Principles of Quantum Mechanics

SECOND EDITION

SOLUTION MANUAL

R. Shankar

COMPILED BY YEMI BUKKY

+234(0)8057474928; +234(0)8064974071

(jjbukky@yahoo.com)

Department of Physics,

Federal University of Technology,

Minna, NG

Nigeria

PRINCIPLES OF QUANTUM MECHANICS

BY R. SHANKAR

SECOND EDITION

SOLUTIONS

COMPILED BY YEMI BUKKY (*jjbukky@yahoo.com*)

Department of Physics,

Federal University of Technology, Minna, NG

Nigeria

+2348057474928

Homework 1 (1.1.1 ~ 1.4.2)

Shanshan Cao

1.1.1

a. $|0\rangle$ is unique.

Proof: (i) $|v\rangle + |0\rangle = |v\rangle$ Where $|v\rangle$ is an arbitrary state ket.

(ii) $|v\rangle + |0'\rangle = |v\rangle$

Set $|v\rangle = |0'\rangle$ in (i) $\Rightarrow$ $|0'\rangle + |0\rangle = |0'\rangle$

Set $|v\rangle = |0\rangle$ in (ii) $\Rightarrow$ $|0\rangle + |0'\rangle = |0\rangle$

Therefore, $|0'\rangle = |0'\rangle + |0\rangle$ $\overset{\uparrow \text{Commutativity of vector addition}}{=\!=\!=}$ $|0\rangle + |0'\rangle = |0\rangle$.

b. $0|v\rangle = |0\rangle$

Proof: $1|v\rangle = (1+0)|v\rangle = 1|v\rangle + 0|v\rangle$, where $1|v\rangle = |v\rangle$.

Therefore $|v\rangle = |v\rangle + 0|v\rangle$.

Since $|v\rangle$ is arbitrary here, compared with the definition of $|0\rangle$, which is $|v\rangle + |0\rangle = |v\rangle$ for any $|v\rangle$, plus $|0\rangle$ is unique, we can conclude that $0|v\rangle = |0\rangle$.

c. $|-v\rangle = -|v\rangle$

Proof: $|v\rangle + (-|v\rangle) = 0|v\rangle = |0\rangle$ for arbitrary $|v\rangle$. By definition of $|-v\rangle$, which is $|v\rangle + |-v\rangle = 0$, we can conclude that, $|-v\rangle = -|v\rangle$.

d. $|-v\rangle$ is the unique additive inverse of $|v\rangle$.

Proof: Suppose there exists another vector $|w\rangle$, satisfying $|w\rangle = -|v\rangle$, then

$$\begin{aligned} |w\rangle + |v\rangle &= -|v\rangle + |v\rangle \\ &= (1-1)|v\rangle \\ &= 0|v\rangle \\ &= |0\rangle \end{aligned}$$

Add $|-v\rangle$ on both sides, we have

$|w\rangle + |v\rangle + |-v\rangle = |0\rangle + |-v\rangle$

LHS $= |w\rangle + (|v\rangle + |-v\rangle) = |w\rangle + |0\rangle = |w\rangle$

RHS $= |-v\rangle$

Therefore, $|w\rangle = |-v\rangle$.

1.1.2

Null vector : By definition $|V\rangle + |0\rangle = |V\rangle$ for any $|V\rangle$.

Set $|0\rangle = (a_0, b_0, c_0)$, $|V\rangle = (a, b, c)$, where a, b, c are arbitrary numbers.

Then, $|0\rangle + |V\rangle = (a_0, b_0, c_0) + (a, b, c)$

$= (a_0 + a, b_0 + b, c_0 + c)$

$= |V\rangle$

$= (a, b, c)$

$$\Rightarrow \begin{cases} a_0 + a = a \\ b_0 + b = b \\ c_0 + c = c \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ b_0 = 0 \\ c_0 = 0 \end{cases}.$$

Therefore, the null vector is $(0, 0, 0)$.

Inverse vector :

Suppose the inverse vector of (a, b, c) is $(\bar{a}, \bar{b}, \bar{c})$. By definition

$(a, b, c) + (\bar{a}, \bar{b}, \bar{c}) = |0\rangle = (0, 0, 0)$

$\Rightarrow (a + \bar{a}, b + \bar{b}, c + \bar{c}) = (0, 0, 0)$

$$\Rightarrow \begin{cases} a + \bar{a} = 0 \\ b + \bar{b} = 0 \\ c + \bar{c} = 0 \end{cases} \Rightarrow \begin{cases} \bar{a} = -a \\ \bar{b} = -b \\ \bar{c} = -c \end{cases}$$

Therefore, the inverse vector of $(a, b, c) = (-a, -b, -c)$.

$\{(a, b, 1)\}$ does not form a vector space because

① It violates the closure under addition, ie.

$(a_1, b_1, 1) + (a_2, b_2, 1) = (a_1 + a_2, b_1 + b_2, 2) \notin \{(a, b, 1)\}$;

② It violates the closure under scalar multiplication, ie.

$\lambda (a_1, b_1, 1) = (\lambda a_1, \lambda b_1, \lambda) \notin \{(a, b, 1)\}$, as long as $\lambda \neq 1$.

③ There is no null vector, ie. $(0, 0, 0) \notin \{(a, b, 1)\}$

④ The inverse does not exist, ie. $(-a, -b, -1) \notin \{(a, b, 1)\}$

If you want to prove the property of closure in this problem, please mention that $f(x)+g(x)$ and $\alpha f(x)$ are also periodic functions. That is, $f(0)+g(0) = f(L)+g(L)$, $\alpha f(0) = \alpha f(L)$.

1.1.3

(i) $\{f(x)\}$, $f(0) = f(L) = 0$, form a vector space.

(ii) $\{f(x)\}$, periodic functions obeying $f(0) = f(L)$, form a vector space.

(iii) $\{f(x)\}$, $f(0) = 4$ DO NOT form a vector space;

① If $g(x), h(x) \in \{f(x)\}$, then $g(x)+h(x) \notin \{f(x)\}$, since $g(0)+h(0) = 8 \neq 4$;

② If $g(x) \in \{f(x)\}$, then $\lambda g(x) \notin \{f(x)\}$, since $\lambda g(0) = 4\lambda \neq 4$, as long as $\lambda \neq 1$;

③ No null vector. $g(x) \equiv 0 \notin \{f(x)\}$, since $g(0) = 0 \neq 4$;

④ If $g(x) \in \{f(x)\}$, then the inverse $-g(x) \notin \{f(x)\}$, since $-g(0) = -4 \neq 4$.

1.1.4

Suppose $a_1|1\rangle + a_2|2\rangle + a_3|3\rangle = 0$.

We have $\begin{pmatrix} 0\cdot a_1 + 1\cdot a_2 + (-2)\cdot a_3 & 1\cdot a_1 + 1\cdot a_2 + (-1)\cdot a_3 \\ 0\cdot a_1 + 0\cdot a_2 + 0\cdot a_3 & 0\cdot a_1 + 1\cdot a_2 + (-2)\cdot a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} a_2 - 2a_3 = 0 \\ a_1 + a_2 - a_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = -a_3 \\ a_2 = 2a_3 \end{cases}$$

It is not necessary for a_1, a_2 and a_3 to be 0 together. Therefore, $|1\rangle$, $|2\rangle$ and $|3\rangle$ are linearly DEPENDENT.

1.1.5

Suppose $a_1(1,1,0) + a_2(1,0,1) + a_3(3,2,1) = 0$

Then $\begin{cases} a_1 + a_2 + 3a_3 = 0 \\ a_1 + 2a_3 = 0 \\ a_2 + a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -2a_3 \\ a_2 = -a_3 \end{cases}$

When $a_3 \neq 0$, a_1, a_2, a_3 can have non-zero values. Therefore, $(1,1,0)$, $(1,0,1)$, $(3,2,1)$ are linearly dependent.

Suppose $a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1) = 0$

Then $\begin{cases} a_1 + a_2 = 0 \\ a_1 + a_3 = 0 \\ a_2 + a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$ is the only solution.

Therefore, $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are linearly independent.

1.3.1

Using Gram-Schmidt process here, starting from $\vec{A}$.

$$\vec{e}_1 = \frac{\vec{A}}{|\vec{A}|} = \frac{3\vec{i} + 4\vec{j}}{\sqrt{3^2+4^2}} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

$$\begin{aligned} \vec{e}_2' &= \vec{B} - (\vec{B}\cdot\vec{e}_1)\vec{e}_1 \\ &= (2\vec{i} - 6\vec{j}) - \left(\frac{6}{5} - \frac{24}{5}\right)\left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right) \\ &= \left(2 + \frac{54}{25}\right)\vec{i} + \left(-6 + \frac{72}{25}\right)\vec{j} \\ &= \frac{104}{25}\vec{i} - \frac{78}{25}\vec{j} \end{aligned}$$

$$\vec{e}_2 = \frac{\vec{e}_2'}{|\vec{e}_2'|} = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$$

Therefore, the new basis is $\vec{e}_1 = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$, $\vec{e}_2 = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$.

1.3.2

Step 1: $$|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{3}\begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Step 2-1: $$\begin{aligned} |2'\rangle &= |II\rangle - (\langle 1|II\rangle)|1\rangle \\ &= \begin{pmatrix}0\\1\\2\end{pmatrix} - 0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} \\ &= \begin{pmatrix}0\\1\\2\end{pmatrix} \end{aligned}$$

Step 2-2: $$|2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix}$$

Step 3-1 : $|3'\rangle = |\text{III}\rangle - (\langle 1|\text{III}\rangle)|1\rangle - (\langle 2|\text{III}\rangle)|2\rangle$

$$= \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} - 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(0 + \frac{2}{\sqrt{5}} + \frac{10}{\sqrt{5}}\right) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{12}{5} \\ \frac{24}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

Step 3-2 : $|3\rangle = \frac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}} = \begin{pmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

1.3.3

When $|V\rangle = c|W\rangle$

$$\langle V|V\rangle = |c|^2 \langle W|W\rangle = |c|^2 |W|^2$$

$$\langle W|V\rangle\langle V|W\rangle = (c\langle W|W\rangle)(c^*\langle W|W\rangle)$$

$$= |c|^2 \langle W|W\rangle\langle W|W\rangle$$

$$= |c|^2 |W|^4$$

$$\Rightarrow \langle V|V\rangle = \frac{\langle W|V\rangle\langle V|W\rangle}{|W|^2}.$$

When two arrows are parallel or anti-parallel with each other, the square of their inner product equals to the product of their norms.

1.3.4

$$\text{Re}\langle V|W\rangle \leq |\langle V|W\rangle| \leq |V||W|$$

$$\Rightarrow 2\,\text{Re}\langle V|W\rangle \leq 2|V||W|$$

Add $|V|^2 + |W|^2$ to both sides of the inequality above, we have

$$\langle V|V\rangle + 2\text{Re}\langle V|W\rangle + \langle W|W\rangle \leq |V|^2 + |W|^2 + 2|V||W|$$

$$\text{LHS} = \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle$$

$$= \langle V+W|V+W\rangle$$

$$= |V+W|^2$$

RHS = $(|V|+|W|)^2$

Therefore, $|V+W|^2 \le (|V|+|W|)^2$

$\Rightarrow |V+W| \le |V|+|W|$ (the triangular inequality)

Attention, we are supposed to prove the equality holds ONLY IF $|V\rangle = a|W\rangle$ where a is a real number.

The equality holds only if the two equalities below hold:

① $\langle V|W\rangle = |V|\,|W|$

② $\mathrm{Re}\langle V|W\rangle = |\langle V|W\rangle|$

From the proof process of the Schwarz inequality in Shankar, we know that equality ① holds only if

$$|Z\rangle = |V\rangle - \frac{\langle W|V\rangle}{|W|^2}|W\rangle = 0$$

which means $|V\rangle$ must be able to expressed as $\alpha|W\rangle$, where α is a number. To prove that α must be real, we substitute $|V\rangle = \alpha|W\rangle$ into equality ② above.

$\langle V|W\rangle = \alpha^* \langle V|V\rangle$

To satisfy equality ②, $\langle V|W\rangle$ must be real. Since $\langle V|V\rangle$ is real, α^* must be real. Therefore, α must be a real number.

1.4.1

Given a vector space V^n, one can start with an arbitrary vector $|V\rangle \neq 0$ and construct $n-1$ other vectors orthogonal to this $|V\rangle$ through Gram-Schmidt process. Since these $n-1$ vectors are linearly independent, they span a V^{n-1} subspace. Now we prove that this subspace V^{n-1} is the set of all vectors orthogonal to $|V\rangle$.

① Every vector in V^{n-1} is orthogonal to $|V\rangle$: Since every vector in V^{n-1} can be expressed as a linear combination of the $n-1$ vectors we constructed above — $|W\rangle = \sum_{i=1}^{n-1} a_i |V^i\rangle$, and $\langle V|W\rangle = \sum_{i=1}^{n-1} a_i \langle V|V_i\rangle = 0$, because $\langle V|V_i\rangle = 0$ for each V_i.

② Every vector in V^n but outside V^{n-1} is NOT orthogonal to $|V\rangle$. Since this kind of vector can be expressed as $|U\rangle = a|V\rangle + \sum_{i=1}^{n-1} a_i |V^i\rangle$, where $a \neq 0$. Therefore $\langle V|U\rangle = a\langle V|V\rangle = a \neq 0$.

1.4.2

Since $|V_1^{n_1}\rangle$ and $|V_2^{n_2}\rangle$ are two subspace orthogonal to each other, we can take the n_1 basis vectors of $|V_1^{n_1}\rangle$ and the n_2 basis vectors of $|V_2^{n_2}\rangle$, and put them together. Because these n_1+n_2 vectors are orthogonal to each other, they can span a $|V^{n_1+n_2}\rangle$ subspace. We now prove that this $|V^{n_1+n_2}\rangle$ is nothing but the $|V_1\rangle \oplus |V_2\rangle$.

① Every vector in $|V^{n_1+n_2}\rangle$ is in $|V_1\rangle \oplus |V_2\rangle$:

Since each vector in $|V^{n_1+n_2}\rangle$ can be expressed as $|u\rangle = \sum_{i=1}^{n_1} a_i |v_i\rangle + \sum_{j=1}^{n_2} b_j |w_j\rangle$, where $\{|v_i\rangle\}, \{|w_j\rangle\}$ are basis vectors of $|V_1\rangle$ and $|V_2\rangle$ respectively. Notice that $\sum_{i=1}^{n_1} a_i |v_i\rangle$ is a vector in $|V_1\rangle$, and $\sum_{j=1}^{n_2} b_j |w_j\rangle$ is a vector in $|V_2\rangle$. Therefore, $|u\rangle$ can be expressed as a combination of vectors from $|V_1\rangle$ and $|V_2\rangle$. According to the definition of $|V_1\rangle \oplus |V_2\rangle$ (Definition 12 in Shankar), $|u\rangle \in |V_1\rangle \oplus |V_2\rangle$.

② Every vector in $|V_1\rangle \oplus |V_2\rangle$ is in $|V^{n_1+n_2}\rangle$:

Every vector in $|V_1\rangle \oplus |V_2\rangle$ can be expressed as $|z\rangle = c_1|z_1\rangle + c_2|z_2\rangle$, where $|z_1\rangle \in |V_1^{n_1}\rangle$, $|z_2\rangle \in |V_2^{n_2}\rangle$. Therefore, $|z_1\rangle = \sum_{i=1}^{n_1} p_i |v_i\rangle$, and $|z_2\rangle = \sum_{j=1}^{n_2} q_j |w_j\rangle$. Thus, $|z\rangle = \sum_{i=1}^{n_1} c_1 p_i |v_i\rangle + \sum_{j=1}^{n_2} c_2 q_j |w_j\rangle$, which lies in $|V^{n_1+n_2}\rangle$.

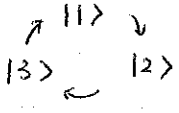
Therefore, the dimension of $|V_1\rangle \oplus |V_2\rangle$ is n_1+n_2.

Homework 2 (1.6.1 ~ 1.8.10)

Shanshan Cao

1.6.1

$U|1\rangle = |2\rangle$, $U|2\rangle = |3\rangle$, $U|3\rangle = |1\rangle$



This is a cyclic permutation of the three vectors.

It is also equivalent to rotation of the coordinate axis along $(1,1,1)$ by $120°$.

1.6.2

(1) $(\Omega\Lambda)^+ = \Lambda^+\Omega^+ = \Lambda\Omega \neq \Omega\Lambda$ not Hermitian

(2) $(\Omega\Lambda+\Lambda\Omega)^+ = (\Omega\Lambda)^+ + (\Lambda\Omega)^+ = \Lambda^+\Omega^+ + \Omega^+\Lambda^+ = \Lambda\Omega + \Omega\Lambda = \Omega\Lambda + \Lambda\Omega$ Hermitian

(3) $[\Omega,\Lambda]^\dagger = (\Omega\Lambda - \Lambda\Omega)^+ = (\Omega\Lambda)^+ - (\Lambda\Omega)^+ = \Lambda^+\Omega^+ - \Omega^+\Lambda^+ = \Lambda\Omega - \Omega\Lambda$
$= -[\Omega,\Lambda]$ anti - Hermitian

(4) $(i[\Omega,\Lambda])^+ = -i[\Omega,\Lambda]^+ = -i\cdot(-[\Omega,\Lambda]) = i[\Omega,\Lambda]$ Hermitian

1.6.3

Suppose U_1, U_2 are unitary. $U_1^+U_1 = U_2^+U_2 = I$.

Therefore, $(U_1U_2)^+(U_1U_2) = U_2^+U_1^+U_1U_2 = U_2^+(U_1^+U_1)U_2 = U_2^+ I U_2$
$= U_2^+U_2 = I$.

Hence, U_1U_2 is unitary.

1.6.4

$\det(U^+U) = \det I = 1$

$\det(U^+U) = \det(U^+)\det U = \det(U^T)^*\det U = (\det U)^*(\det U)$
$= |\det U|^2$

Therefore, $|\det U| = 1$, which indicates that $\det U$ is a complex number of unit modulus.

1.6.5

According to Example 1.6.1, $R(\frac{1}{2}\pi i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Therefore, $R^+R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$.

Hence, $R(\frac{1}{2}\pi i)$ is unitary.

Unitary

1.6.6

① $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^+ \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$

$\det\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right] = \left(\frac{1}{\sqrt{2}}\right)^2(1\cdot 1 - i\cdot i) = 1 = e^{i\theta}$, when $\theta = 2k\pi$ for integer k.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \neq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ Not Hermitian.

② $\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}^+ \cdot \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = I$ Unitary

$\det\left[\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\right] = \left(\frac{1}{2}\right)^2\left[(1+i)^2 - (1-i)^2\right] = \frac{1}{4}(2i+2i) = i$

$= e^{i\theta}$, when $\theta = 2k\pi + \frac{\pi}{2}$ for integer k.

$\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}^+ = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix} \neq \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$ Not Hermitian.

1.7.1

(1) $\mathrm{Tr}(\Omega\Lambda) = \sum_i (\Omega\Lambda)_{ii} = \sum_i\sum_j \Omega_{ij}\Lambda_{ji} = \sum_j\sum_i \Lambda_{ji}\Omega_{ij} = \sum_j (\Lambda\Omega)_{jj} = \mathrm{Tr}(\Lambda\Omega)$.

(2) $\mathrm{Tr}(\Omega\Lambda\Theta) = \sum_i (\Omega\Lambda\Theta)_{ii} = \sum_i\sum_j\sum_k \Omega_{ij}\Lambda_{jk}\Theta_{ki} = \sum_j\sum_k\sum_i \Lambda_{jk}\Theta_{ki}\Omega_{ij}$

$= \sum_j (\Lambda\Theta\Omega)_{jj} = \mathrm{Tr}(\Lambda\Theta\Omega)$

$= \sum_k\sum_i\sum_j \Theta_{ki}\Omega_{ij}\Lambda_{jk} = \sum_k (\Theta\Omega\Lambda)_{kk} = \mathrm{Tr}(\Theta\Omega\Lambda)$

(3) $\mathrm{Tr}(U^+\Omega U) = \mathrm{Tr}(\Omega UU^+) =_{10} \mathrm{Tr}(\Omega I) = \mathrm{Tr}\,\Omega$.

1.7.2

$$\begin{aligned}\det(U^+\Omega U) &= \det U^+ \det\Omega \det U \\ &= \det\Omega\,(\det U^+ \det U) \\ &= \det\Omega \det(U^+U) \\ &= \det\Omega \cdot 1 \\ &= \det\Omega .\end{aligned}$$

1.8.1

(1) $$\det(\Omega - I\omega) = \begin{vmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{vmatrix} = (1-\omega)(2-\omega)(4-\omega) = 0$$

$$\omega = 1, 2, 4$$

$$\omega=1: \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad\Rightarrow\quad |\omega=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\omega=2: \begin{pmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad\Rightarrow\quad |\omega=2\rangle = \frac{1}{\sqrt{30}}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\omega=4: \begin{pmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad\Rightarrow\quad |\omega=4\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

(2) $\Omega^+ \neq \Omega$ NOT Hermitian

$\langle\omega=1|\omega=2\rangle \neq 0$ $\langle\omega=1|\omega=4\rangle \neq 0$ $\langle\omega=2|\omega=4\rangle \neq 0$.

The eigenvectors are not orthogonal.

1.8.2

(1) $\Omega^+ = \Omega$ Hermitian

(2) $$\det(\Omega - I\omega) = \begin{vmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{vmatrix} = -\omega(\omega^2-1) = 0$$

$\omega = 0, 1, -1$.

$\omega = 0$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\omega = 1$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\omega = -1$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(3) $U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad U^+ = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$U^+\Omega U = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1.8.3

(1) $$\det(\Omega - I\omega) = \begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2}-\omega \end{vmatrix} = (1-\omega)\left[\left(\frac{3}{2}-\omega\right)^2 - \frac{1}{4}\right] = 0$$

$\omega = 1$ or $\omega^2 - 3\omega + 2 = 0$

$\Rightarrow \omega_1 = \omega_2 = 1, \quad \omega_3 = 2.$

(2) $\omega = 2$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} x = 0 \\ y + z = 0 \end{matrix}$

Set $y = a$, we have $z = -a$.

Therefore, $|\omega=2\rangle = \frac{1}{\sqrt{2a^2}}\begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$

(3) $\omega = 1$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ x is arbitrary, set it to be b,

$y - z = 0$, Set $y = c$, we have $z = c$

Therefore, $|\omega=1\rangle = \frac{1}{\sqrt{b^2+2c^2}}\begin{pmatrix} b \\ c \\ c \end{pmatrix}.$

1.8.4

(1) $\det(\Omega - I\omega) = \begin{vmatrix} 4-\omega & 1 \\ -1 & 2-\omega \end{vmatrix} = \omega^2 - 6\omega + 9 = 0$

$\omega_1 = \omega_2 = 3$.

(2) $\omega = 3$: $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x + y = 0$

Set $x = a$, we have $y = -a$.

Therefore, $|\omega = 3\rangle = \frac{1}{\sqrt{2a^2}}\begin{pmatrix} a \\ -a \end{pmatrix}$.

This is the only eigenvector we can find.

1.8.5

(1) $\Omega^+\Omega = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

(2) $\det(\Omega - I\omega) = \begin{vmatrix} \cos\theta - \omega & \sin\theta \\ -\sin\theta & \cos\theta - \omega \end{vmatrix} = \omega^2 - 2\omega\cos\theta + 1 = 0$

$\omega = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

(3) $\omega = e^{i\theta}$: $\begin{pmatrix} \cos\theta - e^{i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{i\theta} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$.

$\Rightarrow -ix + y = 0 \Rightarrow |\omega = e^{i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\omega = e^{-i\theta}$: $\begin{pmatrix} \cos\theta - e^{-i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{-i\theta} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$.

$\Rightarrow ix + y = 0 \Rightarrow |\omega = e^{-i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

$\langle \omega = e^{i\theta} | \omega = e^{-i\theta} \rangle = \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1 + i^2) = 0$.

Orthogonal.

(4) $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$

$$U^+\Omega U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

1.8.6

(1) Suppose U is the unitary matrix that transforms Ω in to diagonal matrix.

$\det \Omega = \det(U^+\Omega U) = \det D = \prod_{i=1}^{n} \omega_i$.

Where D is the corresponding diagonal matrix with Ω's eigenvalues ω_i on its diagonal.

(2) $\mathrm{Tr}\,\Omega = \mathrm{Tr}(U^+\Omega U) = \mathrm{Tr}\, D = \sum_{i=1}^{n} \omega_i$.

1.8.7

According to 1.8.6, we have

$$\begin{cases} \omega_1 + \omega_2 = \mathrm{Tr}\,\Omega = 1+1 = 2 \\ \omega_1\omega_2 = \det\Omega = 1\times 1 - 2\times 2 = -3 \end{cases}$$

$$\Rightarrow \begin{cases} \omega_1 = -1 \\ \omega_2 = 3 \end{cases} \text{ or } \begin{cases} \omega_1 = 3 \\ \omega_2 = -1 \end{cases}.$$

Verification.

$$\det(\Omega - I\omega) = \begin{vmatrix} 1-\omega & 2 \\ 2 & 1-\omega \end{vmatrix} = 0$$

$$\Rightarrow \omega^2 - 2\omega - 3 = 0 \Rightarrow \omega_1 = 3,\ \omega_2 = -1.$$

1.8.8

(1) $M^iM^j + M^jM^i = 2\delta^{ij} I \Rightarrow M^iM^i = I$

Therefore, $M^iM^i|\omega\rangle = M^i(\omega|\omega\rangle) = \omega^2|\omega\rangle$

and $M^iM^i|\omega\rangle = I|\omega\rangle = |\omega\rangle$.

