\varphi}{\partial z})^2$

$\because \frac{\partial\varphi}{\partial x}=-\frac{\sin\varphi}{r\sin\theta} \quad \therefore (\frac{\partial\varphi}{\partial x})^2=\frac{\sin^2\varphi}{r^2\sin^2\theta} \because \frac{\partial\varphi}{\partial y}=\frac{\cos\varphi}{r\sin\theta} \qquad \therefore (\frac{\partial\varphi}{\partial y})^2=\frac{\cos^2\varphi}{r^2\sin^2\theta} \because \frac{\partial\varphi}{\partial z}=0 \therefore (\frac{\partial\varphi}{\partial z})^2=0$

$\therefore (\frac{\partial\varphi}{\partial x})^2+(\frac{\partial\varphi}{\partial y})^2+(\frac{\partial\varphi}{\partial z})^2=\frac{1}{r^2\sin^2\theta}$

2.10　求$\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x},\frac{\partial r}{\partial x}\frac{\partial\varphi}{\partial x},\frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial x}$

$\because \frac{\partial r}{\partial x}=\frac{x}{r}=\sin\theta\cos\varphi \qquad \frac{\partial\theta}{\partial x}=\frac{\cos\theta\cos\varphi}{r} \qquad \frac{\partial\varphi}{\partial x}=-\frac{\sin\varphi}{r\sin\theta} \qquad \therefore \frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}=\frac{\sin\theta\cos\theta\cos^2\varphi}{r}$

$\frac{\partial r}{\partial x}\frac{\partial\varphi}{\partial x}=-\frac{\sin\varphi\cos\varphi}{r} \qquad \frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial x}=-\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}$

2.11　求$\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y},\frac{\partial r}{\partial y}\frac{\partial\varphi}{\partial y},\frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial y}$

$\because \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta\sin\varphi \qquad \frac{\partial\theta}{\partial y}=\frac{\cos\theta\sin\varphi}{r} \qquad \frac{\partial\varphi}{\partial y}=\frac{\cos\varphi}{r\sin\theta}$

$\therefore \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}=\frac{\sin\theta\cos\theta\sin^2\varphi}{r} \qquad \frac{\partial r}{\partial y}\frac{\partial\varphi}{\partial y}=\frac{\sin\varphi\cos\varphi}{r} \qquad \frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial y}=\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}$

2.12　求$\frac{\partial r}{\partial z}\frac{\partial\theta}{\partial z},\frac{\partial r}{\partial z}\frac{\partial\varphi}{\partial z},\frac{\partial\theta}{\partial z}\frac{\partial\varphi}{\partial z}$

$\because \frac{\partial r}{\partial z}=\cos\theta \qquad \frac{\partial\theta}{\partial z}=-\frac{\sin\theta}{r} \qquad \frac{\partial\varphi}{\partial z}=0 \qquad \therefore \frac{\partial r}{\partial z}\frac{\partial\theta}{\partial z}=-\frac{\sin\theta\cos\theta}{r} \qquad \frac{\partial r}{\partial z}\frac{\partial\varphi}{\partial z}=0 \qquad \frac{\partial\theta}{\partial z}\frac{\partial\varphi}{\partial z}=0$

2.13　求$\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial\theta}{\partial z},\frac{\partial r}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial\varphi}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial\varphi}{\partial z},\frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial y}+\frac{\partial\theta}{\partial z}\frac{\partial\varphi}{\partial z}$

© 1994-2012 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

$$\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z}=\frac{\sin\theta\cos\theta\cos^2\varphi}{r}+\frac{\sin\theta\cos\theta\sin^2\varphi}{r}-\frac{\sin\theta\cos\theta}{r}=0$$

$$\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z}=-\frac{\sin\varphi\cos\varphi}{r}+\frac{\sin\varphi\cos\varphi}{r}+0=0$$

$$\frac{\partial \theta}{\partial x}\frac{\partial \varphi}{\partial x}+\frac{\partial \theta}{\partial y}\frac{\partial \varphi}{\partial y}+\frac{\partial \theta}{\partial z}\frac{\partial \varphi}{\partial z}=-\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}+\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}+0=0$$

3 结论

将所求得的各式的值代入（4）式，得

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{2}{r}\frac{\partial}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \qquad (5)$$

又 $$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})=\frac{1}{r^2\sin\theta}(\cos\theta\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial^2}{\partial\theta^2})=\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})=\frac{1}{r^2}(2r\frac{\partial}{\partial r}+r^2\frac{\partial^2}{\partial r^2})=\frac{2}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}$$

则（5）式变为

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\varphi^2}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

∴在球坐标系中拉普拉斯算符的表达式为[3] $$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

参考文献：

[1] 梁昆淼.数学物理方法[M].北京:人民教育出版社,1981.511.

[2] 四川大学数学系,高等数学教研室.高等数学（第三版，第二册）[M].北京:高等教育出版社,1996.

[3] 周世勋.量子力学教程[M].北京:高等教育出版社,1979.65.

The Deduction of the Representative of Laplace's Operator in Spherical Coordinate System

SHI Feng-liang, ZHANG Hai-ping

(Physics Department, Tangshan Teachers College, Hebei Tangshan 063000, China)

Abstract: In This paper, The representative of Laplace's Operator of Spherical Coordinate System was obtained from the representative of Laplace's Operator of rectangular Coordinate System by the relationship of x、y、z of rectangular Coordinates and r、θ、Φ of Spherical Coordinate.

Key words: partial derivative; second partial derivative; Laplace's operator

责任编辑、校对：孙海祥

© 1994-2012 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net