$\Rightarrow \omega^2 = 1 \Rightarrow \omega = \pm 1$.

(2) $M^i M^j = -M^j M^i$

$\Rightarrow M^j M^i M^j = -M^j M^j M^i = -M^i$

$\Rightarrow \mathrm{Tr}\, M^i = \mathrm{Tr}(-M^j M^i M^j) = -\mathrm{Tr}(M^j M^i M^j)$.

$= -\mathrm{Tr}(M^i M^j M^j) = -\mathrm{Tr}(M^i I)$

$= -\mathrm{Tr}\, M^i$

$\Rightarrow \mathrm{Tr} M^i = 0$.

(3) According to 1.8.6

$\mathrm{Tr}\, M^i = \sum_{k=1}^{n} \omega_k$, where n is the dimension of the matrix.

Since $\omega_k = \pm 1$, $\mathrm{Tr}\, M^i$ can be zero only if n is even.

* The sum of an odd number of odd numbers is still odd, and cannot be zero.

1.8.9.

$\vec{l} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$

$= \sum_\alpha m_\alpha [\vec{\omega}(\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega})]$

$= \sum_\alpha m_\alpha [\vec{\omega} \cdot r_\alpha^2 - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega})]$

$\Rightarrow l_i = \sum_\alpha m_\alpha [\omega_i r_\alpha^2 - (\vec{r}_\alpha)_i (\vec{r}_\alpha \cdot \vec{\omega})]$

$= \sum_\alpha m_\alpha [\omega_i r_\alpha^2 - (\vec{r}_\alpha)_i \sum_j (\vec{r}_\alpha)_j \omega_j]$

$= \sum_j \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j] \omega_j$

$\equiv \sum_j M_{ij} \omega_j$,

where $M_{ij} = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j]$.

Or $|l\rangle = M |\omega\rangle$.

(1) No. The angular momentum and angular velocity are not parallel unless $|\omega\rangle$ is an eigenvector of M.

(2) $M_{ji}^* = \left(\sum_\alpha m_\alpha [r_\alpha^2 \delta_{ji} - (\vec{r}_\alpha)_j (\vec{r}_\alpha)_i]\right)^*$

Since m_α, r_α^2, $(\vec{r}_\alpha)_j$, $(\vec{r}_\alpha)_i$ are all real,

$M_{ji}^* = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ji} - (\vec{r}_\alpha)_j (\vec{r}_\alpha)_i] = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j] = M_{ij}$.

Hermitian.

(3) Since M is Hermitian, we can always find three eigenvectors which are orthogonal to each other by solving the eigen-problem $M|\omega\rangle = \omega|\omega\rangle$. And these three eigenvectors denote the three directions for $\vec{\omega}$ we want to find in the 3-D Euclidean space.

(4) The complete symmetry of sphere means all directions are equivalent eigendirections. Therefore, the eigenvalues are degenerate.

1.8.10

$$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} = 0$$

Since the two Hermitian matrices commute, they can be diagonalized simultaneously.

Starting from Λ,

$$\det(\Lambda - I\omega) = \begin{vmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{vmatrix} = (2-\omega)(\omega+1)(\omega-3) = 0$$

$\omega = -1, 2, 3$.

$\omega = -1$: $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega = -1\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\omega = 2$: $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega = 2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\omega = 3$: $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega = 3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U^{+}\Lambda U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$U^{+}\Omega U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Homework 3 (1.8.11 ~ 1.10.3)

Shanshan Cao

1.8.11

(1) Equation of motion $\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Set $\begin{vmatrix} -\frac{2k}{m}+\omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m}+\omega^2 \end{vmatrix} = 0$, we have $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{k}{m}}$.

The corresponding eigenvectors are $|\omega_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $|\omega_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Set $\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. $\Lambda^+\begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}\Lambda = \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix}$.

In eigenbasis,

$$\begin{pmatrix} \ddot{x}_I \\ \ddot{x}_{II} \end{pmatrix} = \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix}\begin{pmatrix} x_I \\ x_{II} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_I(t) = x_I(0)\cos\omega_1 t \\ x_{II}(t) = x_{II}(0)\cos\omega_2 t. \end{cases} \quad (1)$$

In this problem, $|1\rangle = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We first transform it into eigenbasis.

$$\begin{pmatrix} x_I(0) \\ x_I(0) \end{pmatrix} = \Lambda^+\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the eigenbasis, the state evolves according to (1),

$$\begin{pmatrix} x_I(t) \\ x_{II}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\omega_1 t \\ \frac{1}{\sqrt{2}}\cos\omega_2 t \end{pmatrix}.$$

Then, we tranform it back to the original basis.

$$|1(t)\rangle = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \Lambda\begin{pmatrix} x_I(t) \\ x_{II}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}\cos\omega_1 t \\ \frac{1}{\sqrt{2}}\cos\omega_2 t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos\omega_1 t + \frac{1}{2}\cos\omega_2 t \\ -\frac{1}{2}\cos\omega_1 t + \frac{1}{2}\cos\omega_2 t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\cos\sqrt{\frac{3k}{m}}t + \frac{1}{2}\cos\sqrt{\frac{k}{m}}t \\ -\frac{1}{2}\cos\sqrt{\frac{3k}{m}}t + \frac{1}{2}\cos\sqrt{\frac{k}{m}}t \end{pmatrix}.$$

(2) By substituting $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Eq.(1.8.31), we can get the same solution.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos\sqrt{\frac{k}{m}}t + \frac{1}{2}\cos\sqrt{\frac{3k}{m}}t \\ \frac{1}{2}\cos\sqrt{\frac{k}{m}}t - \frac{1}{2}\cos\sqrt{\frac{3k}{m}}t \end{pmatrix}.$$

1.8.12

(1) $|\ddot{x}(t)\rangle = \Omega\,|x(t)\rangle$

$|x(t)\rangle = U(t)\,|x(0)\rangle$

$\Rightarrow \frac{d^2}{dt}U(t)|x(0)\rangle = \Omega\,U(t)|x(0)\rangle$

$\Rightarrow \left(\frac{d^2}{dt} - \Omega\right)U(t)|x(0)\rangle = 0$.

Since $|x(0)\rangle$ is arbitrary, we have

$\frac{d^2}{dt^2}U(t) - \Omega\,U(t) = 0$.

(2) From 1.8.11, we know the $\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ can diagonalize Ω, and therefore, it can also diagonalize U.

In this common basis, we have

$$\begin{pmatrix} \ddot{U}_{11}(t) & 0 \\ 0 & \ddot{U}_{22}(t) \end{pmatrix} - \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix}\begin{pmatrix} U_{11}(t) & 0 \\ 0 & U_{22}(t) \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} \ddot{U}_{11}(t) + \omega_1^2 U_{11}(t) = 0 \\ \ddot{U}_{22}(t) + \omega_2^2 U_{22}(t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} U_{11} = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t \\ U_{22} = A_2 \cos\omega_2 t + B_2 \sin\omega_2 t \end{cases}$$

$$|\dot{x}(0)\rangle = \frac{d}{dt}\left[U(t)|x(0)\rangle\right]\Big|_{t=0} = \left(\frac{d}{dt}U(t)\right)\Big|_{t=0}|x(0)\rangle = 0$$

$\Rightarrow \frac{d}{dt}U(t)\Big|_{t=0} = 0 \Rightarrow \dot{U}_{11}(0) = \dot{U}_{22}(0) = 0 \Rightarrow B_1 = B_2 = 0$.

To satisfy that U is unitary, $A_1 = A_2 = 1$.

Therefore, $U = \begin{pmatrix} \cos\omega_1 t & 0 \\ 0 & \cos\omega_2 t \end{pmatrix}$,

which is the same as Eq. (1.8.43).

1.9.1

In the eigenbasis, $\Omega = \begin{pmatrix} \omega_1 & & \\ & \omega_2 & \\ & & \ddots \\ & & & \omega_m \end{pmatrix}$, where ω_i are eigenvalues.

$$f(\Omega) = \sum_{n=0}^{\infty}\Omega^n = \begin{pmatrix} \sum_{n=0}^{\infty}\omega_1^n & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty}\omega_m^n \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\omega_1} & & \\ & \ddots & \\ & & \frac{1}{1-\omega_m} \end{pmatrix} = \frac{1}{1-\Omega}.$$

This equality holds iff $|\omega_i| < 1$ for $i = 1, \ldots, m$.

Therefore, $f(\Omega)$ can be defined as $\frac{1}{1-\Omega}$ if and only if the absolute value of each of Ω's eigenvalue is less than 1.

1.9.2

$$U^\dagger = (e^{iH})^\dagger = e^{-iH^\dagger} = e^{-iH}$$

↑ H is Hermitian

$$U^\dagger U = e^{-iH} e^{iH} = e^{-iH+iH} = 1 \Leftarrow \text{unitary}$$

↑ ☆ This equality only holds for commuting operators.

1.9.3

In the eigenbasis of H, $U = e^{iH} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(i\epsilon_1)^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{(i\epsilon_m)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{i\epsilon_1} & & \\ & \ddots & \\ & & e^{i\epsilon_m} \end{pmatrix}$.

where $\epsilon_1, \cdots, \epsilon_m$ are eigenvalues of H. i.e $H = \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_m \end{pmatrix}$.

Therefore,

$$\det U = \prod_{i=1}^{m} e^{i\epsilon_i} = e^{i\sum_{i=1}^{m}\epsilon_i} = e^{i\,\mathrm{Tr}H}.$$

1.10.1

Since $\delta(x) = \delta(-x)$, we have $\delta(ax) = \delta(|a|x)$.

Therefore,

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \int_{-\infty}^{\infty} \delta(|a|x)\,dx = \int_{-\infty}^{\infty} \delta(|a|x)\cdot \frac{1}{|a|}\,d(|a|x)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(|a|x)\,d(|a|x) \qquad (\text{change } |a|x \text{ to } x)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(x)\,dx.$$

Thus, $\delta(ax) = \delta(x)/|a|$.

1.10.2

Expand $f(x)$ around x_i, where $f(x_i) = 0$:

$$f(x) = f(x_i) + f'(x_i)(x-x_i) + \frac{1}{2}f''(x_i)(x-x_i)^2 + \cdots$$
$$= 0 + f'(x_i)(x-x_i) + O[(x-x_i)^2]$$
$$\approx f'(x_i)(x-x_i)$$

Introduce a test function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\,\delta(f(x))\,dx = \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\delta(f(x))$$
$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\delta\left(\left.\frac{df}{dx}\right|_{x=x_i}(x-x_i)\right)dx$$
$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\delta\left[\left|\left.\frac{df}{dx}\right|_{x=x_i}\right|(x-x_i)\right]dx$$

See 1.10.1 →

$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\frac{\delta(x-x_i)}{\left|\left.df/dx\right|_{x=x_i}\right|}\,dx.$$

Therefore, $\delta(f(x)) = \sum_i \dfrac{\delta(x-x_i)}{\left|\left.df/dx\right|_{x=x_i}\right|}$.

1.10.3

Introduce a test function $g(x)$. * $g(\infty)$ and $g(-\infty)$ are finite.

$$\int_{-\infty}^{\infty} g(x)\,\frac{d}{dx}\theta(x-x')\,dx$$
$$= \int_{-\infty}^{\infty} g(x)\,d\,\theta(x-x')$$
$$= \theta(x-x')\,g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\theta(x-x')\,g'(x)\,dx$$
$$= 1\cdot g(\infty) - 0\cdot g(-\infty) - \int_0^{\infty} g'(x)\,dx$$
$$= g(\infty) - [g(\infty) - g(0)]$$
$$= g(0)$$
$$= \int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx.$$

Therefore, $\frac{d}{dx}\theta(x-x') = \delta(x)$.

Homework 4 (1.10.4 ~ 2.7.1)

Shanshan Cao

1.10.4

We start from Eq.(1.10.55)

$$|\psi(t)\rangle = \sum_{m=1}^{\infty} |m\rangle\langle m|\psi(0)\rangle \cos\omega_m t, \quad \omega_m = \frac{m\pi}{L}$$

$$\psi(x,t) = \langle x|\psi(t)\rangle = \sum_{m=1}^{\infty} \langle x|m\rangle\langle m|\psi(0)\rangle \cos\omega_m t$$

$$\langle x|m\rangle = \psi_m(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{m\pi x}{L} \qquad (\text{Eq. (1.10.50)})$$

$$\langle m|\psi(0)\rangle = \int_0^L \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{m\pi x}{L}\cdot\psi(x,0)\,dx$$

$$= \left(\frac{2}{L}\right)^{\frac{1}{2}}\left[\int_0^{\frac{L}{2}} \frac{2xh}{L}\sin\frac{m\pi x}{L}dx + \int_{\frac{L}{2}}^{L}\frac{2h}{L}(L-x)\sin\frac{m\pi x}{L}dx\right]$$

$$\equiv \left(\frac{2}{L}\right)^{\frac{1}{2}}[\,\mathrm{I}+\mathrm{II}\,]$$

$$\mathrm{I} = \int_0^{\frac{L}{2}} \frac{2xh}{L}\sin\frac{m\pi x}{L}dx$$

$$= -\frac{2h}{L}\cdot\frac{L}{m\pi}\int_0^{\frac{L}{2}} x\,d\cos\frac{m\pi x}{L}$$

$$= -\frac{2h}{L}\cdot\frac{L}{m\pi}\,x\cos\frac{m\pi x}{L}\Big|_0^{\frac{L}{2}} + \frac{2h}{L}\cdot\frac{L}{m\pi}\int_0^{\frac{L}{2}}\cos\frac{m\pi x}{L}dx$$

$$= -\frac{2h}{m\pi}\cdot\frac{L}{2}\cos\frac{m\pi}{2} + \frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi x}{L}\Big|_0^{\frac{L}{2}}$$

$$= -\frac{hL}{m\pi}\cos\frac{m\pi}{2} + \frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}$$

$$\mathrm{II} = \int_{\frac{L}{2}}^{L}\frac{2h}{L}\cdot L\sin\frac{m\pi x}{L}dx - \int_{\frac{L}{2}}^{L}\frac{2hx}{L}\sin\frac{m\pi x}{L}dx$$

$$= -2h\cdot\frac{L}{m\pi}\cos\frac{m\pi x}{L}\Big|_{\frac{L}{2}}^{L} + \frac{2h}{L}\cdot\frac{L}{m\pi}\int_{\frac{L}{2}}^{L} x\,d\cos\frac{m\pi x}{L}$$

$$= \frac{2hL}{m\pi}\cos\frac{m\pi}{2} - \frac{2hL}{m\pi}\cos m\pi + \frac{2h}{L}\cdot\frac{L}{m\pi}\,x\cos\frac{m\pi x}{L}\Big|_{\frac{L}{2}}^{L} - \frac{2h}{m\pi}\int_{\frac{L}{2}}^{L}\cos\frac{m\pi x}{L}dx$$

$$= \frac{2hL}{m\pi}\cos\frac{m\pi}{2} - \frac{2hL}{m\pi}\cos m\pi + \frac{2hL}{m\pi}\cos m\pi - \frac{hL}{m\pi}\cos\frac{m\pi}{2} - \frac{2h}{m\pi}\cdot\frac{L}{m\pi}\sin\frac{m\pi x}{L}\Big|_{\frac{L}{2}}^{L}$$

$$= \frac{hL}{m\pi}\cos\frac{m\pi}{2} - \underbrace{\frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin m\pi}_{0} + \frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}$$

$$\mathrm{I}+\mathrm{II} = \frac{4h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}$$

Therefore,

$$\psi(x,t) = \sum_{m=1}^{\infty}\left(\frac{2}{L}\right)^{\frac{1}{2}}\sin\frac{m\pi x}{L}\cdot\left(\frac{2}{L}\right)^{\frac{1}{2}}\cdot\frac{4h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}\cos\omega_m t$$

$$= \sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t\cdot\left(\frac{8h}{\pi^2 m^2}\right)\sin\left(\frac{\pi m}{2}\right).$$

2.1.1

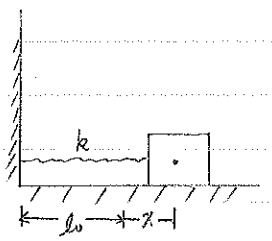
$$T = \frac{1}{2}m\dot{x}^2$$

$$V = \frac{1}{2}kx^2$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0.$$

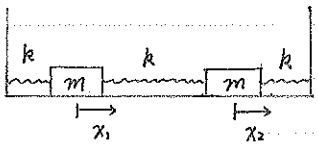


2.1.2

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2-x_1)^2 + \frac{1}{2}kx_2^2$$

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2) - k(x_1^2 - x_1x_2 + x_2^2)$$



$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \qquad \frac{\partial L}{\partial x_1} = -2kx_1 + kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \Rightarrow m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \Rightarrow \ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \quad ①$$

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \qquad \frac{\partial L}{\partial x_2} = kx_1 - 2kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \Rightarrow \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2 \quad ②$$

① and ② are the same as Eqs. (1.8.24) and (1.8.25).

2.1.3

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$

$$V = V(r)$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{\partial V(r)}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \underline{\underline{m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\,\dot{\phi}^2 + \frac{\partial V(r)}{\partial r} = 0}}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \underline{\underline{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0}}$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}, \qquad \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}(mr^2\sin^2\theta\,\dot{\phi}) = 0$$

$$\Rightarrow mr^2\sin^2\theta\,\dot{\phi} = l \qquad (l \text{ is a constant})$$

$$\Rightarrow \underline{\underline{\dot{\phi} = \frac{l}{mr^2\sin^2\theta}}}.$$

Note: $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\ \dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \\ \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{cases}$$

$$T = \frac{1}{2}m\big[\dot r^2\sin^2\theta\cos^2\phi + r^2\dot\theta^2\cos^2\theta\cos^2\phi + r^2\dot\phi^2\sin^2\theta\sin^2\phi + 2r\dot r\dot\theta\sin\theta\cos\theta\cos^2\phi$$
$$- 2r\dot r\sin^2\theta\cos\phi\sin\phi - 2r^2\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi + \dot r^2\sin^2\theta\sin^2\phi +$$
$$r^2\dot\theta^2\cos^2\theta\sin^2\phi + r^2\dot\phi^2\sin^2\theta\cos^2\phi + 2r\dot r\dot\theta\sin\theta\cos\theta\sin^2\phi + 2\dot r r\dot\phi\sin^2\theta\sin\phi\cos\phi$$
$$+ 2r^2\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi + \dot r^2\cos^2\theta + r^2\dot\theta^2\sin^2\theta - 2r\dot r\dot\theta\sin\theta\cos\theta\big]$$
$$= \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2 + r^2\dot\phi^2\sin^2\theta)$$

2.5.1

$$p_s = \frac{\partial T}{\partial \dot q_s} = \sum_i\sum_j T_{ij}(q)\,\dot q_i\,\delta_{js} + \sum_i\sum_j T_{ij}(q)\,\delta_{is}\,\dot q_j$$
$$= \sum_i T_{is}(q)\,\dot q_i + \sum_j T_{sj}(q)\,\dot q_j$$
$$\sum_s p_s\dot q_s = \sum_s\sum_i T_{is}(q)\,\dot q_i\,\dot q_s + \sum_s\sum_j T_{sj}(q)\,\dot q_s\,\dot q_j$$
$$= T + T$$
$$= 2T.$$

2.5.2

$$L = \frac{1}{2}m\dot x^2 - \frac{1}{2}kx^2$$
$$p = \frac{\partial L}{\partial \dot x} = m\dot x$$
$$H = \dot x p - L = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Since L is not an explicit function of t, H is a conservative.

Set $H = E$, where E is a constant, we have

$$\frac{1}{2}kx^2 + \frac{p^2}{2m} = E$$
$$\Rightarrow \frac{x^2}{\frac{2E}{k}} + \frac{p^2}{2mE} = 1.$$

Therefore, if we set $a^2 = 2E/k$, $b^2 = 2mE$, we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{p}{b}\right)^2 = 1.$$

2.5.3

$$L = \frac{1}{2}m(\dot x_1^2 + \dot x_2^2) - k(x_1^2 + x_2^2 - x_1x_2)$$
$$p_1 = \frac{\partial L}{\partial \dot x_1} = m\dot x_1 \quad\Rightarrow\quad \dot x_1 = \frac{p_1}{m}$$
$$p_2 = \frac{\partial L}{\partial \dot x_2} = m\dot x_2 \quad\Rightarrow\quad \dot x_2 = \frac{p_2}{m}$$

$$H = \dot{x}_1 p_1 + \dot{x}_2 p_2 - L$$
$$= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + k(x_1^2 + x_2^2 - x_1 x_2)$$

$$\begin{cases} \dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m} & ① \\ \dot{p}_1 = -\frac{\partial H}{\partial x_1} = -2kx_1 + kx_2 & ② \end{cases}$$

From ①, we know $p_1 = m\dot{x}_1$. Therefore, $\dot{p}_1 = m\ddot{x}_1$. Substitute it into ②, we have $m\ddot{x}_1 = -2kx_1 + kx_2$.

$$\Rightarrow \ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2. \quad ③$$

$$\begin{cases} \dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m} & ④ \\ \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -2kx_2 + kx_1 & ⑤ \end{cases}$$

From ④, we know $p_2 = m\dot{x}_2$. Therefore, $\dot{p}_2 = m\ddot{x}_2$. Substitute it into ⑤, we have $m\ddot{x}_2 = -2kx_2 + kx_1$.

$$\Rightarrow \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2. \quad ⑥$$

2.7.1

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) = -\sum_i \left(\frac{\partial \lambda}{\partial q_i}\frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i}\frac{\partial \omega}{\partial q_i}\right) = -\{\lambda, \omega\}.$$

$$\{\omega, \lambda+\sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\cdot\frac{\partial(\lambda+\sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\cdot\frac{\partial(\lambda+\sigma)}{\partial q_i}\right)$$
$$= \sum_i \left[\frac{\partial \omega}{\partial q_i}\cdot\left(\frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i}\right) - \frac{\partial \omega}{\partial p_i}\cdot\left(\frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i}\right)\right]$$
$$= \sum_i \left[\left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) + \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right)\right]$$
$$= \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) + \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right)$$
$$= \{\omega, \lambda\} + \{\omega, \sigma\}.$$

$$\{\omega, \lambda\sigma\} = \sum_i \left[\frac{\partial \omega}{\partial q_i}\frac{\partial(\lambda\sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial(\lambda\sigma)}{\partial q_i}\right]$$
$$= \sum_i \left[\lambda\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} + \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i}\sigma - \lambda\frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\sigma\right]$$
$$= \lambda\sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right) + \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right)\cdot\sigma$$
$$= \lambda\{\omega, \sigma\} + \{\omega, \lambda\}\sigma.$$

Physics 827: Homework Set No. 1

Due date: Thursday, September 29, 2011, 3:00pm in the grader's office (see course web site)

Total point value of set: 55 points

Problem 1 (5 pts.): Exercise 1.1.2 (Shankar p.3)

Problem 2 (5 pts.): Exercise 1.1.3 (Shankar p.4)

Problem 3 (10 pts.): Orthonormalize the vectors $|I\rangle \leftrightarrow (0,3,4)$, $|II\rangle \leftrightarrow (2,0,0)$, $|III\rangle \leftrightarrow (0,-1,3)$, using the Gram-Schmidt procedure and starting with $|I\rangle$.

Problem 4 (5 pts.): Exercise 1.3.4 (Shankar p.17, proof of triangle inequality)

Problem 5 (15 pts.): Exercise 1.4.1 (Shankar p.18)

Problem 6 (10 pts.): Exercise 1.4.2 (Shankar p.18)

Problem 7 (5 pts.): Exercise 1.6.1 (Shankar p.22)

HW 1

Physics 827 (QM1)

Au2011

Exercise 1.1.2 (p. 3)

The rule given for addition and scalar multiplication is that for the cartesian components of 3-vectors in $\mathbb{V}^3(\mathbb{R})$.

- The null vector is $|0\rangle = (0,0,0)$; it obviously satisfies $|V\rangle + |0\rangle = (v_1, v_2, v_3) + (0,0,0) = (v_1+0, v_2+0, v_3+0) = (v_1, v_2, v_3) = |V\rangle$
- The inverse of $|V\rangle = (a,b,c)$ is $|-V\rangle = (-a,-b,-c) = -(a,b,c) = -|V\rangle$
- The vectors of the form $(a,b,1)$ do not include the null vector (i.e. $\alpha(a,b,1) \notin \mathbb{V}$ for $\alpha = 0$), so $\mathbb{V} = \{(a,b,1) \mid a,b \in \mathbb{R}\}$ is not a vector space.

Exercise 1.1.3 (p. 4)

- $\mathbb{V}_1 = \{f(x) \mid 0 \le x \le L, f(0) = f(L) = 0\}$

  If I add two such functions or multiply such a function with a constant, the result will satisfy the boundary condition ✓

  The null vector $f(x) = 0 \ \forall x$ satisfies the boundary condition ✓

  The inverse $-f(x)$ satisfies the boundary condition ✓

  Distributive, commutative and associative properties are satisfied independent of boundary conditions ✓

  $\Rightarrow \mathbb{V}_1$ is a vector space.

- $\mathbb{V}_2 = \{f(x) \mid 0 \le x \le L, f(0) = f(L)\}$ (periodic functions)

  Adding two periodic functions or multiplying them by a constant results again in periodic functions ✓

  The null function is periodic ✓

- The inverse $-f(x)$ of a periodic function is periodic ✓
- Distribution, commutation and associative properties hold independent of boundary conditions

$\Rightarrow \mathbb{V}_2$ is a vector space

- $\mathbb{V}_3 = \{ f(x) \mid 0 \le x \le L, f(0) = 4 \}$

This is not a vector space since $g(x) = \alpha f(x)$, $\alpha \neq 1$, does not satisfy $g(0) = 4$.

- Exercise 1.3.2 (modified) (p. 16)

Orthonormalize the 3 vectors $|\mathrm{I}\rangle = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $|\mathrm{II}\rangle = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $|\mathrm{III}\rangle = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

$$|1\rangle = \frac{|\mathrm{I}\rangle}{\sqrt{\langle \mathrm{I}|\mathrm{I}\rangle}} = \frac{1}{\sqrt{4+0+0}} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|2'\rangle = |\mathrm{II}\rangle - |1\rangle \underbrace{\langle 1|\mathrm{II}\rangle}_{0} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$ (since $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ are already orthogonal)

$$|2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{1}{\sqrt{0+9+16}} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$$

$$|3'\rangle = |\mathrm{III}\rangle - |1\rangle \underbrace{\langle 1|\mathrm{III}\rangle}_{0} - |2\rangle\langle 2|\mathrm{III}\rangle = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix} \left(0 - \frac{3}{5} + \frac{12}{5}\right)$$

$$= \begin{bmatrix} 0 \\ -1 - \frac{27}{25} \\ 3 - \frac{36}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{52}{25} \\ +\frac{39}{25} \end{bmatrix}$$

$$|3\rangle = \frac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}} = \begin{bmatrix} 0 \\ -\frac{52}{25} \\ +\frac{39}{25} \end{bmatrix} \frac{1}{\frac{1}{25}\sqrt{0 + 4^2 \cdot 13^2 + 3^2 \cdot 13^2}} = \begin{bmatrix} 0 \\ -52 \\ +39 \end{bmatrix} \frac{1}{13 \cdot 5} = \begin{bmatrix} 0 \\ -4/5 \\ 3/5 \end{bmatrix}$$

• Exercise 1.3.4 (p. 17)

Prove $|V+W| \leq |V| + |W|$.

Proof: Since both sides are real and positive, it's sufficient to prove $|V+W|^2 \leq (|V| + |W|)^2$

or $|V|^2 + 2|V||W| + |W|^2 - \langle V+W | V+W \rangle \geq 0$

Work on l.h.s.:

$$|V|^2 + 2|V||W| + |W|^2 - \underbrace{\langle V|V\rangle}_{|V|^2} - \langle V|W\rangle - \underbrace{\langle W|V\rangle}_{=\langle V|W\rangle^*} - \underbrace{\langle W|W\rangle}_{|W|^2}$$

$$= 2\left(|V||W| - \underbrace{\mathrm{Re}\langle V|W\rangle}_{\leq |\langle V|W\rangle|}\right)$$

Since $|\langle V|W\rangle|^2 = \left(\mathrm{Re}\langle V|W\rangle\right)^2 + \left(\mathrm{Im}\langle V|W\rangle\right)^2$

$$\geq \left(\mathrm{Re}\langle V|W\rangle\right)^2$$

$$\geq 2\left(|V||W| - |\langle V|W\rangle|\right) \geq 0 \quad \diamond$$

↑ Schwarz inequality

• Exercise 1.4.1 (p. 18)

In $\mathbb{V}^n$, prove that $\mathbb{V}_\perp^{n-1} = \{|W\rangle \mid \langle V|W\rangle = 0, |V\rangle \neq 0\}$ is a subspace of dimension $n-1$.

Proof: Since $\mathbb{V}^n$ has dimension n, we can find n linearly independent vectors in $\mathbb{V}^n$, $\{|W_i\rangle \mid i=1,\ldots,n\}$. Using the Gram-Schmidt algorithm, we can orthonormalize these vectors to obtain an orthonormal basis

$\{|i\rangle \,|\, i=1,\dots,n\}$. Now choose an arbitrary vector $|V\rangle \neq |0\rangle$ in $\mathbb{V}^n$. We can decompose it in this basis as

$$|V\rangle = \sum_{i=1}^{n} v_i |i\rangle, \qquad (*)$$

where at least one of the components v_i is non-zero. Let us assume without loss of generality that $v_1 \neq 0$. (We can always achieve that by renumbering the basis states accordingly.). Eq. (*) can then be rewritten as

$$|V\rangle = v_1|1\rangle + \sum_{i=2}^{n} v_i|i\rangle \;\Rightarrow\; |1\rangle = \sum_{i=2}^{n} \frac{v_i}{v_1}|i\rangle - \frac{1}{v_1}|V\rangle \qquad (**)$$

Since the basis was orthonormal, $|1\rangle$ is orthogonal to all states $|i\rangle, i=2,\dots,n$. For (**) to hold, $|V\rangle$ must therefore have a part that is orthogonal to all states $|i\rangle, i=2,\dots,n$. Hence $|V\rangle$ is linearly independent of $\{|i\rangle \,|\, i=2,\dots,n\}$, since the part of $|V\rangle$ that is orthogonal to these vectors cannot be obtained from any linear superposition of these vectors.

Let us now look at this new set of n linearly independent vectors given by $\{|V\rangle, |2\rangle, \dots, |n\rangle\}$. Using Gram-Schmidt again, starting from $|V\rangle$, we obtain a new orthonormal basis of $\mathbb{V}^n$,

$\{|1'\rangle, |2'\rangle, \dots, |n'\rangle\}$ whose first unit vector $|1'\rangle = \dfrac{|V\rangle}{\sqrt{\langle V|V\rangle}}$ is in the direction of $|V\rangle$.

We can now decompose any vector $|W\rangle \in \mathbb{V}^n$ as

$$|W\rangle = \sum_{i=1}^{n} w_n |i'\rangle .$$

Any element of the set $\mathbb{V}_\perp = \{|V_\perp\rangle \in \mathbb{V}^n, \langle V_\perp|V\rangle = 0\}$ of vectors $|V_\perp\rangle$ that are orthogonal to $|V\rangle$ are orthogonal to $|1'\rangle$ and thus can be written as

$$|V_\perp\rangle = \sum_{i=2}^{n} v_i |i'\rangle$$

with no component along $|1'\rangle$. The set $\{|i'\rangle | i=2,\dots,n\}$ forms an $(n-1)$-dimensional orthonormal basis, so the set $\mathbb{V}_\perp \equiv \mathbb{V}_\perp^{n-1}$ has dimension $n-1$. (We need to show that this subset of $\mathbb{V}^n$ is a vector space:) It contains the null vector $|0\rangle$ since $\langle 0|V\rangle = 0$ for any $|V\rangle \in \mathbb{V}^n$. For any vector $|V_\perp\rangle \in \mathbb{V}_\perp^{n-1}$, its inverse $|-V_\perp\rangle$ is also in $\mathbb{V}_\perp^{n-1}$ since $\langle -V_\perp|V\rangle = -\langle V_\perp|V\rangle = 0$, so $|-V_\perp\rangle$ is orthogonal to $|V\rangle$. Adding two vectors that are orthogonal to $|V\rangle$ or multiplying them by a constant again produces vectors that are orthogonal to $|V\rangle$, i.e. elements of $\mathbb{V}_\perp^{n-1}$.

Hence $\mathbb{V}_\perp^{n-1}$ satisfies all axioms that define a vector space; it is an $(n-1)$-dimensional subspace of $\mathbb{V}^n$.

Exercise 1.4.2 (p. 18)

- The elements of $V_1^{n_1}$ can be written as

$$|V\rangle = \sum_{i=1}^{n_1} v_i |i\rangle$$

where $\{|i\rangle, i=1,\dots,n_1\}$ is an orthonormal basis of $V_1^{n_1}$.

- The elements of $V_2^{n_2}$ can be written as

$$|W\rangle = \sum_{i=1}^{n_2} w_i |i'\rangle$$

where $\{|i'\rangle, i=1,\dots,n_2\}$ is an orthonormal basis of $V_2^{n_2}$.

Since all elements of $V_1^{n_1}$ are orthogonal to all elements of $V_2^{n_2}$, all states $|i\rangle$ $(i=1,\dots,n_1)$ are orthogonal to all states $|i'\rangle$ $(i=1,\dots,n_2)$.

Hence the set $\{|i\rangle, |j'\rangle, i=1,\dots,n_1; j=1,\dots,n_2\}$ is an orthonormal basis for an (n_1+n_2)-dimensional vector space.

By definition, the elements of $V_1^{n_1} \oplus V_2^{n_2}$ are all linear combinations

$$|Z\rangle = a|V\rangle + b|W\rangle \qquad |V\rangle \in V_1^{n_1},\ |W\rangle \in V_2^{n_2},$$

a, b constants.

These can be written as

$$|Z\rangle = a\sum_{i=1}^{n} v_i |i\rangle + b\sum_{j=1}^{n_2} w_2 |j'\rangle = \sum_{i=1}^{n_1} (a v_i)|i\rangle + \sum_{j=1}^{n_2} (b w_j)|j'\rangle$$

i.e. they can be decomposed in the orthonormal

basis set $\{|i\rangle, |j'\rangle \mid i=1,\dots n_1; j=1,\dots,n_2\}$ which is n_1+n_2 dimensional. Hence the dimensionality of $V_1^{n_1} \oplus V_2^{n_2}$ is not larger than n_1+n_2.

Since the orthogonal basis elements themselves are also elements of $V_1^{n_1} \oplus V_2^{n_2}$, its dimensionality is at least n_1+n_2

Hence $V_1^{n_1} \oplus V_2^{n_2}$ has dimensionality equal to n_1+n_2.

Exercise 1.6.1 (p. 22)

From the columns of the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [\Omega_{ij}]$ we read off the action of $\hat{\Omega}$ on the basis states:

$\hat{\Omega}|1\rangle = |2\rangle$, $\hat{\Omega}|2\rangle = |3\rangle$, $\hat{\Omega}|3\rangle = |1\rangle$.

So $\hat{\Omega}$ cyclically permutes the states $|1\rangle, |2\rangle, |3\rangle$.

This corresponds to a rotation by 120° (⅓ of 360°) around the axis $\frac{|1\rangle + |2\rangle + |3\rangle}{\sqrt{3}}$. ($\frac{1}{\sqrt{3}}$ normalizes the unit vector pointing along this axis.).

Physics 710-712 due October 30, 2009

Problem Set 4

Problem 1: Do exercise 2.5.3 of the text.

Solution: The problem asks us to get the equations of motion using the Hamiltonian method for the system shown in Figure 1.5 (p. 46) of the text. Using x_1 and x_2 shown there as the generalized coordinates, the kinetic energy and potential energy are

$$T = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2, \qquad V = \frac{k}{2}x_1^2 + \frac{k}{2}(x_1 - x_2)^2 + \frac{k}{2}x_2^2,$$

so the Lagrangian is $\mathcal{L} = T - V$ and thus the generalized momenta are

$$\begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial T}{\partial \dot{x}_1} = m\dot{x}_1, \qquad \Rightarrow \qquad \dot{x}_1 = \frac{p_1}{m}, \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\partial T}{\partial \dot{x}_2} = m\dot{x}_2, \qquad \Rightarrow \qquad \dot{x}_2 = \frac{p_2}{m}, \end{aligned}$$

so the Hamiltonian is

$$\mathcal{H} = T + V = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}x_1^2 + \frac{k}{2}(x_1 - x_2)^2 + \frac{k}{2}x_2^2,$$

and Hamilton's equations are

$$\begin{aligned} \dot{x}_1 &= \frac{\partial \mathcal{H}}{\partial p_1} = \frac{p_1}{m}, \qquad \dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = -2kx_1 + kx_2, \\ \dot{x}_2 &= \frac{\partial \mathcal{H}}{\partial p_2} = \frac{p_2}{m}, \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = -2kx_2 + kx_1. \end{aligned}$$

Problem 2: Do exercise 2.7.2 of the text.

Solution: (i):

$$\begin{aligned} \{q_i, q_j\} &:= \sum_k \left(\frac{\partial q_i}{\partial q_k}\cdot\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\cdot\frac{\partial q_j}{\partial q_k} \right) = \sum_k \left(\frac{\partial q_i}{\partial q_k}\cdot 0 - 0\cdot\frac{\partial q_j}{\partial q_k} \right) = 0 \\ \{p_i, p_j\} &:= \sum_k \left(\frac{\partial p_i}{\partial q_k}\cdot\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\cdot\frac{\partial p_j}{\partial q_k} \right) = \sum_k \left(0\cdot\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\cdot 0 \right) = 0 \\ \{q_i, p_j\} &:= \sum_k \left(\frac{\partial q_i}{\partial q_k}\cdot\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\cdot\frac{\partial p_j}{\partial q_k} \right) = \sum_k (\delta_{ik}\delta_{jk} - 0\cdot 0) = \delta_{ij}, \end{aligned}$$

and

$$\begin{aligned} \{q_i, \mathcal{H}\} &:= \sum_k \left(\frac{\partial q_i}{\partial q_k}\cdot\frac{\partial \mathcal{H}}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\cdot\frac{\partial \mathcal{H}}{\partial q_k} \right) = \sum_k \left(\delta_{ik}\cdot\frac{\partial \mathcal{H}}{\partial p_k} - 0\cdot\frac{\partial \mathcal{H}}{\partial q_k} \right) = \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i, \\ \{p_i, \mathcal{H}\} &:= \sum_k \left(\frac{\partial p_i}{\partial q_k}\cdot\frac{\partial \mathcal{H}}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\cdot\frac{\partial \mathcal{H}}{\partial q_k} \right) = \sum_k \left(0\cdot\frac{\partial \mathcal{H}}{\partial p_k} - \delta_{ik}\cdot\frac{\partial \mathcal{H}}{\partial q_k} \right) = -\frac{\partial \mathcal{H}}{\partial q_i} = \dot{p}_i, \end{aligned}$$

where in the last steps I used Hamilton's equations.

(ii): The Hamiltonian given is $\mathcal{H} = p_x^2 + p_y^2 + ax^2 + by^2$. If $a = b$, $\mathcal{H}$ has a

symmetry under simultaneous rotations in the x-y and p_x-p_y planes, under which ℓ_z (the generator) is conserved. Therefore $\{\ell_z, \mathcal{H}\} = 0$.

We check this as follows:

$$\{\ell_z, \mathcal{H}\} = \sum_k \left(\frac{\partial \ell_z}{\partial q_k} \cdot \frac{\partial \mathcal{H}}{\partial p_k} - \frac{\partial \ell_z}{\partial p_k} \cdot \frac{\partial \mathcal{H}}{\partial q_k} \right) = \frac{\partial \ell_z}{\partial x} \cdot \frac{\partial \mathcal{H}}{\partial p_x} + \frac{\partial \ell_z}{\partial y} \cdot \frac{\partial \mathcal{H}}{\partial p_y} - \frac{\partial \ell_z}{\partial p_x} \cdot \frac{\partial \mathcal{H}}{\partial x} - \frac{\partial \ell_z}{\partial p_y} \cdot \frac{\partial \mathcal{H}}{\partial y}.$$

But

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial p_k} &= 2p_k, & \frac{\partial \mathcal{H}}{\partial x_k} &= \left(\frac{\partial \mathcal{H}}{\partial x} \ , \ \frac{\partial \mathcal{H}}{\partial y} \right) = (2ax \ , \ 2by), \\ \frac{\partial \ell_z}{\partial p_k} &= \frac{\partial (x p_y - y p_x)}{\partial p_k} = \left(\frac{\partial \ell_z}{\partial p_x} \ , \ \frac{\partial \ell_z}{\partial p_y} \right) = (-y \ , \ x), & \frac{\partial \ell_z}{\partial q_k} &= \left(\frac{\partial \ell_z}{\partial x} \ , \ \frac{\partial \ell_z}{\partial y} \right) = (p_y \ , \ -p_x), \end{aligned}$$

so

$$\{\ell_z \ , \ \mathcal{H}\} = p_y \cdot 2p_x + (-p_x) \cdot 2p_y - (-y) \cdot 2ax - x \cdot 2by = 2xy(a-b)$$

which vanishes if $a = b$.

Problem 3: Do exercise 2.8.1 of the text.

Solution: Since $g = p_1 + p_2$, it generates the infinitesimal transformations

$$\begin{aligned} \delta x_1 &= +\epsilon \frac{\partial g}{\partial p_1} = +\epsilon, & \delta p_1 &= -\epsilon \frac{\partial g}{\partial x_1} = 0, \\ \delta x_2 &= +\epsilon \frac{\partial g}{\partial p_2} = +\epsilon, & \delta p_2 &= -\epsilon \frac{\partial g}{\partial x_2} = 0. \end{aligned}$$

So, to order ϵ, these give the canonical transformations $x_i \to \bar{x}_i(x_j, p_j)$ and $p_i \to \bar{p}_i(x_j, p_j)$ with

$$\begin{aligned} \bar{x}_1 &= x_1 + \epsilon, & \bar{p}_1 &= p_1, \\ \bar{x}_2 &= x_2 + \epsilon, & \bar{p}_2 &= p_2, \end{aligned}$$

which is precisely a spatial translation of the whole system by an amount ϵ.

PHY 614 : Quantum Mechanics

Problem Set No. 1 : **Due : September 09 ,2011**

Solutions to all the problems will be posted on the website after the due date.

1. (20 points) The lagrangian for a charged particle restricted to move on a two dimensional plane with coordinates x and y in the presence of a magnetic field in the z direction perpendicular to the plane is given by

$$L_1 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{eB}{2}(x\dot{y} - y\dot{x}) \tag{1}$$

   (a) Calculate the conjugate momenta.

   (b) Calculate the Poisson bracket $\{\dot{x}, \dot{y}\}_{PB}$.

   (c) Calculate the hamiltonian H_1 which follows from L_1.

   (d) Obtain the equations of motion which follow from L_1.

   (e) Now consider a different lagrangian

$$L_2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + eB(x\dot{y}) \tag{2}$$

   Obtain the equations of motion which follow from L_2

   (f) Calculate the hamiltonian which follows from L_2

   (g) Obtain the equations of motion which follow from the hamiltonians H_1 and H_2

   (h) Explain why the equations of motion which follow from L_1 and L_2 are identical.

2. (10 points) Exercise (2.7.1) of Shankar. (Ignore the last sentence "*Note the similarity* $\cdots$)

3. (10 points) Exercise (2.2.4) of Shankar

4. (10 points) Exercise (2.7.8) of Shankar

5. (10 points) Exercise (2.8.2) of Shankar

6. (10 points) Exercise (2.8.3) of Shankar

7. (10 points) Exercise (2.8.4) of Shankar

8. (10 points) Exercise (2.8.7) of Shankar

PHY-614 (2011)
PROBLEM SET #1

Problem 1

$$L_1 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{eB}{2}(x\dot{y} - y\dot{x}) \quad (7)$$

(a) $$P_x = m\dot{x} - \frac{eB}{2}y$$
$$P_y = m\dot{y} + \frac{eB}{2}x \quad (8)$$

(b) $$\dot{x} = \frac{1}{m}\left(P_x + \frac{eB}{2}y\right)$$
$$\dot{y} = \frac{1}{m}\left(P_y - \frac{eB}{2}x\right) \quad (9)$$

(c) $$\{\dot{x}, \dot{y}\}_{PB} = \frac{1}{m}\left\{P_x + \frac{eB}{2}y,\ P_y - \frac{eB}{2}x\right\}$$

$$= -\frac{eB}{2m}\{P_x, x\} + \frac{eB}{2m}\{y, P_y\}$$

$$= -\frac{eB}{2m}(-1) + \frac{eB}{2m}(1)$$

$$= \frac{eB}{m} \quad (10)$$

(c) $$H_1 = P_x\dot{x} + P_y\dot{y} - L$$

$$= \frac{P_x}{m}\left(P_x + \frac{eB}{2}y\right) + \frac{P_y}{m}\left(P_y - \frac{eB}{2}x\right)$$

$$- \frac{eB}{2m}x\left(P_y - \frac{eB}{2}x\right)$$

$$+ \frac{eB}{2m}y\left(P_x + \frac{eB}{2}y\right)$$

$$- \frac{1}{2m}\left(P_x + \frac{eB}{2}y\right)^2$$

$$= - \frac{1}{2m}\left(P_y - \frac{eB}{2}x\right)^2$$

This leads to

$$H_1 = \frac{1}{2m}\left(P_x + \frac{eB}{2}y\right)^2 + \frac{1}{2m}\left(P_y - \frac{eB}{2}x\right)^2 \quad (11)$$

(d) Equations of motion from Lagrangian

$$\frac{dP_x}{dt} = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{eB}{2}y\right) = +\frac{eB}{2}\dot{y}$$

or

$$m\frac{d^2x}{dt^2} = eB\dot{y} \quad (12)$$

$$\frac{dP_y}{dt} = \frac{\partial L}{\partial y} \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{eB}{2}x\right) = -\frac{eB}{2}\dot{x}$$

$$m\frac{d^2y}{dt^2} = -eB\dot{x} \quad (13)$$

(e) For Lagrangian

$$L_2 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + eB\,x\,\dot{y} \quad (14)$$

$$P_x = m\dot{x}$$
$$P_y = m\dot{y} + eBx \quad (15)$$

Two eqns of motion are

$$\frac{dP_x}{dt} = \frac{\partial L}{\partial x} \Rightarrow m\frac{d^2x}{dt^2} = eB\dot{y} \quad (16)$$

$$\frac{dP_y}{dt} = \frac{\partial L}{\partial y} \Rightarrow m\frac{d^2y}{dt^2} = -eB\dot{x}$$

(f) The Hamiltonian is now

$$H_2 = \dot{x}P_x + \dot{y}P_y - L$$

$$= \frac{P_x^2}{m} + \frac{P_y}{m}(P_y - eBx) - \frac{P_x^2}{2m} - \frac{1}{2m}(P_y - eBx)^2 - \frac{eBx}{m}(P_y - eBx)$$

$$\boxed{H_2 = \frac{P_x^2}{2m} + \frac{(P_y - eBx)^2}{2m}} \quad (17)$$

(g) Equations of motion from Hamiltonian H_1

$$\dot{x} = \frac{\partial H_1}{\partial P_x} \Rightarrow \dot{x} = \frac{1}{m}\left(P_x + \frac{eB}{2}y\right)$$

$$\dot{y} = \frac{\partial H_1}{\partial P_y} \Rightarrow \dot{y} = \frac{1}{m}\left(P_x - \frac{eB}{2}x\right) \quad (18)$$

which are same as eqn (9)

$$\dot{P}_x = -\frac{\partial H_1}{\partial x} \Rightarrow \frac{dP_x}{dt} = \frac{eB}{2m}\left(P_x - \frac{eB}{2}x\right)$$

which is same as (12), after using (9) or (18)

Similarly for $\frac{dP_y}{dt}$

(b) Note

$$L_1 - L_2 = -\frac{eB}{2}(x\dot{y} + y\dot{x})$$

$$= \frac{d}{dt}\left(-\frac{eB}{2}xy\right) \quad (19)$$

which is a total derivative —

This explains why we get the same dynamics

This is related to the fact that the Lagrangian involves the vector potential rather than the field strength. The term is

$$e\vec{V}\cdot\vec{A} \quad (20)$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$

In our case

$$B = \partial_x A_y - \partial_y A_x \quad (21)$$

If we use a different vector potential $\vec{A}'$

$$\vec{A}' = \vec{A} + \vec{\nabla}\alpha \quad (22)$$

α being some function, we get the same $\vec{B}$

This corresponds to a gauge transformation.

L_1 is the Lagrangian in the gauge

$$A_x^{(1)} = -\frac{eB}{2} y \qquad (23)$$
$$A_y^{(1)} = \frac{eB}{2} x$$

where L_2 is the Lagrangian in the gauge

$$A_y^{(2)} = eBx \qquad (24)$$
$$A_x^{(2)} = 0$$

They are related by a gauge transformation

$$A_i^{(2)} = A_i^{(1)} + \partial_i \left(\frac{eBxy}{2} \right) \qquad (25)$$

PHY 614 - 2011
PROBLEM SET #1

Problem 2 (2.7.4)

(i) $\{\omega, \lambda\} = \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \lambda}{\partial q_i}\frac{\partial \omega}{\partial p_i}$

$= -\left(\frac{\partial \lambda}{\partial q_i}\frac{\partial \omega}{\partial p_i} - \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i}\right)$

$= -\{\lambda, \omega\}$

(ii) $\{\omega, \lambda + \sigma\} = \frac{\partial \omega}{\partial q_i}\frac{\partial}{\partial p_i}(\lambda+\sigma) - \frac{\partial \omega}{\partial p_i}\frac{\partial}{\partial q_i}(\lambda+\sigma)$

$= \{\omega, \lambda\} + \{\omega, \sigma\}$

(iii) $\{\omega, \lambda\sigma\} = \frac{\partial \omega}{\partial q_i}\left(\lambda\frac{\partial \sigma}{\partial p_i} + \frac{\partial \lambda}{\partial p_i}\sigma\right)$

$- \frac{\partial \omega}{\partial p_i}\left(\lambda\frac{\partial \sigma}{\partial q_i} + \frac{\partial \lambda}{\partial q_i}\sigma\right)$

$= \lambda\left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right)$

$+ \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right)\sigma$

$= \lambda\{\omega, \sigma\} + \{\omega, \lambda\}\sigma$

Problem 3 : (2.7.4)

$$\{\bar{X}, \bar{Y}\} = \{X\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta\}$$

$$= \sin\theta\cos\theta\{x,x\} - \sin^2\theta\{y,x\} + \cos^2\theta\{x,y\} - \sin\theta\{y,y\}$$

$$= (\cos^2\theta + \sin^2\theta)\{x,y\}$$

$$= 0$$

Since $\{x,y\} = 0$

$$\{\bar{X}, \bar{P}_x\} = \{x\cos\theta - y\sin\theta,\ P_x\cos\theta - P_y\sin\theta\}$$

$$= \cos^2\theta\{x, P_x\} - \sin\theta\cos\theta\{y, P_x\} - \sin\theta\cos\theta\{x, P_y\} + \sin^2\theta\{y, P_y\}$$

$$= \cos^2\theta + \sin^2\theta = 1$$

Similarly $\{\bar{X}, \bar{P}_y\} = 0$

$\{\bar{Y}, \bar{P}_y\} = 1$

$\{\bar{Y}, \bar{P}_x\} = 0$

This proves that the transformation is canonical

Problem 4 (2.7.8)

(1) Since $q_i = q_i(\overline{q}_j)$

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial q_i}{\partial \overline{q}_j} \frac{d\overline{q}_j}{dt}$$

(2) $$\Rightarrow \frac{\partial \dot{q}_i}{\partial \dot{\overline{q}}_j} = \frac{\partial q_i}{\partial \overline{q}_j}$$

(3) By definition

$$\overline{p}_i = \left[\frac{\partial \mathcal{L}(\overline{q}, \dot{\overline{q}})}{\partial \dot{\overline{q}}_i}\right]_{\overline{q}_i}$$

However $\mathcal{L}$ is by definition

$$\mathcal{L}(\overline{q}, \dot{\overline{q}}) = \mathcal{L}(q, \dot{q})$$

Thus

$$\overline{p}_i = \left[\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\overline{q}}_i}\right]_{\overline{q}_i}$$

$$= \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{\overline{q}}_i}\right]_{\overline{q}_i} = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)_{\overline{q}} \left(\frac{\partial q_j}{\partial \overline{q}_i}\right)$$

However since $\overline{q} = \overline{q}(q)$, holding $\overline{q}$ constant is same as holding q constant. Thus

$$\overline{p}_i = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)_q \frac{\partial q_j}{\partial \overline{q}_i} = \frac{\partial q_j}{\partial \overline{q}_i} p_j$$

Problem 5: (2.8.2)

If the generator is $g(P_i, q_i)$

$$\delta P_i = \varepsilon \{P_i, g\} = -\varepsilon \frac{\partial g}{\partial q_i} \quad (26)$$

$$\delta q_i = \varepsilon \{q_i, g\} = \varepsilon \frac{\partial g}{\partial q_i}$$

Then

$$\{q_i', P_j'\}$$
$$= \{q_i + \delta q_i, P_j + \delta P_j\}$$

$$= \{q_i, P_j\} + \varepsilon \{\{q_i, g\}, P_j\} \quad (27)$$
$$+ \varepsilon \{q_i, \{P_j, g\}\}$$

Now use the Jacobi identity

$$0 = \{q_i, \{P_j, g\}\} + \{P_j, \{g, q_i\}\} + \{g, \{q_i, P_j\}\} \quad (28)$$

Then use antisymmetry of Poisson bracket and $\{q_i, P_j\} = \delta_{ij}$ to get

$$0 = \{q_i, \{P_j, g\}\} + \{\{q_i, g\}, P_j\} + \{g, \delta_{ij}\}.$$

However $\{g, \delta_{ij}\} = 0$

Thus

$$0 = \{q_i, \{P_j, g\}\} + \{\{q_i, g\}, P_j\} \quad (29)$$

Use this in eqn (27) to get

$$\{q_i', P_j'\} = \{q_i, P_j\} = \delta_{ij}$$

One may similarly verify that

$$\{q_i', q_j'\} = \{P_i', P_j'\} = 0$$

Thus transformation is canonical

~~12~~
11

Problem 6 (2.83)

The infinitesimal transformations for such a transformation are

$$\delta x = -\varepsilon y$$
$$\delta p_x = 0$$
$$\delta y = \varepsilon x$$
$$\delta p_y = 0$$

~~Suppose there is a g such that $\delta R = \varepsilon \{R, g\}$.~~

Then

$$\{x + \delta x, p_x + \delta p_x\}$$
$$= \{x - \varepsilon y, p_x\} = 1$$

$$\{x + \delta x, p_y + \delta p_y\}$$
$$= \{x - \varepsilon y, p_y\} = -\varepsilon$$

Thus $\{x', p_y'\} \neq 0$. Hence the transformation is not canonical

Suppose there is a $g(x, y; p_x, p_y)$

such that

$$\delta H = \varepsilon \{H, g\}$$

$$\{H, g\} = \frac{\partial H}{\partial x}\frac{\partial g}{\partial P_x} + \frac{\partial H}{\partial y}\cdot\frac{\partial g}{\partial P_y} - \frac{\partial H}{\partial P_x}\frac{\partial g}{\partial x} - \frac{\partial H}{\partial P_y}\cdot\frac{\partial g}{\partial y}$$

$$= m\omega^2\left(x\frac{\partial g}{\partial P_x} + y\frac{\partial g}{\partial P_y}\right) - \frac{1}{m}\left(P_x\frac{\partial g}{\partial x} + P_y\frac{\partial g}{\partial y}\right)$$

The first set of terms which multiply $m\omega^2$ comes from variation of H due to variation of x, the second set from variation of p.
Since $\delta P_x = \delta P_y = 0$ we must have $g(P_x, P_y)$ only. Since $\delta H = 0$ we must have

$$x\frac{\partial g}{\partial P_x} + y\frac{\partial g}{\partial P_y} = 0$$

The only solution of this equation is

$$g = \text{constant}.$$

Problem 7 (2.8.4)

For infinitesimal rotations in (x,p) plane

$$\delta x = \varepsilon p$$
$$\delta p = -\varepsilon x$$

This is clearly a canonical transformation

$$\{x', p'\} = \{x,p\} + \varepsilon\{\delta x, p\} + \varepsilon\{x, \delta p\} + O(\epsilon^2)$$
$$= \{x,p\} = 1$$

For finite transformations

$$x' = \cos\theta \cdot x + \sin\theta\, p$$
$$p' = -\sin\theta\, x + \cos\theta\, p$$

$$\{x', p'\} = \cos^2\theta\{x,p\} - \sin^2\theta\{p,x\}$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1$$

so that this is canonical

If $g(x,p)$ is the generator

$$\delta x = \epsilon\{x, g\} = \varepsilon\frac{\partial g}{\partial p} = \varepsilon p \Rightarrow \frac{\partial g}{\partial p} = p$$
$$\delta p = \varepsilon\{p, g\} = -\varepsilon\frac{\partial g}{\partial x} = -\varepsilon x \Rightarrow \frac{\partial g}{\partial x} = x$$

The soln of these 2 equations is

$$g(x,p) = \frac{1}{2}(p^2 + x^2)$$

ie the Hamiltonian itself.

In fact the canonical transformation is just time evolution with $\theta = t$

Problem 8: (2.8.7)

The total energy can be easily calculated to be

$$E = \frac{1}{2} m\omega^2 (A^2 + B^2)$$

Imposing the initial and final conditions

$$x(0) = x_1 = A$$

$$x(T) = x_2 = A \cos \omega T + B \sin \omega T$$

Thus $A = x_1$

$$B = \frac{x_2 - x_1 \cos \omega T}{\sin \omega T}$$

The action is given by

$$S = \int_0^T dt \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right]$$

Substituting for A, B and evaluating the integral we get the answer easily

Physics 710-711-712 November 16, 2009

Problem Set 5

Problem 1: Do exercise 4.2.1 of the text.

(1)

Solution: The possible outcomes are $L_z = \{1, 0, -1\}$, which are the eigenvalues of L_z.

(2)

Solution: $L_z|\psi\rangle = 1 \cdot |\psi\rangle$ implies

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(Note that I have normalized $|\psi\rangle$!) Then

$$\begin{aligned}
\langle L_x \rangle &= \langle \psi | L_x | \psi \rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0. \\
\langle L_x^2 \rangle &= \langle \psi | L_x^2 | \psi \rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}. \\
\Delta L_x &= \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \sqrt{\left(\frac{1}{2}\right) - 0^2} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

(3)

Solution: The characteristic equation for L_x is

$$0 = \det(L_x - \lambda) = \det \begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} = \lambda - \lambda^3, \qquad \Rightarrow \qquad \lambda \in \{1, 0, -1\}.$$

The corresponding eigenvectors, $|\lambda\rangle$, then satisfy

$$0 = (L_x - \lambda)|\lambda\rangle = \begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\lambda a + \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} - \lambda b + \frac{c}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} - \lambda a \end{pmatrix}$$

where we have parameterized the components of $|\lambda\rangle$ by $(a \;\; b \;\; c)$. For $\lambda = 1$, we can solve for b and c in terms of a, giving $b = \sqrt{2}a$ and $c = a$. We then determine a by normalizing $|\lambda = 1\rangle$:

$$|\lambda = 1\rangle = \begin{pmatrix} a \\ \sqrt{2}a \\ a \end{pmatrix}, \qquad \Rightarrow \qquad 1 = \langle \lambda = 1 | \lambda = 1 \rangle = \begin{pmatrix} a^* & \sqrt{2}a^* & a^* \end{pmatrix} \begin{pmatrix} a \\ \sqrt{2}a \\ a \end{pmatrix} = 4|a|^2, \qquad \Rightarrow \qquad a = \frac{1}{2}$$

(where I have chosen the arbitrary phase to be 1). Thus, and doing the same thing for $\lambda = 0$ and $\lambda = -1$, gives

$$|\lambda = 1\rangle = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}, \qquad |\lambda = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}, \qquad |\lambda = -1\rangle = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}.$$

(4)

Solution: The possible outcomes are $L_x = \{1, 0, -1\}$, which are the eigenvalues of L_x. $|\psi\rangle$ is the normalized eigenstate of L_z with eigenvalue $L_z = -1$, which is

$$|\psi\rangle = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

So (here $\mathcal{P}$ stands for "probability of"):

$$\begin{aligned}
\mathcal{P}(L_x = 1) &= |\langle\lambda = 1|\psi\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right|^2 = \frac{1}{4},\\
\mathcal{P}(L_x = 0) &= |\langle\lambda = 0|\psi\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right|^2 = \frac{1}{2},\\
\mathcal{P}(L_x = -1) &= |\langle\lambda = -1|\psi\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right|^2 = \frac{1}{4}.
\end{aligned}$$

(5)

Solution:

$$L_z^2 = \begin{pmatrix}1 & & \\ & 0 & \\ & & 1\end{pmatrix}, \qquad \Rightarrow \qquad \text{the possible outcomes are } L_z^2 = \{0, 1\}.$$

An eigenbasis of the $L_z^2 = 1$ eigenspace is $\{|a\rangle, |b\rangle\}$ with

$$|a\rangle = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \qquad |b\rangle = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Therefore, upon measuring $L_z^2 = 1$, the state collapses to

$$|\psi\rangle \longrightarrow |\psi'\rangle = \frac{(|a\rangle\langle a| + |b\rangle\langle b|)|\psi\rangle}{|(|a\rangle\langle a| + |b\rangle\langle b|)|\psi\rangle|}.$$

But

$$[|a\rangle\langle a| + |b\rangle\langle b|]\,|\psi\rangle = \left[\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\right]\frac{1}{2}\begin{pmatrix}1\\ 1\\ \sqrt{2}\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\frac{1}{2} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}1\\ 0\\ \sqrt{2}\end{pmatrix},$$

has norm

$$\sqrt{\frac{1}{2}\begin{pmatrix}1 & 0 & \sqrt{2}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}} = \frac{\sqrt{3}}{2},$$

so

$$|\psi'\rangle = \frac{1}{\sqrt{3}/2}\frac{1}{2}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}.$$

The probability of $L_z^2 = +1$ is

$$\begin{aligned}\mathcal{P}(L_z^2 = 1) &= \langle\psi|\left(|a\rangle\langle a| + |b\rangle\langle b|\right)|\psi\rangle = |\langle a|\psi\rangle|^2 + |\langle b|\psi\rangle|^2 \\ &= \left|\frac{1}{2}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}\right|^2 + \left|\frac{1}{2}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.\end{aligned}$$

If we measured L_z the posible outcomes are the eigenvalues L_z, $\{0, \pm 1\}$, with probabilities

$$\mathcal{P}(L_z = 1) = \left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}|\psi'\rangle\right|^2 = \left|\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{3}.$$

$$\mathcal{P}(L_z = 0) = \left|\begin{pmatrix}0 & 1 & 0\end{pmatrix}|\psi'\rangle\right|^2 = \left|\frac{1}{\sqrt{3}}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right|^2 = 0.$$

$$\mathcal{P}(L_z = -1) = \left|\begin{pmatrix}0 & 0 & 1\end{pmatrix}|\psi'\rangle\right|^2 = \left|\frac{1}{\sqrt{3}}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right|^2 = \frac{2}{3}.$$

(6)

Solution: In the L_z eigenbasis

$$|L_z = 1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad |L_z = 0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad |L_z = -1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

write the unknown state as

$$|\psi\rangle = \begin{pmatrix}a\\b\\c\end{pmatrix}.$$

Then

$$\mathcal{P}(L_z = 1) = \frac{1}{4} = |\langle L_z = 1|\psi\rangle|^2 = \left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}\right|^2 = |a|^2,$$

$$\mathcal{P}(L_z = 1) = \frac{1}{2} = |\langle L_z = 0|\psi\rangle|^2 = \left|\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}\right|^2 = |b|^2,$$

$$\mathcal{P}(L_z = 1) = \frac{1}{4} = |\langle L_z = -1|\psi\rangle|^2 = \left|\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}\right|^2 = |c|^2.$$

The most general solution to these three equations is then

$$a = \frac{1}{2}e^{i\delta_1}, \qquad b = \frac{1}{\sqrt{2}}e^{i\delta_2}, \qquad c = \frac{1}{2}e^{i\delta_3},$$

for some arbitrary phases δ_i, which gives the desired answer.

The δ_i phase factors are *not* irrelevant. For example

$$\begin{aligned}
\mathcal{P}(L_x = 0) &= |\langle\lambda = 0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\frac{1}{2}\begin{pmatrix}e^{i\delta_1}\\ \sqrt{2}e^{i\delta_2}\\ e^{i\delta_3}\end{pmatrix}\right|^2 = \left|\frac{e^{i\delta_1}}{2\sqrt{2}} - \frac{e^{i\delta_3}}{2\sqrt{2}}\right|^2 \\
&= \frac{1}{8}\left(e^{i\delta_1} - e^{i\delta_3}\right)\left(e^{-i\delta_1} - e^{-i\delta_3}\right) = \frac{1}{8}\left(1 - e^{i(\delta_3-\delta_1)} - e^{-i(\delta_3-\delta_1)} + 1\right) \\
&= \frac{1}{4}\left(1 - \cos(\delta_3 - \delta_1)\right),
\end{aligned}$$

so something measurable (a probability) depends on the difference of the phases.

Problem 2: Do exercise 4.2.2 of the text.
Solution:

$$\begin{aligned}
\langle P\rangle &= \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} dx \langle\psi|x\rangle\langle x|P|\psi\rangle \\
&= \int_{-\infty}^{\infty} dx\, \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) = -i\hbar\int_{-\infty}^{\infty} dx\, \psi(x)\frac{d\psi(x)}{dx} \\
&= -\frac{i\hbar}{2}\int_{-\infty}^{\infty} dx\, \frac{d}{dx}\left(\psi(x)^2\right) = -\frac{i\hbar}{2}\,\psi^2\Big|_{-\infty}^{\infty} = 0
\end{aligned}$$

if $\psi \to 0$ as $|x| \to \infty$.

Alternatively, use the k-basis:

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} dk \langle\psi|k\rangle\langle k|P|\psi\rangle = \int_{-\infty}^{\infty} dk\, \hbar k\, \langle\psi|k\rangle\langle k|\psi\rangle = \int_{-\infty}^{\infty} dk\, \hbar k\, \psi^*(k)\psi(k).$$

But

$$\psi(k) = \langle k|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle k|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{ikx}\psi(x),$$

therefore

$$\psi(-k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\psi(x) = \psi^*(k)$$

since $\psi(x)$ is real. So

$$\langle P\rangle = \int_{-\infty}^{\infty} dk\, \hbar k\psi^*(k)\psi(k) = \int_{-\infty}^{\infty} dk\, \hbar k\psi(-k)\psi(k).$$

and under the change of variables $k \to -k$, this becomes

$$\langle P\rangle = \int_{-\infty}^{\infty} dk\, \hbar(-k)\psi(k)\psi(-k) = -\langle P\rangle,$$

and so $\langle P\rangle = 0$.

Problem 3: Do exercise 2.4.3 of the text.
Solution:

$$\begin{aligned}
\left\langle e^{ip_0x/\hbar}\psi\middle|P\middle|e^{ip_0x/\hbar}\psi\right\rangle &= \int_{-\infty}^{\infty}dx\,\langle e^{ip_0x/\hbar}\psi|x\rangle\langle x|P\middle|e^{ip_0x/\hbar}\psi\right\rangle = \int_{-\infty}^{\infty}dx\,\left(e^{ip_0x/\hbar}\psi(x)\right)^*(-i\hbar)\frac{d}{dx}\left(e^{ip_0x/\hbar}\psi(x)\right)\\
&= -i\hbar\int_{-\infty}^{\infty}dx\,\psi^*(x)e^{-ip_0x/\hbar}\left[\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi(x)+e^{ip_0x/\hbar}\frac{d\psi}{dx}\right]\\
&= \int_{-\infty}^{\infty}dx\,\psi^*(x)\,p_0\,\psi(x) - i\hbar\int_{-\infty}^{\infty}dx\,\psi^*(x)\frac{d\psi}{dx}\\
&= p_0\left[\int_{-\infty}^{\infty}dx\,\langle\psi|x\rangle\langle x|\psi\rangle\right]+\left[\int_{-\infty}^{\infty}dx\,\langle\psi|x\rangle\langle x|P|\psi\rangle\right] = p_0\langle\psi|\psi\rangle+\langle\psi|P|\psi\rangle = p_0+\langle P\rangle.
\end{aligned}$$

P4410 Problem Set 3 Solutions

Shankar 4.2.1

(1) The possible values one can obtain if L_z is measured are its eigenvalues.

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \boxed{\text{Eigenvalues are } \{-1, 0, 1\}}$$

(2) The state in which $L_z = 1$ is the coresponding eigenvector

$$|L_z; 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In $|L_z;1\rangle$, What is $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?

$$\langle L_x \rangle_{z1} = \langle L_z;1 | L_x | L_z;1 \rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \boxed{0 = \langle L_x \rangle_{z,1}}$$

$$\langle L_x^2 \rangle_{z,1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{1/2 = \langle L_x^2 \rangle_{z,1}}$$

$$\Delta L_x = \left[\langle L_z;1 | (L_x - \overset{0}{\cancel{\langle L_x \rangle}})^2 | L_z;1 \rangle \right]^{1/2}$$

$$\boxed{\Delta L_x = \frac{1}{\sqrt{2}}}$$

Shankar 4.2.1 continued

(3) Find normalized eigenstates and the eigenvalues of L_x in L_z basis

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues are $\{-1, 0, 1\}$

The eigenvectors are $\left\{\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\}$

and normalizing

$$\left\{\frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\}$$

Writing in terms of L_z basis:

$$-1: \quad \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$0: \quad -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+1: \quad \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

4.2.1 cont'd

(4) Particle is in state $|L_z=-1\rangle \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$P(L_x=-1) = \left|\langle L_x=-1 | L_z=-1\rangle\right|^2 = \left|\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{4}$$

$$P(L_x=0) = \left|\langle L_x=0 | L_z=-1\rangle\right|^2 = \left|\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2}$$

$$P(L_x=-1) = \left|\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{4}$$

(5) If $L_z^2=1$, $L_z=\pm 1$. If this is an ideal measurement, it leaves the particle in a state where the $L_z=0$ component has been "knocked out":

$|\psi\rangle \longrightarrow \begin{pmatrix} 1/2 \\ 0 \\ \sqrt{1/2} \end{pmatrix}$ But this needs to be normalized, so

the state becomes $\begin{pmatrix} 1/2 \\ 0 \\ \sqrt{1/2} \end{pmatrix} \cdot \left[\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right]^{-1/2} = \frac{2}{\sqrt{3}}\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}$

Probability of this is $P(L_x=1)+P(L_x=-1) = \frac{1}{4}+\frac{1}{2} = \boxed{\frac{3}{4}}$

(6) The <u>overall</u> phase is irrelevant physically, but the <u>relative</u> phase of the three terms is important.

(2) Shankar 4.2.3

In x basis, $\langle p \rangle = \int dx\, \psi^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi(x)$

so $-i\hbar \int dx\, \psi^*(x) e^{\frac{-ip_0 x}{\hbar}} \frac{d}{dx} \psi(x) e^{\frac{+ip_0 x}{\hbar}}$

$= -i\hbar \int dx\, \psi^*(x) e^{\frac{-ip_0 x}{\hbar}} \left[\frac{+ip_0}{\hbar} \psi(x) e^{\frac{+ip_0 x}{\hbar}} + e^{\frac{+ip_0 x}{\hbar}} \frac{d}{dx} \psi(x) \right]$

$= -i\hbar \left[\int dx \left(\frac{ip_0}{\hbar} e^{\frac{-ip_0 x}{\hbar}} \psi^*(x) e^{\frac{ip_0 x}{\hbar}} \psi(x) \right) + \int dx \left(\psi^*(x) e^{\frac{-ip_0 x}{\hbar}} e^{\frac{ip_0 x}{\hbar}} \frac{d}{dx} \psi(x) \right) \right]$

$= p_0 \underbrace{\int dx\, \psi^*(x) \psi(x)}_{=1} - \underbrace{i\hbar \int dx\, \psi^*(x) \frac{d}{dx} \psi(x)}_{= \langle p \rangle}$

$= p_0 + \langle p \rangle$ QED.

(3) Shankar 5.1.1

$E = \frac{p^2}{2m} \Rightarrow p = \alpha \sqrt{2mE}$; clearly $|p\rangle = |E, \alpha\rangle$.

$dp = \frac{\alpha m}{\sqrt{2mE}} dE$

so: $\int_{-\infty}^{\infty} dp\, |p\rangle\langle p| e^{-iEt/\hbar}$

$= \int_{-\infty}^{0} dp\, |p\rangle\langle p| e^{-iEt/\hbar} + \int_{0}^{\infty} dp\, |p\rangle\langle p| e^{-iEt/\hbar}$ change integration variable:

$= -\int_{0}^{+\infty} dE \frac{-m}{\sqrt{2mE}} |E,-1\rangle\langle E,-1| e^{\frac{-iEt}{\hbar}} + \int_{0}^{\infty} dE \frac{m}{\sqrt{2mE}} |E,+1\rangle\langle E,+1| e^{\frac{-iEt}{\hbar}} = \sum_{\alpha=\pm 1} \int_{0}^{\infty} dE \frac{m}{\sqrt{2mE}} |E,\alpha\rangle\langle E,\alpha| e^{\frac{-iEt}{\hbar}}$

QED.

(4) Shankar 5.2.1

Let $|1\rangle$ = original ground state

$|1'\rangle$ = ground state of new well.

If well extends from $\frac{L}{2}$ to $\frac{L}{2}$ before,

$-L$ to L after

then $\langle x|1\rangle = \begin{cases} \sqrt{\frac{2}{L}}\cos\frac{\pi x}{L} & |x|<\frac{L}{2} \\ 0 & |x|>\frac{L}{2} \end{cases}$

$\langle x|1'\rangle = \begin{cases} \sqrt{\frac{1}{L}}\cos\frac{\pi x}{2L} & |x|<L \\ 0 & |x|>L \end{cases}$

Prob. of finding particle in $|1'\rangle$ is $|\langle 1|1'\rangle|^2$

$$= \left| \int_{-\infty}^{\infty} dx \langle 1|x\rangle\langle x|1'\rangle \right|^2 = \left| \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \frac{\sqrt{2}}{L}\cos\frac{\pi x}{L}\cos\frac{\pi x}{2L} \right|^2$$

Let $\xi = \frac{\pi x}{2L}$, $dx = \frac{2L}{\pi}d\xi$

$$= \left| \int_{-\pi/4}^{\pi/4} d\xi \frac{2L}{\pi}\frac{\sqrt{2}}{L}\cos 2\xi \cos\xi \right|^2$$

$$= \frac{8}{\pi^2}\left| \int_{-\pi/4}^{\pi/4} d\xi \cos 2\xi \cos\xi \right|^2$$

$$= \frac{8}{\pi^2}\left|\frac{\sqrt{8}}{3}\right|^2 = \left(\frac{8}{3\pi}\right)^2$$

(5) Shankar 5.22 part (a)

(a) Show that for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$, where E_0 is the lowest energy eigenvalue

Expand $|\psi\rangle$ in the eigenbasis of H

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n |\psi_n\rangle, \quad H|\psi_n\rangle = E_n|\psi_n\rangle$$

Then $$H|\psi\rangle = \sum_{n=0}^{\infty} a_n H|\psi_n\rangle = \sum_{n=0}^{\infty} a_n E_n |\psi_n\rangle$$

Taking the inner product with $\langle\psi|$

$$\langle\psi|H|\psi\rangle = \sum_{n=0}^{\infty} a_n E_n \langle\psi|\psi_n\rangle$$

$$= \sum_{n=0}^{\infty} a_n^2 E_n = a_0^2 E_0 + a_1^2 E_1 + a_2^2 E_2 + \cdots$$

If E_0 is the lowest energy eigenvalue...

$E_0 \leq E_n$, for $n > 0$

Then we can shrink all E_n, $n>0$ to E_0 and we have the inequality

$$\langle\psi|H|\psi\rangle \geq E_0 \left(\sum_{n=0}^{\infty} a_n^2\right)$$

$\sum_{n=0}^{\infty} a_n^2 = 1$ since $|\psi\rangle$ is normalized

$$\Rightarrow \langle\psi|H|\psi\rangle \geq E_0$$

(6)

10.1.1

(1) $[\Omega_1^{(1)} \otimes I^{(2)}, I^{(1)} \otimes \Lambda_2^{(2)}] \underbrace{|\omega_1\rangle \otimes |\omega_2\rangle}_{\text{Test state}}$

$= \Omega_1^{(1)} \otimes I^{(2)} (I^{(1)} \otimes \Lambda_2^{(2)}) |\omega_1\rangle \otimes |\omega_2\rangle$
$- (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} \otimes I^{(2)}) |\omega_1\rangle \otimes |\omega_2\rangle$

$= \Omega_1^{(1)} \otimes I^{(2)} (I^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle$
$- (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} |\omega_1\rangle \otimes I^{(2)} |\omega_2\rangle$

$= (\Omega_1^{(1)} \otimes I^{(2)}) |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle$
$- (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} |\omega_1\rangle \otimes |\omega_2\rangle$

$= \Omega_1^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle - \Omega_1^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle$
$= 0$ ✓

(2) $(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\Theta_1^{(1)} \otimes \Lambda_2^{(2)}) \underbrace{|\omega_1\rangle \otimes |\omega_2\rangle}_{\text{test state}}$

$(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\Theta_1^{(1)} |\omega_1\rangle \otimes \Lambda_2^{(2)} |\omega_2\rangle)$

$\Omega_1^{(1)} \Theta_1^{(1)} |\omega_1\rangle \otimes \Gamma_2^{(2)} \Lambda_2^{(2)} |\omega_2\rangle$

$= (\Omega \Theta)_1^{(1)} \otimes (\Gamma \Lambda)_2^{(2)}$

(3) If $[\Omega_1^{(1)}, \Lambda_1^{(1)}] = \Gamma_1^{(1)}$

Show $[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}] = \Gamma_1^{(1)} \otimes I^{(2)}$

$$\left(\left(\Omega_1^{(1)} \otimes I^{(2)}\right)\left(\Lambda_1^{(1)} \otimes I^{(2)}\right) - \left(\Lambda_1^{(1)} \otimes I^{(2)}\right)\left(\Omega_1^{(1)} \otimes I^{(2)}\right)\right)|\omega_1\rangle \otimes |\omega_2\rangle$$

$$= \left(\Omega_1^{(1)}\Lambda_1^{(1)} - \Lambda_1\Omega_1^{(1)}\right)|\omega_1\rangle \otimes I^{(2)}|\omega_2\rangle$$

$$= \left(\Gamma_1^{(1)} \otimes I^{(2)}\right)|\omega_1\rangle \otimes |\omega_2\rangle$$ ✓ (By the given commutation relation)

(4)

$$= \left(\Omega_1^{(1)\times(2)} + \Omega_2^{(1)\times(2)}\right)^2$$

$$= \left(\Omega^{2(1)\times(2)} + \Omega_1^{(1)\times(2)}\Omega_2^{(1)\times(2)} + \Omega_2^{(1)\times(2)}\Omega_1^{(1)\times(2)} - \Omega_2^{2(1)\times(2)}\right)|\omega_1\rangle \otimes |\omega_2\rangle$$

$$= \left(\Omega^2\right)^{(1)}|\omega_1\rangle \otimes I^{(2)}|\omega_2\rangle + \Omega_1^{(1)}|\omega_1\rangle \otimes \Omega_2^{(2)}|\omega_2\rangle + \Omega_1^{(1)}|\omega_1\rangle \otimes \Omega_2^{(2)}|\omega_2\rangle + I^{(1)}|\omega_1\rangle \otimes \left(\Omega_2^2\right)^{(2)}$$

$$= \left(\Omega_1^2\right)^{(1)} \otimes I^{(2)} + I^{(1)} \otimes \left(\Omega_2^2\right)^{(2)} + 2\,\Omega_1^{(1)} \otimes \Omega_2^{(2)}$$ ✓

Homework Set #5

S4.2.3

$$\psi'(x) = e^{i/\hbar p_0 x}\,\psi(x)$$

$$\langle \psi' | P | \psi' \rangle = \int dx\, \psi'(x)^* \frac{\hbar}{i}\frac{d}{dx}\,\psi'(x)$$

$$= \int dx\, \psi'(x)^* \frac{\hbar}{i}\frac{d}{dx}\, e^{i/\hbar p_0 x}\,\psi(x)$$

$$= \int dx\, \psi'(x)^*\, e^{i/\hbar p_0 x}\left(p_0 + \frac{\hbar}{i}\frac{d}{dx}\right)\psi(x)$$

$$= \int dx\, \psi(x)^* \left(p_0 + \frac{\hbar}{i}\frac{d}{dx}\right)\psi(x)$$

$$= p_0 + \langle \psi | P | \psi \rangle$$

S 5.1.1

$$1 = \int_{-\infty}^{+\infty} dp \; |p><p|$$

$$= \int_{-\infty}^{0} dp \; \left|E = \frac{p^2}{2m}, -\right\rangle\left\langle E = \frac{p^2}{2m}, -\right|$$

$$+ \int_{0}^{\infty} dp \; \left|E = \frac{p^2}{2m}, +\right\rangle\left\langle E = \frac{p^2}{2m}, +\right|$$

$$= \int_{0}^{\infty} dE \left|\frac{dp}{dE}\right| \sum_{\epsilon=\pm} |E,\epsilon><E,\epsilon|$$

$$= \sum_{\epsilon=\pm} \int_{0}^{\infty} dE \sqrt{\frac{m}{2E}} \; |E,\epsilon><E,\epsilon|$$

§5.1.2 $\langle x|H|E\rangle = E\langle x|E\rangle$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) = E\,\psi_E(x)$$

The most general solution is

$$\psi_E(x) = A_+ e^{+\frac{i}{\hbar}\sqrt{2mE}\,x} + A_- e^{-\frac{i}{\hbar}\sqrt{2mE}\,x}$$

$$\therefore \langle x|E\rangle = \beta\langle x|E,+\rangle + \gamma\langle x|E,-\rangle$$

$$\beta = A_+\sqrt{2\pi\hbar}$$

$$\gamma = A_-\sqrt{2\pi\hbar}$$

$$\therefore |E\rangle = \beta|E,+\rangle + \gamma|E,-\rangle$$

§5.1.3

$$U(t)\,\psi(x,0) = \exp\left(\frac{i}{\hbar}t\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\frac{1}{\pi^{1/4}}e^{-x^2/2}$$

$$= \frac{1}{\pi^{1/4}}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m!\,k!}\left(\frac{i}{\hbar}t\frac{\hbar^2}{2m}\right)^l\left(\frac{-1}{2}\right)^k\frac{d^{2l}}{dx^{2l}}x^{2k}$$

$$\frac{d^p}{dx^p}x^q = 0 \quad \text{if } p > q$$

$$= q(q-1)\cdots(q-p+1)\,x^{q-p} \quad \text{if } p \le q$$

$$= \frac{1}{\pi^{1/4}}\sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\frac{1}{l!\,k!}\left(\frac{itk}{2m}\right)^l\left(\frac{-1}{2}\right)^k\frac{(2k)!}{(2k-2l)!}x^{2(k-l)}$$

$$m = k-l \qquad k = l+m$$

$$= \frac{1}{\pi^{1/4}}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(2l+2m)!}{l!\,(l+m)!\,(2m)!}\left(\frac{-1}{2}\right)^m\left(\frac{-it\hbar}{4m}\right)^l x^{2m}$$

$$\frac{(2p)!}{p!} = \frac{(2p)(2p-1)(2p-2)\cdots 2\cdot 1}{p\cdot(p-1)\cdots 1}$$

$$= 2^p (2p-1)(2p-3)\cdots 1$$

$$= 2^{2p}\left(p-\tfrac{1}{2}\right)\left(p-\tfrac{3}{2}\right)\cdots \tfrac{3}{2}\cdot\tfrac{1}{2}$$

$$U(t)\psi(x,0) = \frac{1}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}\left(-\frac{1}{2}\right)^n \times$$

$$\times \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{-i\hbar t}{4m}\right)^l 2^{2(l+n)}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\left(l+n-\frac{1}{2}\right)$$

$$= \frac{1}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{1}{n!\,2^{2n}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n-\frac{1}{2}\right)}x^{2n}\left(\frac{-1}{2}\right)^n 2^{2n}$$

$$\times \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{i\hbar t}{m}\right)^l \frac{1}{2}\cdot\frac{3}{2}\cdots\left(l+n-\tfrac{1}{2}\right)$$

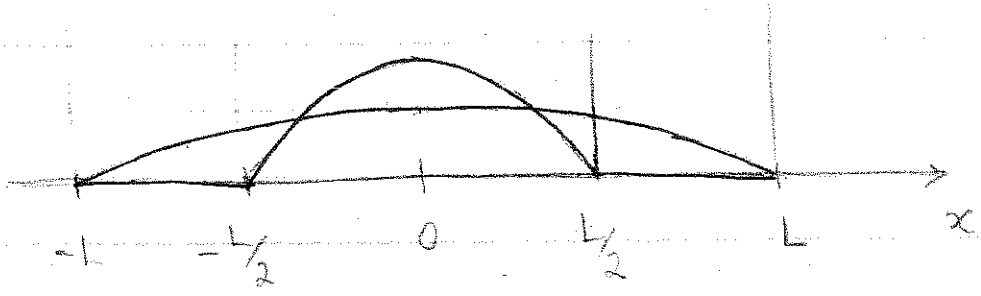
$$U(t)\psi(x,0) = \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \left(\frac{-1}{2}\right)^n \cdot$$

$$\left[1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left(\frac{-i\cdot t\hbar}{m}\right)^l \left(n+\tfrac{1}{2}\right)\left(n+\tfrac{3}{2}\right)\cdots\left(l+n-\tfrac{1}{2}\right)\right]$$

$$= \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \left(\frac{-1}{2}\right)^n \left(1+\frac{it\hbar}{m}\right)^{-n-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\pi}\left(1+\frac{it\hbar}{m}\right)^{1/2}} \exp\left[\frac{-1}{2}\cdot\frac{x^2}{\left(1+\frac{it\hbar}{m}\right)}\right]$$

S.B.2.1



$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right) \quad \text{for } |x| \leq \frac{L}{2}$$

$$= 0 \quad \text{for } \frac{L}{2} \leq |x| \leq L$$

$$\psi'(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{\pi}{2L}(x + L)\right)$$

$$P = |\langle \psi' | \psi \rangle|^2 = \left| \int_{-L}^{+L} dx \; \psi'(x)^* \, \psi(x) \right|^2$$

$$= \left(\sqrt{\frac{2}{L}} \int_{-L/2}^{L/2} dx \; \sin\left(\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right) \sin\left(\frac{\pi}{2L}(x + L)\right) \right)^2$$

$$= \left(\frac{\sqrt{2}}{L} \int_{0}^{L} dx' \; \sin\left(\frac{\pi x'}{L}\right) \sin \frac{\pi}{2L}\left(x' + \frac{L}{2}\right) \right)^2$$

$$\phi = \left(\frac{\sqrt{2}}{L} \int_0^{\pi} d\theta \; \frac{L}{\pi} \sin\theta \; \sin\left(\frac{1}{2}\theta + \frac{\pi}{4}\right) \right)^2$$

$$\theta = \frac{\pi x'}{L}$$

$$\phi = \left(\frac{\sqrt{2}}{\pi} \int_0^{\pi} d\theta \; \frac{1}{2}\left[\cos\left(\frac{1}{2}\theta - \frac{\pi}{4}\right) - \cos\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) \right] \right)^2$$

$$= \left(\frac{1}{\sqrt{2}\pi} \left[2\sin\left(\frac{1}{2}\theta - \frac{\pi}{4}\right) - \frac{2}{3}\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) \right]_0^{\pi} \right)$$

$$= \left[\frac{1}{\sqrt{2}\pi} \left(2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \frac{2}{3}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \right) \right]^2$$

$$= \left[\frac{2}{\pi}\left(1 + \frac{1}{3}\right) \right]^2 = \left(\frac{8}{3\pi}\right)^2$$

S5.2.2 a

$$H|E_n\rangle = E_n|E_n\rangle$$

E_0 is the lowest energy eigenvalue

$$\langle E_n|E_{n'}\rangle = \delta_{nn'}$$

$$\sum_{n=0}^{\infty} |E_n\rangle\langle E_{n'}| = 1$$

$$H = H\sum_{n=0}^{\infty} |E_n\rangle\langle E_n|$$

$$= \sum_{n=0}^{\infty} E_n |E_n\rangle\langle E_n|$$

$$\langle\psi|H|\psi\rangle = \sum_{n=0}^{\infty} E_n \langle\psi|E_n\rangle\langle E_n|\psi\rangle$$

$$= \sum_{n=0}^{\infty} E_n |\langle\psi|E_n\rangle|^2$$

$$= E_0 + \sum_{n=0}^{\infty} (E_n - E_0)|\langle\psi|E_n\rangle|^2$$

≥ 0 $\quad$ ≥ 0

since $\sum_{n=0}^{\infty} |\langle\psi|E_n\rangle|^2 = \langle\psi|\psi\rangle = 1$

Note: $\langle\psi|H|\psi\rangle = E_0$ only if $|\psi\rangle = |E_0\rangle$

b.

$$\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

$$\int_{-\infty}^{+\infty} dx \, |\psi_\alpha(x)|^2 = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} \, dx = 1$$

$$V(x) = -|V(x)| \quad \text{for all } x$$

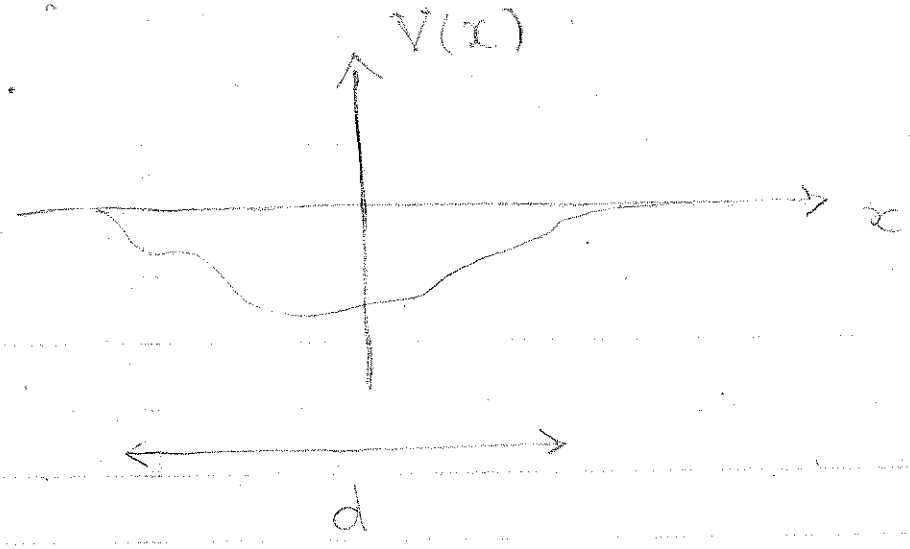
$$E_\alpha = \langle \psi_\alpha | H | \psi_\alpha \rangle$$

$$= \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2/2} \left(-\frac{k^2}{2m} \frac{d^2}{dx^2} - |V(x)| \right) e^{-\alpha x^2/2}$$

$$= \frac{1}{2m} \langle \psi_\alpha | P^2 | \psi_\alpha \rangle - \langle \psi_\alpha | |V(x)| | \psi_\alpha \rangle$$

$$\langle \psi_\alpha | P^2 | \psi_\alpha \rangle = \frac{k^2}{2\Delta^2} \Big|_{\Delta = \frac{1}{\sqrt{\alpha}}} = \frac{\hbar^2 \alpha}{2}$$

since $\psi_\alpha(x)$ is a Gaussian



$$\langle \psi_\alpha | V(x) | \psi_\alpha \rangle = \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\, V(x)$$

$$\to \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx\, V(x) \qquad \text{for} \quad \alpha^{-1} \gg d$$

$$\alpha \ll d^{-1}$$

$$= -\sqrt{\alpha}\, C \qquad \text{with}$$

$$C = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx\, |V(x)| \,, \qquad C < 0 \,,$$

$$E_\alpha \to \frac{\hbar^2 \alpha}{4m} - \sqrt{\alpha}\, C < 0 \qquad \text{for } \alpha \text{ sufficiently small}$$

$$E_0 \leq \langle \psi_\alpha | H | \psi_\alpha \rangle$$

$$\therefore \; E_0 < 0$$

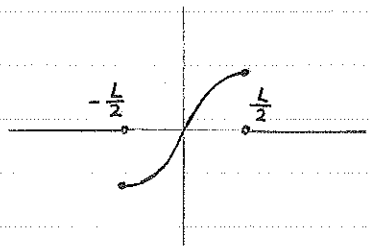
$\therefore$ there is at least one bound state.

Homework 6

Shanshan Cao

5.1.4

The answer is already given in Shankar. Moreover, it is important to realize that this cannot be a solution of Schrödinger Equation. Solutions of Schrödinger Equation should satisfy the following:



① the wavefunction is continuous;

② the first order derivative of the wavefunction is usually continuous for finite potentials.

Therefore, the given wavefunction is not physical (not continuous at $x = \pm \frac{L}{2}$.

5.2.1

Initial state: $\psi_i(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \qquad \left(-\frac{L}{2} \le x \le \frac{L}{2}\right)$

It is undisturbed under a sudden expansion of the box from L to 2L.

New ground state: $\phi_1(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right) \qquad (-L \le x \le L)$

The probability of finding the particle in $\phi_1(x)$ is

$$P = |\langle \phi_1 | \psi_i \rangle|^2$$

$$\langle \phi_1 | \psi_i \rangle = \int_{-\infty}^{\infty} \phi_1^*(x)\, \psi_i(x)\, dx$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{\frac{1}{L}} \cdot \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{8}{3\pi}.$$

Therefore, $P = \left(\frac{8}{3\pi}\right)^2$.

5.2.2

(a) $|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle$, where ϕ_i is an eigenstate of H with eigenvalue E_i.

$$\langle \psi | H | \psi \rangle = \sum_i \sum_j \alpha_i^* \alpha_j \langle \phi_i^* | H | \phi_j \rangle = \sum_i \sum_j \alpha_i^* \alpha_j E_j \langle \phi_i^* | \phi_j \rangle$$

$$= \sum_i \sum_j \alpha_i^* \alpha_j E_j \delta_{ij} = \sum_i |\alpha_i|^2 E_i \ge E_0 \sum_i |\alpha_i|^2 = E_0$$

Therefore, $\langle \psi | H | \psi \rangle \ge E_0$ 76

(b) Consider $\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2/2}$

$$\frac{d}{dx}\psi_\alpha(x) = -\frac{\alpha}{2}\cdot 2x \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2/2} = -\alpha x \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2/2}$$

$$\frac{d^2}{dx^2}\psi_\alpha(x) = -\alpha\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2/2} + \alpha^2 x^2 \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2/2}$$

$$\langle E(\alpha)\rangle = \langle \psi_\alpha(x)| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - |V(x)|\,|\psi_\alpha(x)\rangle$$

$$= \langle \psi_\alpha(x)| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi_\alpha(x)\rangle - \langle \psi_\alpha(x)|\,|V(x)|\,|\psi_\alpha(x)\rangle$$

$$= \frac{\hbar^2}{2m}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\alpha e^{-\alpha x^2}dx - \int_{-\infty}^{\infty}\alpha^2 x^2 e^{-\alpha x^2}dx\right) - \int_{-\infty}^{\infty}|V(x)|\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha x^2}dx$$

$$= \frac{\hbar^2}{2m}\left(\frac{\alpha}{2}\right) - \int_{-\infty}^{\infty}|V(x)|\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha x^2}dx$$

Try to find the minimum value for $\langle E(\alpha)\rangle$.

$$\frac{d}{d\alpha}\langle E(\alpha)\rangle = 0$$

$$\Rightarrow \frac{\hbar^2}{4m} - \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}|V(x)|\left(\frac{1}{2\alpha} - x^2\right)e^{-\alpha x^2}dx = 0$$

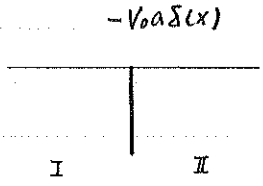
$$\Rightarrow \alpha = \frac{\int_{-\infty}^{\infty}|V(x)|\,\psi_\alpha^2(x)\,dx}{\frac{\hbar^2}{2m} + 2\int_{-\infty}^{\infty}|V(x)|\,x^2\psi_\alpha^2(x)dx}$$

$$\langle E(\alpha)\rangle = \frac{\int_{-\infty}^{\infty}|V(x)|\,\psi_\alpha^2(x)\,dx}{2 + \frac{8m}{\hbar^2}\int_{-\infty}^{\infty}|V(x)|\,x^2\psi_\alpha^2(x)\,dx} - \int_{-\infty}^{\infty}|V(x)|\,\psi_\alpha^2(x)\,dx < 0$$

According to part (a), $E_0 \le \langle E(\alpha)\rangle < 0$. Therefore, there exists at least a bound state with $E < 0$.

5.2.3

* Boundary condition for δ - potential.



$$\int_{-\epsilon}^{\epsilon}\psi''dx = \psi'(x)\Big|_{-\epsilon}^{\epsilon} = \psi_{II}'(\epsilon) - \psi_I'(-\epsilon) \quad (1)$$

Meanwhile,

since $\psi'' + \frac{2m}{\hbar^2}[E + V_0 a\delta(x)]\psi = 0$

$$\int_{-\epsilon}^{\epsilon}\psi''dx = -\int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}[E + V_0 a\delta(x)]\psi\,dx$$

$$= -\int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}E\psi\,dx - \int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}V_0 a\delta(x)\psi\,dx$$

$$= 0 - \frac{2m}{\hbar^2}V_0 a\psi(x)\Big|_{-\epsilon}^{\epsilon} \quad (2)$$

↑ $\psi(x)$ is continuous ↑ δ-function integral

Take the limit $\epsilon \to 0$

(1) → $\lim_{\epsilon\to 0}\int\psi''dx = \psi_{II}'(0) - \psi_I'(0)$

(2) → $\lim_{\epsilon\to 0}\int\psi''dx = -\frac{2m}{\hbar^2}V_0 a\psi(0)$

Therefore, the boundary conditions for δ - potential are:

$$\begin{cases} \psi_{II}(0) = \psi_I(0) & (3) \\ \psi_{II}'(0) - \psi_I'(0) = -\frac{2m}{\hbar^2}V_0 a\psi(0) & (4) \end{cases}$$

Let's solve the problem 5.2.3.

I. $\psi'' + \frac{2m}{\hbar^2}E\psi = 0$, set $-\frac{2m}{\hbar^2}E = k^2$ $(E<0)$

$\Rightarrow \psi = Ae^{kx} + Be^{-kx}$

Since $\psi \to 0$ as $x \to -\infty$, B should be 0.

Therefore, $\psi_I(x) = Ae^{kx}$ (5)

II. $\psi'' + \frac{2m}{\hbar^2}E\psi = 0$, set $-\frac{2m}{\hbar^2}E = k^2$ $(E<0)$

$\Rightarrow \psi = Ce^{kx} + De^{-kx}$

Since $\psi \to 0$ as $x \to \infty$, C should be 0.

Therefore, $\psi_{II}(x) = De^{-kx}$ (6)

Substitute (5) and (6) into the boundary conditions (3) and (4), we have,

$$\begin{cases} D = A \\ -kD - kA = -\frac{2m}{\hbar^2}V_0 aD \end{cases}$$

$\Rightarrow k = \frac{m}{\hbar^2}V_0 a$

$\Rightarrow E = -\frac{\hbar^2}{2m}k^2 = -\frac{ma^2V_0^2}{2\hbar^2}$.

This is the only solution for E. (Only one bound state.)

5.2.4

The energy of a particle in a box is given by

$$E_n = \frac{\hbar^2\pi^2n^2}{2mL^2}$$

$$F = -\frac{\partial E_n}{\partial L} = \frac{\hbar^2\pi^2n^2}{mL^3} = \frac{2E_n}{L}.$$

From the classical point of view, when a particle hits the wall,

$\Delta P = P - (-P) = 2P$ (change of the momentum).

The average time inteval between the particle hits the same wall twice:

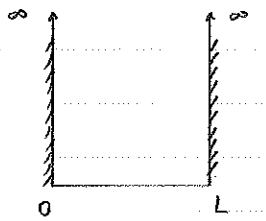
$\overline{\Delta t} = \frac{2L}{v}$.

Considering $E = \frac{P^2}{2m} = \frac{1}{2}mv^2$, and $\overline{F} = \frac{\Delta P}{\overline{\Delta t}}$, we have

$\overline{F} = \frac{Pv}{L} = \frac{2E}{L}$.

This agrees with the quantum result above.

5.2.5



Between 0 and L, we have:

$$\psi'' + \frac{2m}{\hbar^2}E\psi = 0.$$

Set $k^2 = \frac{2m}{\hbar^2}E$,

$$\psi'' + k^2\psi = 0,$$

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Boundary conditions:

$$\psi(0) = 0 \Rightarrow B = 0$$

$$\psi(L) = 0 \Rightarrow A\sin(kL) = 0 \Rightarrow kL = n\pi \Rightarrow k = \frac{n\pi}{L},\quad n = 1,2,3,\cdots$$

Therefore,

$$E_n = \frac{\hbar^2k^2}{2m} = \frac{n^2\hbar^2\pi^2}{2mL^2}.$$

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right).$$

Normalization:

$$1 = \int_0^L \psi^*(x)\psi(x)\,dx = A^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = A^2\cdot\frac{L}{2}$$

$$\Rightarrow A = \sqrt{\frac{2}{L}}.$$

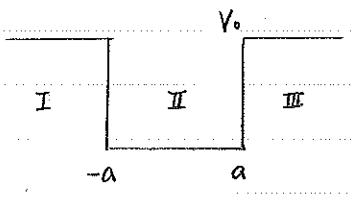
$$\Rightarrow \psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right).$$

5.2.6

(1) Ⅱ: $\psi'' + \frac{2m}{\hbar^2}E\psi = 0$

$$k^2 = \frac{2m}{\hbar^2}E,$$

$$\Rightarrow \psi_{Ⅱ}(x) = A\cos kx + B\sin kx$$



Ⅰ and Ⅲ: $\psi'' + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$

$$k^2 = -\frac{2m}{\hbar^2}(E - V_0)$$

$$\Rightarrow \psi(x) = Ce^{kx} + De^{-kx}$$

Since $\psi_Ⅰ(x) \to 0$ as $x \to -\infty$; $\psi_Ⅲ(x) \to 0$ as $x \to \infty$, we have

$$\psi_Ⅰ(x) = Ce^{kx},\quad \psi_Ⅲ(x) = De^{-kx}.$$

* For even solution:

$$\begin{cases} \psi_Ⅱ(x) = A\cos kx \\ \psi_Ⅰ(x) = De^{-kx} \end{cases}$$

Boundary conditions:

$$\begin{cases} \psi_{II}(a) = \psi_{III}(a) \\ \psi_{II}'(a) = \psi_{III}'(a) \end{cases} \Rightarrow \begin{cases} A\cos ka = De^{-ka} \\ -kA\sin ka = -kDe^{-ka} \end{cases}$$

$$\Rightarrow k\tan ka = k.$$

* For odd solution:

$$\begin{cases} \psi_{II}(x) = B\sin kx \\ \psi_{III}(x) = De^{-kx} \end{cases}$$

Boundary conditions:

$$\begin{cases} \psi_{II}(a) = \psi_{III}(a) \\ \psi_{II}'(a) = \psi_{III}'(a) \end{cases} \Rightarrow \begin{cases} B\sin ka = De^{-ka} \\ kB\cos ka = -kDe^{-ka} \end{cases}$$

$$\Rightarrow k\cot ka = -k.$$

$$k^2 + k^2 = \frac{2m}{\hbar^2}E - \frac{2m}{\hbar^2}(E - V_0) = \frac{2m}{\hbar^2}V_0$$

* When $V_0 \to \infty$, $k \to \infty$ while k keeps the same.

Therefore, $\tan ka$, $\cot ka \to \infty$

$ka = \frac{n\pi}{2}$, $n = 1, 2, 3 \cdots$

Thus, $E = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}\cdot\frac{n^2\pi^2}{4a^2} = \frac{n^2\pi^2\hbar^2}{8ma^2}$. This is the energy level in the box with length $2a$.

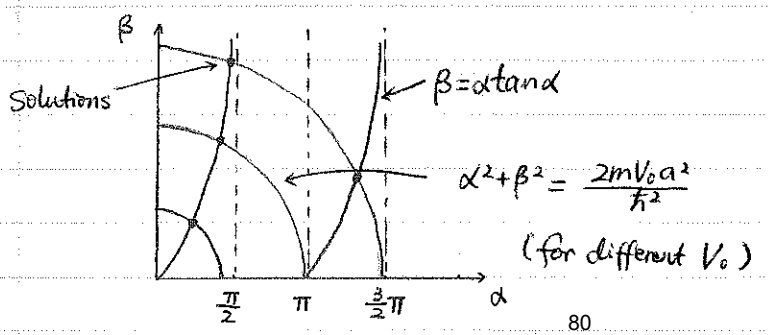
(2)

$\beta = ka$, $\alpha = ka$

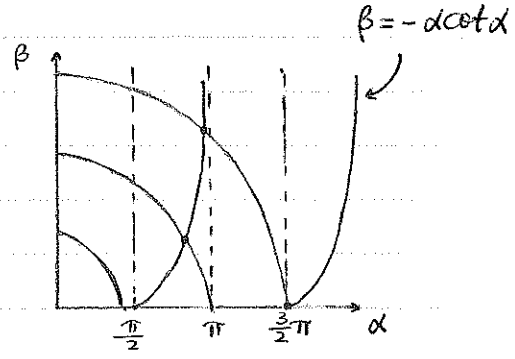
$\Rightarrow \beta = \alpha\tan\alpha$ for even solution;

$\Rightarrow \beta = -\alpha\cot\alpha$ for odd solution.

$$\alpha^2 + \beta^2 = a^2(k^2 + k^2) = \frac{2mV_0a^2}{\hbar^2}.$$



Even Solutions



Odd Solutions

(3) From part (2), we see that there always exists at least one intersection between $\alpha^2+\beta^2 = \frac{2mV_0a^2}{\hbar^2}$ and $\beta = \alpha\tan\alpha$, no matter how small V_0 is. Therefore, there is always one even solution. This accords with Exercise 5.2.2 b.

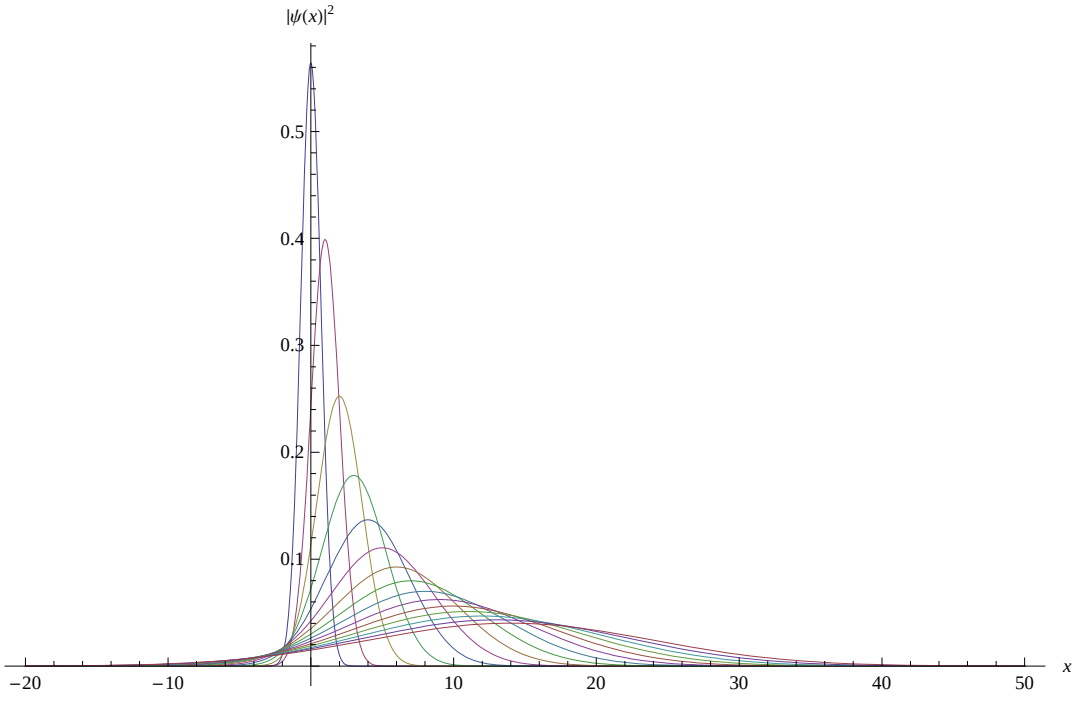
However, we cannot find an intersection between $\alpha^2+\beta^2 = \frac{2mV_0a^2}{\hbar^2}$ and $\beta = -\alpha\cot\alpha$ unless $\sqrt{\frac{2mV_0a^2}{\hbar^2}} \geq \frac{\pi}{2}$. Therefore, there is no odd solution unless $V_0 \geq \hbar^2\pi^2/8ma^2$. When V_0 just meets this requirement, we have the intersection $(\alpha,\beta) = (\frac{\pi}{2}, 0)$. Hence,

$$0 = \beta = ka = \sqrt{-\frac{2m}{\hbar^2}(E-V_0)}\cdot a$$
$$\Rightarrow E = V_0 = \frac{\hbar^2\pi^2}{8ma^2}.$$

Problem 2

In[3]:=
```
Psi[x_, t_] := (Pi (Δ^2 + hbar^2 t^2 / (m^2 Δ^2)))^(-1 / 2)
  Exp[(-(x - (p0 / m) t)^2) / (Δ^2 + hbar^2 t^2 / (m^2 Δ^2))];
Plot[Evaluate[Table[Psi[x, t] /. {p0 → 1, Δ → 1, hbar → 1, m → 1},
  {t, 0, 14}]], {x, -20, 50},
 AxesLabel → {"x", "|ψ(x)|^2"}, PlotRange → All]
```

Out[4]=



Problem Set 7

Problem 1: Exercise 5.3.1

The Hamiltonian is

$$H = \frac{1}{2m}P^2 + V_r(X) - iV_i$$

where V_r is a real function and V_i a real constant. Therefore

$$H^\dagger = \frac{1}{2m}(P^\dagger)^2 + V_r(X^\dagger) - (i)^* V_i = \frac{1}{2m}P^2 + V_r(X) + iV_i \neq H,$$

so H is *not* Hermitian.

Derivation of the continuity equation. Schrodinger's equation and its complex conjugate in this case read

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\psi + V_r\psi - iV_i\psi, \\ -i\hbar\frac{\partial\psi^*}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\psi^* + V_r\psi^* + iV_i\psi^*. \end{aligned}$$

Multiplying the first by ψ^* and teh second by ψ and taking the difference, then dividing by $i\hbar$ gives

$$\frac{\partial P}{\partial t} = -\vec{\nabla}\cdot\vec{j} - \frac{2}{\hbar}V_i P,$$

where, as before, $P = |\psi|^2$ and $\vec{j} = \hbar(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla})/(2mi)$ are the probability density and current, respectively. Integrating this over all space, the $\vec{\nabla}\cdot\vec{j}$ term vanishes (by the divergence theorem, since we assume $\vec{j} \to 0$ at infinity), giving

$$\frac{d\mathcal{P}}{dt} = -\frac{2}{\hbar}V_i\,\mathcal{P},$$

where $\mathcal{P} = \int d^3x P$ is the total probability. (I can pull V_i out of the integral since it is assumed constant in the problem.) Integrating this differential equation gives

$$\mathcal{P}(t) = \mathcal{P}(0)\,e^{-2V_i t/\hbar}.$$

Problem 2: Exercise 5.3.4

With primes denoting derivative with respect to x,

$$\begin{aligned} j &:= \frac{\hbar}{2mi}\left[\psi^*\psi' - \psi(\psi^*)'\right] \\ &= \frac{\hbar}{2mi}\big[(A^*e^{-ixp/\hbar} + B^*e^{ixp/\hbar})(Ae^{ixp/\hbar} + Be^{-ixp/\hbar})' \\ &\qquad - (Ae^{ixp/\hbar} + Be^{-ixp/\hbar})(A^*e^{-ixp/\hbar} + B^*e^{ixp/\hbar})'\big] \\ &= \frac{1}{2mi}\big[(A^*e^{-ixp/\hbar} + B^*e^{ixp/\hbar})(ipAe^{ixp/\hbar} - ipBe^{-ixp/\hbar}) \\ &\qquad - (Ae^{ixp/\hbar} + Be^{-ixp/\hbar})(-ipA^*e^{-ixp/\hbar} + ipB^*e^{ixp/\hbar})\big] \\ &= \frac{p}{2m}\big[|A|^2 + AB^*e^{2ixp/\hbar} - A^*Be^{-2ixp/\hbar} - |B|^2 \\ &\qquad + |A|^2 + A^*Be^{-2ixp/\hbar} - AB^*e^{2ixp/\hbar} - |B|^2\big] \\ &= \frac{p}{m}\big[|A|^2 - |B|^2\big]. \end{aligned}$$

Problem 3: Exercise 5.4.2

(a) For $x < 0$, $V = 0$, so the general solution of the energy eigenstate equation is $\psi_< = Ae^{ikx} + Be^{-ikx}$ where $\hbar k = \sqrt{2mE}$. Similarly, for $x > 0$, $\psi_> = Ce^{ikx} + De^{-ikx}$. Scattering boundary conditions means we set $D = 0$ (no incoming particles from $x = +\infty$).

Now we need to figure out the boundary conditions at $x = 0$. Look at the time-independent Schrodinger equation,

$$-\frac{\hbar^2}{2m}\psi'' + V_0 a\delta(x)\psi = E\psi. \tag{1}$$

Since the potential has an infinite jump in it, ψ will be continuous, but ψ' may have a finite jump. To see how big the ψ' jump is, integrate (1) from $x = -\epsilon$ to $x = +\epsilon$ to get

$$\frac{\hbar^2}{2m}\left[\psi'(-\epsilon) - \psi'(\epsilon)\right] + V_0 a\psi(0) = E\int_{-\epsilon}^{\epsilon} dx\,\psi.$$

In the limit as $\epsilon \to 0$, the right hand side vanishes since ψ is continuous, from which we learn that

$$\psi'_>(0) - \psi'_<(0) = (2maV_0/\hbar^2)\psi(0).$$

Applying this boundary condition along with continuity of ψ to $\psi_<$ and $\psi_>$ gives the two conditions

$$\begin{aligned} A + B &= C \\ ikC - ikA + ikB &= (2maV_0/\hbar^2)C. \end{aligned}$$

Dividing through by A, and solving for B/A and C/A gives $B/A = maV_0/(ik\hbar^2 - maV_0)$ and $C/A = ik\hbar^2/(ik\hbar^2 - maV_0)$. Since $R = |B/A|^2$ and $T = |C/A|^2$, we get

$$R = \frac{m^2a^2V_0^2}{k^2\hbar^4 + m^2a^2V_0^2}, \qquad T = \frac{k^2\hbar^4}{k^2\hbar^4 + m^2a^2V_0^2}.$$

(b) Call $x < -a$ region I, $|x| < a$ region II, and $x > a$ region III. Then solving for the energy eigenstates, $\hbar\psi'' = 2m(E - V(x))\psi$, of energy $0 < E \leq V_0$ in each region gives $\psi_I = Ae^{ikx} + Be^{-ikx}$, $\psi_{II} = Ce^{-\kappa x} + De^{\kappa x}$, and $\psi_{III} = Ee^{ikx} + Fe^{-ikx}$, with $\hbar\kappa = \sqrt{2m(V_0 - E)}$ and $\hbar k = \sqrt{2mE}$. Scattering boundary conditions means we set $F = 0$ (no incoming particles from $x = +\infty$). The incoming wave has amplitude A, the reflected has amplitude B, the transmitted amplitude E. Therefore $R = |B/A|^2$ and $T = |E/A|^2$, so we only need to solve for B/A and E/A.

The boundary conditions at $x = \pm a$ are that ψ and ψ' are continuous, implying

$$\begin{aligned} Ae^{-ika} + Be^{ika} &= Ce^{\kappa a} + De^{-\kappa a}, \\ ikAe^{-ika} - ikBe^{ika} &= -\kappa Ce^{\kappa a} + \kappa De^{-\kappa a}, \\ Ee^{ika} &= Ce^{-\kappa a} + De^{\kappa a}, \\ ikEe^{ika} &= -\kappa Ce^{-\kappa a} + \kappa De^{\kappa a}. \end{aligned}$$

Dividing by A and eliminating C/A and D/A gives

$$\begin{aligned} \frac{B}{A} &= \frac{e^{-2iak}(e^{4a\kappa} - 1)(k^2 + \kappa^2)}{(e^{4a\kappa} - 1)(k^2 - \kappa^2) + 2i(e^{4a\kappa} + 1)k\kappa}, \\ \frac{E}{A} &= \frac{4ie^{2a(\kappa - ik)}k\kappa}{(e^{4a\kappa} - 1)(k^2 - \kappa^2) + 2i(e^{4a\kappa} + 1)k\kappa}, \end{aligned}$$

so

$$R = \frac{(e^{4a\kappa} - 1)^2(k^2 + \kappa^2)^2}{(e^{4a\kappa} - 1)^2(k^2 - \kappa^2)^2 + 4(e^{4a\kappa} + 1)^2k^2\kappa^2},$$

and $T = 1 - R$.

Physics 711 January 15, 2010

Problem Set 8

Problem 1: Exercise 7.3.1

Plug the power series expansion $\psi = \sum_{n=0}^{\infty} c_n y^n$ into the equation $\psi'' + (2\varepsilon - y^2)\psi = 0$ to get

$$\sum_{n=0}^{\infty} c_n \left[n(n-1)y^{n-2} + (2\varepsilon - y^2)y^n \right] = 0.$$

Shift $n \to n+2$ in the first term, and $n \to n-2$ in the third term to get

$$\sum_{n=0}^{\infty} y^n \left[(n+2)(n+1)c_{n+2} + 2\varepsilon c_n - c_{n-2} \right] = 0$$

with the convention that

$$c_{-2} = c_{-1} = 0.$$

This implies

$$c_{n+2} = \frac{c_{n-2}}{(n+1)(n+2)} - \frac{2\varepsilon c_n}{(n+1)(n+2)}$$

for all $n \geq 0$.

Problem 3: Exercise 7.3.7

In the momentum basis $|\psi\rangle \to \psi(p)$, $P \to p$, and $X \to i\hbar(d/dp)$, so the energy eigenvalue equation

$$\left(\frac{1}{2m} P^2 + \frac{m\omega^2}{2} X^2 \right) |E\rangle = E|E\rangle$$

becomes

$$\frac{1}{2m} p^2 \psi(p) - \frac{m\omega^2}{2} \hbar^2 \psi''(p) = E\psi(p).$$

Compare this to the position-basis equation

$$\frac{m\omega^2}{2} x^2 \psi(x) - \frac{1}{2m} \hbar^2 \psi''(x) = E\psi(x).$$

These are the same equations with the substitutions $x \leftrightarrow p$ and $m \leftrightarrow 1/(m\omega^2)$.

Problem 2: Exercise 7.3.5

$$\begin{aligned}
\langle n|X|n\rangle &= \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\cdot x\cdot\psi_n(x) \;=\; \int_{-\infty}^{\infty} dx\, x\cdot\psi_n^2(x) \quad \text{since } \psi_n(x) \text{ is real} \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } x \text{ is odd and } \psi_n^2(x) \text{ is even.}
\end{aligned}$$

$$\begin{aligned}
\langle n|P|n\rangle &= \int_{-\infty}^{\infty} dx\, \psi_n^*(x)(-i\hbar)\frac{d}{dx}\psi_n(x) \;=\; (-i\hbar)\int_{-\infty}^{\infty} dx\, \psi_n\psi_n' \;=\; -\frac{i\hbar}{2}\int_{-\infty}^{\infty} dx\, (\psi_n^2)' \\
&= -\frac{i\hbar}{2}\,\psi_n^2\big|_{-\infty}^{\infty} \;=\; 0 \qquad\qquad\qquad\qquad \text{since } \psi_n \to 0 \text{ as } |x| \to \infty.
\end{aligned}$$

$$\begin{aligned}
\langle 1|X^2|1\rangle &= \int_{-\infty}^{\infty} dx\, \psi_1^* x^2 \psi_1 \;=\; \left(\frac{m\omega}{4\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx\, x^2\left[2x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]^2 e^{-m\omega x^2/\hbar} \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, x^4 e^{-m\omega x^2/\hbar} \;=\; \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\frac{3\sqrt{\pi}}{4}\left(\frac{m\omega}{\hbar}\right)^{-5/2} \\
&= \frac{3\hbar}{2m\omega}.
\end{aligned}$$

$$\begin{aligned}
\langle 1|P^2|1\rangle &= \int_{-\infty}^{\infty} dx\, \psi_1^*(-i\hbar)^2\psi_1'' \;=\; -\hbar^2\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, xe^{-m\omega x^2/2\hbar}\left(xe^{-m\omega x^2/2\hbar}\right)'' \\
&= -\hbar^2\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, x^2\left(\frac{m\omega}{\hbar}\right)\left[\left(\frac{m\omega}{\hbar}\right)x^2 - 3\right]e^{-m\omega x^2/\hbar} \\
&= -\hbar^2\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{5/2}\cdot(-1)\frac{3\sqrt{\pi}}{4}\left(\frac{m\omega}{\hbar}\right)^{-3/2} \;=\; \frac{3m\omega\hbar}{2}.
\end{aligned}$$

$$\Delta X^2 \;=\; \langle 0|X^2|0\rangle \;=\; \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} dx\, x^2 e^{-m\omega x^2/\hbar} \;=\; \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{\sqrt{\pi}}{2}\left(\frac{m\omega}{\hbar}\right)^{-3/2} \;=\; \frac{\hbar}{2m\omega}.$$

$$\begin{aligned}
\Delta P^2 \;=\; \langle 0|P^2|0\rangle &= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-m\omega x^2/2\hbar}\left(e^{-m\omega x^2/2\hbar}\right)'' \\
&= -\frac{\hbar^2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, \left(\frac{m\omega}{\hbar}x^2 - 1\right)e^{-m\omega x^2/\hbar} \\
&= -\frac{\hbar^2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\cdot(-1)\frac{\sqrt{\pi}}{2}\left(\frac{m\omega}{\hbar}\right)^{-1/2} \;=\; \frac{m\omega\hbar}{2}.
\end{aligned}$$

$$\therefore \Delta X\,\Delta P \;=\; \sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{m\omega\hbar}{2}} \;=\; \frac{\hbar}{2}.$$

PHY 6645 - Quantum Mechanics I - Fall 2011
Homework set # 8, due October 19

1. Let $H(\lambda)$ be the Hamiltonian of a system which depends explcitly upon a parameter λ. Consider the eigenvalues $E_n(\lambda)$ and eigenstates $|E_n(\lambda)>$ of $H(\lambda)$. Show that

$$< E_n(\lambda)|\frac{dH}{d\lambda}|E_n(\lambda)>=\frac{dE_n}{d\lambda} \quad , \tag{0.1}$$

for all n.

2. Show that the free particle one-dimensional Schrödinger equation is invariant with respect to Galilian transformations. Do this by showing that, when the transformation $x' = x - vt$, $t' = t$ is applied, the transformed wavefunction $\psi'(x',t') = f(x,t)\psi(x,t)$, with $f(x,t)$ a phase factor, is a solution with respect to the primed variables if $\psi(x,t)$ is a solution with respect to the unprimed ones. Find the phase factor $f(x,t)$, and show that the travelling wave solution $\psi(x,t) = Ae^{i(xt-\omega t)}$ transforms as expected.

3. Problems 7.3.1, 7.3.4 and 7.3.6 in Shankar's book.

1. $$H(\lambda)\,|E_n(\lambda)\rangle = E_n(\lambda)\,|E_n(\lambda)\rangle$$

$$\frac{d}{d\lambda}H(\lambda)\,|E_n(\lambda)\rangle + H(\lambda)\,\frac{d}{d\lambda}|E_n(\lambda)\rangle$$

$$= \frac{dE_n}{d\lambda}\,|E_n(\lambda)\rangle + E_n(\lambda)\,\frac{d}{d\lambda}\,|E_n(\lambda)\rangle$$

$$\langle E_n(\lambda)\,|\,\frac{dH(\lambda)}{d\lambda}\,|E_n(\lambda)\rangle + \langle E_n(\lambda)|H(\lambda)\frac{d}{d\lambda}\,|E_n(\lambda)\rangle$$

$$= \frac{dE_n}{d\lambda} + E_n(\lambda)\,\langle E_n(\lambda)\,|\,\frac{d}{d\lambda}\,|E_n(\lambda)\rangle$$

$$\langle E_n(\lambda)\,|\,H(\lambda) = \langle E_n(\lambda)\,|\,E_n(\lambda)$$

$$\therefore \quad \frac{dE_n}{d\lambda} = \langle E_n(\lambda)\,|\,\frac{dH}{d\lambda}\,|E_n(\lambda)\rangle$$

2.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) = +i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'(x',t')$$

$$= -\frac{\hbar^2}{2m}\left(\underbrace{\frac{\partial x}{\partial x'}}_{=1}\frac{\partial}{\partial x} + \underbrace{\frac{\partial t}{\partial x'}}_{=0}\frac{\partial}{\partial t}\right)^2 f(x,t)\,\psi(x,t)$$

since $x = x' + vt'$ $\quad t = t'$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} f(x,t)\,\psi(x,t)$$

$$i\hbar\frac{\partial}{\partial t'}\psi'(x',t')$$

$$= i\hbar\left(\frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\right) f(x,t)\,\psi(x,t)$$

$$= i\hbar\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) f(x,t)\,\psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'(x',t') = i\hbar\frac{\partial}{\partial t'}\psi'(x',t')$$

for any $\psi(x,t)$ which solves the Schrödinger equation, provided

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 f}{\partial x^2}\psi + 2\frac{\partial f}{\partial x}\frac{\partial \psi}{\partial x}\right)$$

$$= i\hbar\left[\psi\left(v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right) + vf\frac{\partial \psi}{\partial x}\right]$$

for any such ψ... We must have:

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial x^2} = i\hbar\left(v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right) & \text{and} \\ -\frac{\hbar^2}{m}\frac{\partial f}{\partial x} = i\hbar v f \end{cases}$$

$$f(x,t) = f(t)\, e^{-\frac{i}{\hbar}mvx}$$

$$\therefore \quad \frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \frac{i\hbar}{2m}\frac{\partial^2 f}{\partial x^2}$$

$$= \left[-v\left(\frac{-i}{\hbar}mv\right) + \frac{i\hbar}{2m}\left(\frac{-i}{\hbar}mv\right)^2\right] f$$

$$= \frac{i}{2\hbar}mv^2 f$$

$$f(x,t) = c\, e^{-\frac{i}{\hbar} m v x + \frac{i}{\hbar} \frac{1}{2} m v^2 t}$$

$\uparrow$

constant $c = 1$

since $f(x,t) = 1$ when $v = 0$

$$\therefore \quad \psi'(x',t') = e^{-\frac{i}{\hbar} m v x + \frac{i}{\hbar} \frac{1}{2} m v^2 t} \psi(x,t)$$

v = boost velocity

Consider $\psi(x,t) = A\, e^{i(kx - \omega t)}$

In this state, the particle has

momentum $p = \hbar k$

energy $E = \hbar \omega$

ψ satisfies the Schrodinger equation

provided $E = \frac{1}{2m} p^2$

The velocity of the particle is

$$v_p = \frac{1}{m} p \qquad E = \frac{1}{2m} v_p^2$$

$$\psi'(x',t') = A\, e^{+\frac{i}{\hbar}\left(m v_p - m v\right)x - \frac{i}{\hbar}\left(\frac{1}{2} m v_p^2 - \frac{1}{2} m v^2\right)t}$$

$$= A\, e^{i/\hbar\, m (v_p - v)(x' + v t') - \frac{i}{\hbar}\left(\frac{1}{2} m v_p^2 - \frac{1}{2} m v^2\right)t'}$$

$$= A\, e^{i/\hbar\, m (v_p - v) x' - i/\hbar\, \frac{m}{2}\left(v_p^2 + v^2 - v_p v\right)t'}$$

$$= A\, e^{i/\hbar\, m v_p' x' - i/\hbar\, m/2\, v_p'^2 t'}$$ ✓

with $v_p' = v_p - v$

S7.3.1

$$\psi'' + (2\epsilon - y^2)\psi = 0$$

$$\psi(y) = \sum_{n=0}^{\infty} A_n y^n$$

$$\sum_{n=0}^{\infty} A_n \left[n(n-1) y^{n-2} + 2\epsilon y^n - y^{n+2} \right] = 0$$

$$\sum_{n=0}^{\infty} y^n \left[A_{n+2}(n+2)(n+1) + 2\epsilon A_n - A_{n-2} \right] = 0$$

with $A_{-1} = A_{-2} = 0$

$$A_{n+2} = \frac{A_{n-2} - 2\epsilon A_n}{(n+2)(n+1)}$$

$$A_2 = \frac{-2\epsilon A_0}{2 \cdot 1}, \quad A_4 = \frac{A_0 - 2\epsilon A_2}{4 \cdot 3}$$

$$A_4 = A_0 \left(\frac{1}{4 \cdot 3} + \frac{4\epsilon^2}{4 \cdot 3 \cdot 2 \cdot 1} \right)$$

$$= A_0 \frac{1}{4 \cdot 3} \left(1 + \frac{(2\epsilon)^2}{2 \cdot 1} \right)$$

$$A_6 = \frac{1}{6.5}\left(A_2 - 2\epsilon A_4\right)$$

$$= \frac{A_0}{6.5}\left(\frac{-2\epsilon}{2.1} - 2\epsilon\left(\frac{1}{4.3} + \frac{4\epsilon^2}{4.3.2.1}\right)\right)$$

$$= \frac{A_0}{6.5}(-2\epsilon)\left(\frac{1}{2.1} + \frac{1}{4.3} + \frac{4\epsilon^2}{4.3.2.1}\right)$$

$$A_{2p} = \frac{1}{(2p)(2p-1)}\left(A_{2p-4} - 2\epsilon A_{2p-2}\right)$$

$$= \frac{1}{(2p)(2p-1)}\left[\frac{1}{(2p-4)(2p-5)}\left(A_{2p-8} - 2\epsilon A_{2p-6}\right)\right.$$

$$\left. - \frac{1}{(2p-2)(2p-3)}\left(A_{2p-6} - 2\epsilon A_{2p-4}\right)\right]$$

$$= \ldots$$

Indeed, the solution does not follow so readily.

§ 7.3.4

$$\Delta = \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

$$\langle n'|X|n\rangle = \int_{-\infty}^{+\infty} dx\; H_{n'}\left(\frac{x}{\Delta}\right) H_n\left(\frac{x}{\Delta}\right) \frac{x\, e^{-\left(\frac{x}{\Delta}\right)^2}}{\sqrt{\pi}\,\Delta\left(2^{n+n'} n!\, n'!\right)^{1/2}}$$

$$= \frac{\Delta}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\; H_{n'}(y)\, H_n(y)\, y\, e^{-y^2}$$

$$= \frac{\Delta}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\; \frac{1}{2} e^{y^2} H_{n'}(y) \left[H_{n+1}(y) + 2n\, H_{n-1}(y)\right]$$

$$= \frac{1}{2}\Delta\left[\delta_{n',n+1}\sqrt{2(n+1)} + 2n\,\delta_{n',n-1}\,\frac{1}{\sqrt{2n}}\right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\delta_{n',n+1}\sqrt{n+1} + \delta_{n',n-1}\sqrt{n}\right]$$

$$\langle n'|P|n\rangle = \int_{-\infty}^{+\infty} dx\ \psi_{n'}(x)\ \frac{\hbar}{i}\frac{d}{dx}\ \psi_n(x)$$

$$= \frac{\hbar}{i}\frac{\Delta^{-1}}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\ H_{n'}(y)\, e^{-\frac{1}{2}y^2}\ \frac{d}{dy}\ H_n(y)\, e^{-\frac{1}{2}y^2}$$

$$= \frac{\hbar}{i}\frac{\Delta^{-1}}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\ H_{n'}(y)\, e^{-y^2}\ \underbrace{\left[H'_n(y) - y\, H_n(y)\right]}$$

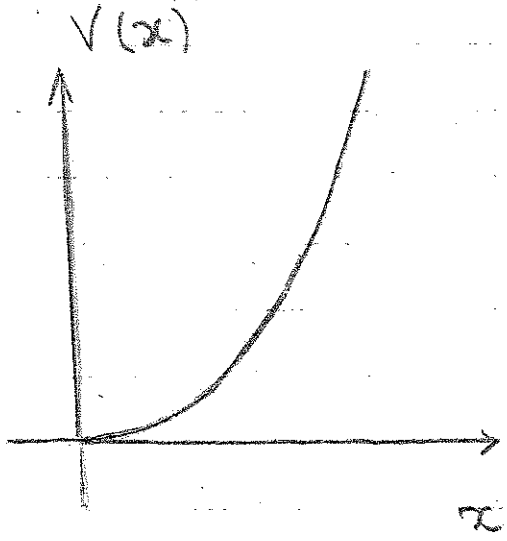
$$2n\, H_{n-1}(y) - \left(\frac{1}{2}H_{n+1}(y) + n\, H_{n-1}(y)\right)$$

$$= n\, H_{n-1}(y) - \frac{1}{2}H_{n+1}(y)$$

$$= \frac{\hbar}{i}\left[\ \delta_{n',n-1}\ n\ \frac{1}{\sqrt{2n}} - \frac{1}{2}\,\delta_{n',n+1}\sqrt{2(n+1)}\ \right]\Delta^{-1}$$

$$= \left(\frac{m\omega k}{2}\right)^{1/2}\frac{1}{i}\left[\ \delta_{n',n-1}\sqrt{n} - \delta_{n',n+1}\sqrt{n+1}\ \right]$$

S 7.3.6



$$V(x) = \frac{1}{2} m\omega^2 x^2 \quad \text{for } x > 0$$

$$= \infty \quad \text{for } x \leq 0$$

$$\psi(x=0) = 0$$

$$\left[-\frac{k^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right] \psi_n(x) = E_n \psi_n(x)$$

Clearly

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\,\Delta\, 2^n n!}} H_n\left(\frac{x}{\Delta}\right) e^{-\frac{1}{2}\frac{x^2}{\Delta^2}} \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x < 0$$

for n odd , are solutions.

There are no other solutions.

Indeed let us assume there is another eigenfunction $\psi'(x)$ of eigenvalue E'.

$$\psi'(x) = \psi_{E'}(x) \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x < 0$$

Consider the wavefunction

$$\psi''(x) = \psi_{E'}(x)\frac{1}{\sqrt{2}} \quad \text{for } x > 0$$

$$= -\psi_{E'}(-x)\frac{1}{\sqrt{2}} \quad \text{for } x < 0$$

$\psi''(x)$ is a eigenfunction of the harmonic oscillator hamiltonian with eigenvalue E'.

But then $\psi'(x)$ is not a new solution.

Homework 8

Shanshan Cao

7.3.1

Set $\psi = \sum_{n=0}^{\infty} a_n y^n$, we have $\psi'' = \sum_{n=0}^{\infty} a_n n(n-1) y^{n-2} = \sum_{n=2}^{\infty} a_n n(n-1) y^{n-2}$.

Therefore,

$$\begin{aligned} 0 &= \psi'' + (2\epsilon - y^2)\psi \\ &= \sum_{n=2}^{\infty} a_n n(n-1) y^{n-2} + 2\epsilon \sum_{n=0}^{\infty} a_n y^n - y^2 \sum_{n=0}^{\infty} a_n y^n \\ &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) y^n + 2\epsilon \sum_{n=0}^{\infty} a_n y^n - \sum_{n=2}^{\infty} a_{n-2} y^n \\ &= (2a_2 + 2\epsilon a_0) + (6a_3 + 2\epsilon a_1) y + \sum_{n=2}^{\infty} \left[a_{n+2}(n+2)(n+1) + 2\epsilon a_n - a_{n-2} \right] y^n \end{aligned}$$

$$\Rightarrow \begin{cases} a_2 = -\epsilon a_0 \\ a_3 = -\frac{1}{3}\epsilon a_1 \\ a_{n+2} = \dfrac{a_{n-2} - 2\epsilon a_n}{(n+2)(n+1)}, \qquad n \geq 2 \end{cases}$$

where a_0 and a_1 are arbitrary constants (may be determined by normalization).

7.3.2

Eqn. (7.3.15) $\quad C_{n+2} = C_n \dfrac{2n+1-2\epsilon}{(n+2)(n+1)}$

For H_3, $\epsilon_3 = 3 + \frac{1}{2} = \frac{7}{2}$

$$C_3 = C_1 \frac{2\times 1 + 1 - 7}{(1+2)(1+1)} = -\frac{2}{3} C_1$$

$$C_5 = C_3 \frac{2\times 3 + 1 - 7}{(3+2)(3+1)} = 0$$

$$C_7 = C_9 = C_{11} = \cdots = 0$$

If we choose $C_1 = -12$, $C_2 = 0$, we have

$C_3 = 8$, $C_4 = C_6 = C_8 = \cdots = 0$.

Therefore, $H_3(y) = -12y + 8y^3$.

For H_4, $\epsilon_4 = 4 + \frac{1}{2} = \frac{9}{2}$

$$C_2 = C_0 \frac{0 + 1 - 9}{2\times 1} = -4 C_0$$

$$C_4 = C_2 \frac{4+1-9}{4\times 3} = -\frac{1}{3}C_2 = \frac{4}{3}C_0$$
$$C_6 = C_4 \frac{8+1-9}{6\times 5} = 0$$

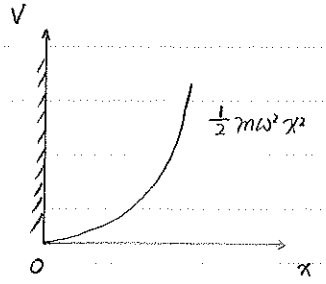
$C_8 = C_{10} = C_{12} = \cdots = 0$

If we choose $C_0 = 12$, $C_1 = 0$, we have

$C_2 = -3$, $C_4 = 16$, $C_3 = C_5 = C_7 = \cdots = 0$.

Therefore, $H_4(y) = 12 - 48y^2 + 16y^4$.

7.3.6



This is similary to the regular harmonic oscillator problem, but with only the odd solutions, so that

$\psi(0) = 0$, (boundary condition)

required by $V(x \le 0) = \infty$.

The normalization factor changes:

For the regular harmonic oscillator, $\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$;

but here, $\int_0^{\infty} \psi'^* \psi' dx = 1$.

Therefore $\psi'_n = \sqrt{2}\,\psi_n$, where ψ_n is the eigenfunctions of the regular harmonic oscillator, and is given by Eqn. (7.3.22).

To sum up, for this problem,

$$\psi_n = \sqrt{2}\left\{\frac{m\omega}{\pi\hbar\, 2^{2(2n+1)}[(2n+1)!]^2}\right\}^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_{2n+1}\left[\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x\right],$$

$$E_n = \left(2n+1+\frac{1}{2}\right)\hbar\omega = \left(2n+\frac{3}{2}\right)\hbar\omega.$$

$n = 0, 1, 2, 3, \cdots$.

7.4.3

For a circular orbit:

$$F = \frac{\partial V}{\partial r} = kar^{k-1}$$

$$F = mv^2/r$$

$$\Rightarrow mv^2/r = kar^{k-1}$$

$$\Rightarrow mv^2 = kar^k$$

$$\Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}kar^2 = \frac{k}{2}V.$$

Therefore, $c(k) = \frac{1}{2}k$.

For the quantum oscillator:

$$\langle T \rangle = \frac{1}{2m}\langle p^2 \rangle$$

$$\langle p^2 \rangle = -\frac{m\omega\hbar}{2}\langle n|(a^+ - a)^2|n\rangle$$

$$= -\frac{m\omega\hbar}{2}\langle n|a^+a^+ - a^+a - aa^+ + aa|n\rangle$$

$$\langle n|a^+a^+|n\rangle = \sqrt{n}\cdot\sqrt{n+1}\,\langle n-1|n+1\rangle = 0$$

$$\langle n|a^+a|n\rangle = \sqrt{n}\cdot\sqrt{n}\,\langle n-1|n-1\rangle = n$$

$$\langle n|aa^+|n\rangle = \sqrt{n+1}\cdot\sqrt{n+1}\,\langle n+1|n+1\rangle = n+1$$

$$\langle n|aa|n\rangle = \sqrt{n+1}\cdot\sqrt{n}\,\langle n+1|n-1\rangle = 0$$

$$\Rightarrow \langle p^2 \rangle = \frac{m\omega\hbar}{2}(n + n + 1) = \frac{m\omega\hbar^2}{2}(2n+1)$$

$$\Rightarrow \langle T \rangle = \frac{\hbar\omega}{4}(2n+1)$$

$$\langle V \rangle = \frac{1}{2}m\omega^2\langle x^2 \rangle$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}\langle n|(a + a^+)^2|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|aa + aa^+ + a^+a + a^+a^+|n\rangle$$

$$= \frac{\hbar}{2m\omega}(n + n + 1)$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

$$\Rightarrow \langle T \rangle = \frac{\hbar\omega}{4}(2n+1)$$

Therefore, $\langle T \rangle = \langle V \rangle$.

7.4.5.

(1) $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle$

$\hat{H}|0\rangle = \frac{1}{2}\hbar\omega\,|0\rangle$

$\hat{H}|1\rangle = \frac{3}{2}\hbar\omega\,|0\rangle$

$\Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\hat{H}t/\hbar}\,|0\rangle + e^{-i\hat{H}t/\hbar}\,|1\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i\omega t}{2}}|0\rangle + e^{-\frac{3i\omega t}{2}}|1\rangle\right).$

(2) $\langle X(0)\rangle = \frac{1}{2}\left(\langle 0| + \langle 1|\right)\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+)\left(|0\rangle + |1\rangle\right)$

$= \frac{1}{2}\cdot\sqrt{\frac{\hbar}{2m\omega}}\,\left(\langle 0| + \langle 1|\right)\,a+a^+\,\left(|0\rangle + |1\rangle\right)$

$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|a|1\rangle + \langle 1|a^+|0\rangle\right)$

$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\,(1+1)$

$= \sqrt{\frac{\hbar}{2m\omega}}.$

$\langle P(0)\rangle = \frac{1}{2}\cdot i\cdot\sqrt{\frac{m\omega\hbar}{2}}\,\left(\langle 0| + \langle 1|\right)\,a^+ - a\,\left(|0\rangle + |1\rangle\right)$

$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left(-\langle 0|a|1\rangle + \langle 1|a^+|0\rangle\right)$

$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\,(-1+1)$

$= 0.$

$\langle X(t)\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|e^{\frac{i\omega t}{2}} + \langle 1|e^{\frac{3i\omega t}{2}}\right)\,a+a^+\,\left(e^{-\frac{i\omega t}{2}}|0\rangle + e^{-\frac{3i\omega t}{2}}|1\rangle\right)$

$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t}\langle 0|a|1\rangle + e^{i\omega t}\langle 1|a^+|0\rangle\right)$

$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t} + e^{i\omega t}\right)$

$= \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t.$

$\langle P(t)\rangle = \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left(\langle 0|e^{\frac{i\omega t}{2}} + \langle 1|e^{\frac{3i\omega t}{\hbar}}\right)\,a^+ - a\,\left(e^{-\frac{i\omega t}{2}}|0\rangle + e^{-\frac{3i\omega t}{2}}|1\rangle\right)$

$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left(-e^{-i\omega t}\langle 0|a|1\rangle + e^{i\omega t}\langle 1|a^+|0\rangle\right)$

$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$= -\sqrt{\frac{m\omega\hbar}{2}}\sin\omega t.$

(3) Ehrenfest's theorem :

$$\langle \dot{O} \rangle = \frac{1}{i\hbar} \langle [O, H] \rangle .$$

$$\langle \dot{X} \rangle = \frac{1}{i\hbar} \langle [X, H] \rangle$$

$$= \frac{1}{i\hbar} \langle [X, \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2] \rangle$$

$$= \frac{1}{i\hbar} \cdot \frac{1}{2m} \langle [X, P^2] \rangle$$

$$= \frac{1}{i\hbar} \cdot \frac{1}{2m} \langle 2i\hbar P \rangle$$

$$= \frac{1}{m} \langle P \rangle \qquad (1)$$

$$\langle \dot{P} \rangle = \frac{1}{i\hbar} \langle [P, H] \rangle$$

$$= \frac{1}{i\hbar} \langle [P, \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2] \rangle$$

$$= \frac{1}{i\hbar} \langle [P, \frac{1}{2} m\omega^2 X^2] \rangle$$

$$= \frac{1}{i\hbar} \langle -\frac{1}{2} m\omega^2 \cdot 2i\hbar \cdot X \rangle$$

$$= -m\omega^2 \langle X \rangle \qquad (2)$$

Combining (1) and (2), we have

$$\langle \ddot{X} \rangle = \frac{1}{m} \langle \dot{P} \rangle = -\omega^2 \langle X \rangle$$

$$\Rightarrow \langle \ddot{X} \rangle + \omega^2 \langle X \rangle = 0$$

$$\Rightarrow \langle X \rangle = A\cos\omega t + B\sin\omega t$$

$$\Rightarrow \langle P \rangle = m\langle \dot{X} \rangle = -m\omega A\sin\omega t + m\omega B\cos\omega t$$

Consider the initial condition :

$$\begin{cases} \langle X(0) \rangle = \sqrt{\frac{\hbar}{2m\omega}} \\ \langle P(0) \rangle = 0 \end{cases}$$

we have $A = \sqrt{\frac{\hbar}{2m\omega}}$, $B = 0$.

Therefore,

$$\begin{cases} \langle X(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t \\ \langle P(t)\rangle = -\sqrt{\frac{m\hbar\omega}{2}}\sin\omega t \end{cases},$$

which is the same as the result in Part (2).

7.4.6

Use Ehrenfest's theorem:

$$\langle \dot{a}(t)\rangle = \frac{1}{i\hbar}\langle [a, H]\rangle$$

$$= \frac{1}{i\hbar}\langle [a, (a^+a + \tfrac{1}{2})\hbar\omega]\rangle$$

$$= -i\omega\langle a\rangle$$

$$\Rightarrow \langle a(t)\rangle = \langle a(0)\rangle e^{-i\omega t}.$$

* $[a, a^+] = 1$

$[a, a] = [a^+, a^+] = 0.$

$$\langle \dot{a}^+(t)\rangle = \frac{1}{i\hbar}\langle [a^+, H]\rangle$$

$$= \frac{1}{i\hbar}\langle [a^+, (a^+a + \tfrac{1}{2})\hbar\omega]\rangle$$

$$= \frac{1}{i\hbar}\cdot(-\hbar\omega)\langle a^+\rangle$$

$$= i\omega\langle a^+\rangle$$

$$\Rightarrow \langle a^+(t)\rangle = \langle a^+(0)\rangle e^{i\omega t}.$$

7.4.8

(1)
$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$
$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$
$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

(2)
$$\{l_x, l_y\} = \{ypz - zpy,\ zpx - xpz\}$$
$$= \left(\frac{\partial l_x}{\partial x}\frac{\partial l_y}{\partial p_x} - \frac{\partial l_x}{\partial p_x}\frac{\partial l_y}{\partial x}\right) + \left(\frac{\partial l_x}{\partial y}\frac{\partial l_y}{\partial p_y} - \frac{\partial l_x}{\partial p_y}\frac{\partial l_y}{\partial y}\right)$$
$$+ \left(\frac{\partial l_x}{\partial z}\frac{\partial l_y}{\partial p_z} - \frac{\partial l_x}{\partial p_z}\frac{\partial l_y}{\partial z}\right)$$

$$= (0-0) + (0-0) + (p_y x - y p_x)$$
$$= x p_y - y p_x$$
$$= l_z .$$

(3) $[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y , \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$
$$= [\hat{y}\hat{p}_z , \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z , \hat{x}\hat{p}_z] - [\hat{z}\hat{p}_y , \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y , \hat{x}\hat{p}_z]$$
$$= \hat{y}[\hat{p}_z , \hat{z}]\hat{p}_x - 0 - 0 + \hat{x}[\hat{z} , \hat{p}_z]\hat{p}_y$$
$$= -i\hbar \hat{y}\hat{p}_x + i\hbar \hat{x}\hat{p}_y$$
$$= i\hbar (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$$
$$= i\hbar \hat{L}_z .$$

7.5.2

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\,x + i\sqrt{\frac{1}{2m\omega\hbar}}\,p$$

In X-basis : $a = \sqrt{\frac{m\omega}{2\hbar}}\,x + i\sqrt{\frac{1}{2m\omega\hbar}}\,(-i\hbar\frac{d}{dx})$
$$= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x + \sqrt{\frac{\hbar}{m\omega}}\,\frac{d}{dx}\right)$$
$$= \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)$$

where $y \equiv \sqrt{\frac{m\omega}{\hbar}}\,x$.

Therefore, we have

$$\frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)\alpha_n H_n(y) e^{-y^2/2} = \sqrt{n}\,\alpha_{n-1} H_{n-1}(y) e^{-y^2/2}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(y\alpha_n H_n(y) e^{-y^2/2} + \alpha_n H_n'(y) e^{-y^2/2} - y\alpha_n H_n(y) e^{-y^2/2}\right) = \sqrt{n}\,\alpha_{n-1} H_{n-1}(y) e^{-y^2/2}$$

$$\Rightarrow H_n'(y) = \sqrt{2n}\,\frac{\alpha_{n-1}}{\alpha_n} H_{n-1}(y) ,$$

where

$$\frac{\alpha_{n-1}}{\alpha_n} = \left\{\frac{2^{2n}(n!)^2}{2^{2(n-1)}[(n-1)!]^2}\right\}^{\frac{1}{4}} = \sqrt{2n} .$$

Therefore, $H_n'(y) = 2n\, H_{n-1}(y)$.

7.5.3

$$a + a^\dagger = \sqrt{2}\,y$$

$$(a+a^\dagger)|n\rangle = \sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle$$

In X-basis,

$$\sqrt{2}\,y\,\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x) + \sqrt{n+1}\,\psi_{n+1}(x)$$

$$\Rightarrow \sqrt{2}\,y\,\alpha_n H_n(y) e^{-y^2/2} = \sqrt{n}\,\alpha_{n-1} H_{n-1}(y) e^{-y^2/2} + \sqrt{n+1}\,\alpha_{n+1} H_{n+1}(y) e^{-y^2/2}$$

$$\alpha_n = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{\frac{1}{4}}$$

$$\Rightarrow \sqrt{2}y \left(\frac{1}{2^{2n}(n!)^2}\right)^{\frac{1}{4}} H_n(y) = \sqrt{n}\left\{\frac{1}{2^{2(n-1)}[(n-1)!]^2}\right\}^{\frac{1}{4}} H_{n-1}(y) + \sqrt{n+1}\cdot \left\{\frac{1}{2^{2(n+1)}[(n+1)!]^2}\right\}^{\frac{1}{4}} H_{n+1}(y)$$

$$\Rightarrow 2y\,H_n(y) = 2n\,H_{n-1}(y) + H_{n+1}(y)$$

$$\Rightarrow H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}(y).$$

Physics 827: Homework Set No. 8

Due date: Friday, December 2, 2011, 1:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 110 points

Problem 1 (5 pts.): Exercise 7.4.1 (Shankar, p. 212)

Problem 2 (10 pts.): Exercise 7.4.2 (Shankar, p. 212)

Problem 3 (30 pts.): Exercise 7.4.5 (Shankar, p. 212)

Problem 4 (15 pts.): Exercise 7.4.6 (Shankar, p. 212)

Problem 5 (50 pts.): Exercise 7.5.4 (Shankar, p. 219)

Exercise 7.4.1 (p. 212) (5pts.)

$$\langle n'|\hat{X}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n'|\hat{a}^+ + \hat{a}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\langle n'|n+1\rangle + \sqrt{n}\,\langle n'|n-1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right) \checkmark$$

$$\langle n'|\hat{P}|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\,\langle n'|\hat{a}^+ - \hat{a}|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right) \checkmark$$

Exercise 7.4.2 (p. 212) (10pts.)

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^+ + \hat{a}) \qquad \hat{P} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^+ - \hat{a})$$

From class

$$\langle n|\hat{X}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{n,n+1} + \sqrt{n}\,\delta_{n,n-1}\right) = 0$$

$$\langle n|\hat{P}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\,\delta_{n,n+1} - \sqrt{n}\,\delta_{n,n-1}\right) = 0$$

Next

$$\langle n|\hat{X}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(\hat{a}^+ + \hat{a})^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(\hat{a}^+)^2 + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+ + \hat{a}^2|n\rangle$$

($(\hat{a}^+)^2 \sim \delta_{n,n+2} = 0$; $\hat{a}^2 \sim \delta_{n,n-2} = 0$)

$$= \frac{\hbar}{2m\omega}\left(\langle n|\hat{a}^+\hat{a}|n\rangle + \langle n|\hat{a}\hat{a}^+|n\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\left(\langle n|\hat{a}^+\sqrt{n}\,|n-1\rangle + \langle n|\hat{a}\sqrt{n+1}\,|n+1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\left(\sqrt{n}\cdot\sqrt{n}\,\langle n|n\rangle + \sqrt{n+1}\,\sqrt{n+1}\,\langle n|n\rangle\right) = \frac{(2n+1)\hbar}{2m\omega}$$

$$\langle n|\hat{P}^2|n\rangle = -\frac{m\hbar\omega}{2}\langle n|(\hat{a}^+ - \hat{a})^2|n\rangle = -\frac{m\hbar\omega}{2}\langle n|\hat{a}^{+2} - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ + \hat{a}^2|n\rangle$$

($\hat{a}^{+2}|n\rangle \to 0$; $\hat{a}^2|n\rangle \to 0$)

$$= +\frac{m\hbar\omega}{2}\langle n|\hat{a}^+\hat{a} + \hat{a}\hat{a}^+|n\rangle = (2n+1)\frac{m\hbar\omega}{2}$$

$$\Rightarrow \Delta P\cdot\Delta X = \sqrt{\left(\langle\hat{X}^2\rangle - \underbrace{\langle\hat{X}\rangle^2}_{0}\right)\left(\langle\hat{P}^2\rangle - \underbrace{\langle\hat{P}\rangle^2}_{0}\right)}$$

$$= (2n+1)\sqrt{\frac{\hbar}{2m\omega}\cdot\frac{m\hbar\omega}{2}} = \left(n+\tfrac{1}{2}\right)\hbar$$

Exercise 7.4.5 (p.212) (30pts)

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

(1) $$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{i(\hat{a}^+\hat{a}+\frac{1}{2})\omega t}|\psi(0)\rangle$$

$$= e^{-i\omega t/2}\sum_{n=0}^{\infty}\frac{(i\omega t)^n}{n!}(\hat{a}^+\hat{a})^n\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$$

(5pts) Since $\hat{a}^+\hat{a}|n\rangle = n|n\rangle \Rightarrow (\hat{a}^+\hat{a})^n|0\rangle = 0^n|0\rangle = 0 \quad (n\geq 1)$

$$(\hat{a}^+\hat{a})^n|1\rangle = 1^n|1\rangle = |1\rangle \quad (n\geq 0)$$

$$= \frac{e^{-i\omega t/2}}{\sqrt{2}}|0\rangle + \frac{e^{-i\omega t/2}}{\sqrt{2}}\underbrace{\sum_{n=0}^{\infty}\frac{(i\omega t)^n}{n!}}_{e^{i\omega t}}|1\rangle = \frac{e^{-i\omega t/2}}{\sqrt{2}}|0\rangle + \frac{e^{-3i\omega t/2}}{\sqrt{2}}|1\rangle.$$

(2) $$\langle\psi(0)|\hat{X}|\psi(0)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\left(\langle 0|+\langle 1|\right)\left(\hat{a}^+ + \hat{a}\right)\left(|0\rangle+|1\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big(\underbrace{\langle 0|\hat{a}^+|0\rangle}_{0} + \underbrace{\langle 1|\hat{a}^+|0\rangle}_{1} + \underbrace{\langle 0|\hat{a}^+|1\rangle}_{0} + \underbrace{\langle 1|\hat{a}^+|1\rangle}_{0}$$
$$+ \underbrace{\langle 0|\hat{a}|0\rangle}_{0} + \underbrace{\langle 1|\hat{a}|0\rangle}_{0} + \underbrace{\langle 0|\hat{a}|1\rangle}_{1} + \underbrace{\langle 1|\hat{a}|1\rangle}_{0}\Big)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} = \langle\hat{X}\rangle(0) \quad (\neq 0!)$$

$$\langle\psi(0)|\hat{P}|\psi(0)\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\frac{1}{2}\left(\langle 0|+\langle 1|\right)\left(\hat{a}^+ - \hat{a}\right)\left(|0\rangle+|1\rangle\right)$$

(10pts.)
$$= i\sqrt{\frac{m\hbar\omega}{2}}\frac{1}{2}\left(0+1+0+0 - (0+0+1+0)\right) = 0 = \langle\hat{P}\rangle(0)$$

$$\langle\psi(t)|\hat{X}|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\left(\langle 0|e^{i\omega t/2} + \langle 1|e^{3i\omega t/2}\right)\left(\hat{a}^+ + \hat{a}\right)\left(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big(e^{i\omega t}\underbrace{\langle 1|\hat{a}^+|0\rangle}_{1} + e^{-i\omega t}\underbrace{\langle 0|\hat{a}|1\rangle}_{1}\Big)$$

(all other matrix elements are zero, see above)

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = \langle\hat{X}\rangle(t)$$

$$\langle\psi(t)|\hat{P}|\psi(0)\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\frac{1}{2}\left(\langle 0|e^{i\omega t/2} + \langle 1|e^{3i\omega t/2}\right)(\hat{a}^+ - \hat{a})\left(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right)$$

$$= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(e^{i\omega t}\langle 1|\hat{a}^+|0\rangle - e^{-i\omega t}\langle 0|\hat{a}|1\rangle\right) \quad \text{(all others are zero)}$$

$$= -\sqrt{\frac{m\hbar\omega}{2}}\sin(\omega t) = \langle\hat{P}\rangle(t)$$

(3) $\frac{d}{dt}\langle\hat{X}\rangle(t) = \frac{i}{\hbar}\langle[\hat{H},\hat{X}]\rangle(t) = \frac{i}{\hbar}\langle\psi(t)|\left[(\hat{a}^+\hat{a}+\frac{1}{2})\hbar\omega, \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^+ + a)\right]|\psi(t)\rangle$

$$= i\omega\sqrt{\frac{\hbar}{2m\omega}}\langle\psi(t)|\underbrace{[\hat{a}^+\hat{a}, \hat{a}^+ + \hat{a}]}_{\hat{a}^+ - \hat{a}\ \text{(see Exercise 7.4.6)}}|\psi(t)\rangle$$

$$= \frac{i}{m}\sqrt{\frac{m\hbar\omega}{2}}\langle\psi(t)|\hat{a}^+ - \hat{a}|\psi(t)\rangle = \frac{\langle\hat{P}\rangle(t)}{m} \qquad (\stackrel{1}{=})$$

(15 pts.)

$\frac{d}{dt}\langle\hat{P}\rangle(t) = \frac{i}{\hbar}\langle[\hat{H},\hat{P}]\rangle(t) = \frac{i}{\hbar}\langle\psi(t)|\left[(\hat{a}^+\hat{a}+\frac{1}{2})\hbar\omega, i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^+ - \hat{a})\right]|\psi(t)\rangle$

$$= -\omega\sqrt{\frac{m\hbar\omega}{2}}\langle\psi(t)|\underbrace{[\hat{a}^+\hat{a}, \hat{a}^+ - \hat{a}]}_{\hat{a}^+ + \hat{a}\ \text{(see Exercise 7.4.6)}}|\psi(t)\rangle$$

$$= -m\omega^2\sqrt{\frac{\hbar}{2m\omega}}\langle\psi(t)|\hat{a}^+ + \hat{a}|\psi(t)\rangle = -m\omega^2\langle\hat{X}\rangle(t) \qquad (\stackrel{2}{=})$$

We solve this by taking another derivative:

$$\frac{d^2\langle\hat{X}\rangle(t)}{dt^2} \stackrel{(1)}{=} \frac{1}{m}\frac{d\langle\hat{P}\rangle(t)}{dt} \stackrel{(2)}{=} -\omega^2\langle\hat{X}\rangle(t) \qquad (\stackrel{3}{=})$$

$$\frac{d^2\langle\hat{P}\rangle(t)}{dt^2} \stackrel{(2)}{=} -m\omega^2\frac{d\langle\hat{X}\rangle(t)}{dt} \stackrel{(1)}{=} -\omega^2\langle\hat{P}\rangle(t) \qquad (\stackrel{4}{=})$$

$(\stackrel{3}{=}) \Rightarrow \langle\hat{X}\rangle(t) = Ae^{i\omega t} + Be^{-i\omega t}$ general solution

$$\left.\begin{aligned}\langle\hat{X}\rangle(0) &= \sqrt{\frac{\hbar}{2m\omega}} = A + B\\ \frac{d}{dt}\langle\hat{X}\rangle\Big|_{t=0} &= \frac{\langle\hat{P}\rangle(0)}{m} = 0 = i\omega(A-B)\end{aligned}\right\}\ \text{initial values}$$

$$\Rightarrow A = B = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \Rightarrow \langle\hat{X}\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{2}(e^{i\omega t} + e^{-i\omega t})\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) \quad \text{as in (2)} \checkmark$$

$(\overset{4}{=}) \Rightarrow \langle \hat{P} \rangle(t) = Ce^{i\omega t} + De^{-i\omega t}$ general solution

$\langle \hat{P} \rangle(0) = 0 = C + D$

$\left.\frac{d\langle P \rangle}{dt}\right|_{t=0} = -m\omega^2 \langle \hat{X} \rangle(0) = -m\omega^2 \sqrt{\frac{\hbar}{2m\omega}} = -\omega\sqrt{\frac{\hbar m\omega}{2}}$ } initial values

$= i\omega(C - D)$

$\Rightarrow C = -D \Rightarrow 2i\omega C = -\omega\sqrt{\frac{\hbar m\omega}{2}}$

$\Rightarrow C = \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}$

$\Rightarrow \underbrace{\langle \hat{P} \rangle(t)} = \sqrt{\frac{\hbar m\omega}{2}} \frac{i}{2}\left(e^{i\omega t} - e^{-i\omega t}\right) = \underbrace{-\sqrt{\frac{\hbar m\omega}{2}} \sin\omega t}$

as in part (2) ✓

Exercise 7.4.6 (p. 212) (15 pts.)

$$\langle \hat{a} \rangle(t) = \langle \psi(t) | \hat{a} | \psi(t) \rangle = \langle \psi(0) | \hat{U}^\dagger(t) \hat{a} \hat{U}(t) | \psi(0) \rangle$$

$$= \langle \psi(0) | e^{i\hat{H}t/\hbar} \hat{a} e^{-i\hat{H}t/\hbar} | \psi(0) \rangle$$

$$= \langle \psi(0) | e^{i(\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega t} \hat{a} e^{-i(\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega t} | \psi(0) \rangle$$

Now we know that

$$e^{\hat{L}} \hat{M} e^{-\hat{L}} = \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{L}, \hat{M}]_{(n)}$$

Here $\hat{L} = i\hat{H}t/\hbar$ and $\hat{M} = \hat{a}$, so $[\hat{L}, \hat{M}]_{(n)} = \left(\frac{it}{\hbar}\right)^n [\hat{H}, \hat{a}]_{(n)}$.

$$[\hat{H}, \hat{a}]_{(0)} = \hat{a}$$

$$[\hat{H}, \hat{a}]_{(1)} \overset{\hbar\omega}{=} [\hat{a}^\dagger\hat{a}, \hat{a}] = \left(\hat{a}^\dagger \underbrace{[\hat{a}, \hat{a}]}_{0} + \underbrace{[\hat{a}^\dagger, \hat{a}]}_{-1} \hat{a}\right)\hbar\omega = -\hbar\omega\hat{a}$$

$$[\hat{H}, \hat{a}]_{(2)} = [\hat{H}, [\hat{H}, \hat{a}]_{(1)}] = -[\hat{H}, \hbar\omega\hat{a}] = +(\hbar\omega)^2 \hat{a}$$

$$\Rightarrow [\hat{H}, \hat{a}]_{(n)} = (-\hbar\omega)^n \hat{a}$$

Therefore

$$e^{i(\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega t} \hat{a} e^{-i(\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega t} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar}(-\hbar\omega)\right)^n \hat{a}$$

$$= \hat{a} \sum_{n=0}^{\infty} (-it\omega)^n = e^{-i\omega t} \hat{a}$$

and

$$\langle \hat{a} \rangle(t) = \langle \psi(0) | e^{-i\omega t} \hat{a} | \psi(0) \rangle = e^{-i\omega t} \langle \psi(0) | \hat{a} | \psi(0) \rangle$$

$$= e^{-i\omega t} \langle \hat{a} \rangle(0)$$

✓

Similarly we show

$$e^{i(\hat{a}^+\hat{a}+\frac{1}{2})\omega t}\hat{a}^+ e^{-i(\hat{a}^+\hat{a}+\frac{1}{2})\omega t} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{it}{\hbar}\right)^n [\hat{H},\hat{a}^+]_{(n)}$$

where

$$[\hat{H},\hat{a}^+]_{(0)} = \hat{a}^+,$$

$$[\hat{H},\hat{a}^+]_{(1)} = \hbar\omega\left[\hat{a}^+\hat{a},\hat{a}^+\right] = \hbar\omega\left(\hat{a}^+\underbrace{[\hat{a},\hat{a}^+]}_{1} + \underbrace{[\hat{a}^+,\hat{a}^+]}_{0}\hat{a}\right) = \hbar\omega\hat{a}^+,$$

$$[\hat{H},\hat{a}^+]_{(2)} = \hbar\omega[\hat{H},\hat{a}^+] = (\hbar\omega)^2\hat{a}^+,$$

$$\vdots$$

$$[\hat{H},\hat{a}^+]_{(n)} = (\hbar\omega)^n\hat{a}^+$$

thus

$$e^{i(\hat{a}^+\hat{a}+\frac{1}{2})\omega t}\hat{a}^+ e^{-i(\hat{a}^+\hat{a}+\frac{1}{2})\omega t} = \sum_{n=0}^{\infty}\frac{(i\omega t)^n}{n!}\hat{a}^+ = e^{i\omega t}\hat{a}^+$$

and

$$\underbrace{\langle\hat{a}^+\rangle(t)} = \langle\psi(0)|e^{i\hat{H}t/\hbar}\hat{a}^+ e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \underbrace{e^{i\omega t}\langle\hat{a}^+\rangle(0)} \quad \checkmark$$

Exercise 7.5.4 (p. 219) (50 pts.)

For a system at temperature T ($\beta = \frac{1}{k_B T}$), the Boltzmann distribution

$$P(i) = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \equiv \frac{1}{Z} e^{-\beta E_i}$$

gives the probability to find it in a state i of energy E_i.

(1) Average energy of the system:

$$\bar{E} = \sum_i E_i P(i) = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} = \frac{1}{Z} \sum_i E_i e^{-\beta E_i}$$

(5 pts.)
$$\frac{\partial}{\partial \beta} \ln Z = \frac{1}{Z} \frac{\partial Z}{\partial \beta} = \frac{1}{Z} \frac{\partial}{\partial \beta} \left(\sum_i e^{-\beta E_i} \right) = -\frac{1}{Z} \sum_i E_i e^{-\beta E_i}$$

$$\Rightarrow \boxed{\bar{E} = -\frac{\partial}{\partial \beta} \ln Z}$$

(2) If the system is a classical oscillator, the position of the particle in the oscillator potential is described by a continuous variable x, and its momentum by a continuous variable p. A state of the particle is completely specified by specifying x and p:

$$i \longmapsto (x,p) \quad ; \quad \sum_i \longmapsto \int dx\, dp \quad ; \quad E_i \longmapsto E(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

(10 pts) The "partition function" Z_{cl} is then given by

$$Z_{cl} = \int dx\, dp\, e^{-\beta\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right)} = \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta m \omega^2}{2} x^2} \cdot \int_{-\infty}^{\infty} dp\, e^{-\frac{\beta}{2m} p^2}$$

$$= \sqrt{\frac{2\pi}{\beta m \omega^2}} \cdot \sqrt{\frac{2m\pi}{\beta}} = \frac{2\pi}{\beta\omega} \checkmark$$

The mean energy of the oscillator is

$$\bar{E}_{cl} = -\frac{\partial}{\partial \beta} \ln Z_{cl} = -\frac{\partial}{\partial \beta} \ln \frac{1}{\beta} = \frac{\partial}{\partial \beta} \ln \beta = \frac{1}{\beta} = k_B T \checkmark$$

A system of n (non-interacting) classical oscillators has mean thermal energy of $\frac{1}{2}k_BT$ per degree of freedom (x and p (for each dimension (x,y,z)) count as two independent degrees of freedom).

(3) Now look at the corresponding quantum oscillator.
Its states are labeled by the quantum number $n = 0, 1, 2, \ldots$, counting the number of phonon excitations above the ground state:

$$Z_{qu} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n \quad \text{(geometric series)}$$

$$= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega}-1} \quad \text{(partition function)}$$

and

$$\bar{E}_{qu} = -\frac{\partial}{\partial\beta}\ln Z_{qu} = -\frac{1}{Z_{qu}}\frac{\partial}{\partial\beta}Z_{qu} = -\frac{1}{Z_{qu}}\cdot\frac{-e^{\beta\hbar\omega/2}\hbar\omega e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}$$

(10pts)

$$= -\frac{1}{Z_{qu}}\left[e^{\beta\hbar\omega/2}\left(\frac{-\hbar\omega e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}\right) + \frac{\frac{\hbar\omega}{2}e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega}-1}\right]$$

$$= -\frac{e^{\beta\hbar\omega/2}}{Z_{qu}}\hbar\omega\,\frac{-e^{\beta\hbar\omega} + \frac{1}{2}(e^{\beta\hbar\omega}-1)}{(e^{\beta\hbar\omega}-1)^2} =$$

$$= -\frac{\hbar\omega}{2}\,\frac{e^{\beta\hbar\omega/2}}{Z_{qu}}\,\frac{-e^{\beta\hbar\omega}-1}{(e^{\beta\hbar\omega}-1)^2} =$$

$$= +\frac{\hbar\omega}{2}\left(e^{\beta\hbar\omega}-1\right)\frac{e^{\beta\hbar\omega}-1+2}{(e^{\beta\hbar\omega}-1)^2} = \frac{\hbar\omega}{2} + \hbar\omega\,\frac{1}{e^{\beta\hbar\omega}-1}$$

$$= \hbar\omega\left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right] \quad \checkmark$$

$f_{BE}(\hbar\omega)$ Bose-Einstein distr.

$\frac{\hbar\omega}{2}$ is the ground state energy, $\hbar\omega f_{BE}(\hbar\omega)$ is the thermal excitation energy.

(4) For $T \to \infty$, $\beta \to 0$. In this limit we can Taylor-expand the Bose-Einstein distribution:

$$\bar{E}_{qu} \xrightarrow[\beta \to 0]{T \to \infty} \hbar\omega \left[\frac{1}{2} + \frac{1}{1 + \beta\hbar\omega + \frac{1}{2}(\beta\hbar\omega)^2 + \dots - 1} \right]$$

$$= \hbar\omega \left[\frac{1}{2} + \frac{1}{\beta\hbar\omega} \frac{1}{1 + \frac{1}{2}(\beta\hbar\omega) + \dots} \right]$$

$$\xrightarrow{\beta \to 0} \frac{\hbar\omega}{2} + \frac{1}{\beta}\left(1 - \frac{1}{2}(\beta\hbar\omega) \pm \dots\right)$$

(10 pts)

$$\xrightarrow{\beta \to 0} \frac{1}{\beta}\left(1 + O(\beta\hbar\omega)\right) = k_B T \left(1 + O\left(\frac{\hbar\omega}{k_B T}\right)\right)$$

So we obtain the classical limit as $T \to \infty$, and corrections to the classical limit are of order $\hbar\omega / k_B T$, which is small as long as the thermal energy scale $k_B T$ is much larger than the phonon energy $\hbar\omega$, $k_B T \gg \hbar\omega$ or $\boxed{T \gg \frac{\hbar\omega}{k_B}}$ ✓

(5) For small oscillations we can model a crystall with N_0 atoms as $3N_0$ decoupled oscillators of different frequencies ω_i.

If these oscillators can be treated classically, the mean energy of the crystal is

$$\bar{E}^{cl.}_{crystal} = 3N_0 k_B T$$

since the mean energy stored in each oscillator is $k_B T$ independent of its frequency.

Defining the specific heat as

$$C(T) = \frac{1}{N_0}\frac{\partial \bar{E}}{\partial T}$$

We find

$$\boxed{C_{cl}(T) = \frac{1}{N_0}\frac{\partial}{\partial T}\left(3N_0 k_B T\right) = 3k_B} \checkmark$$

independent of T and any crystal parameters, i.e. the same for all crystals made from monatomic molecules (that have no internal degrees of freedom, such as molecular vibrations and rotations).

If we follow Einstein and treat the oscillators

(15 pts.) quantum mechanically (even though we can't duplicate his achievement of having done this in 1907, 16 years before the theory of quantum mechanics was discovered!), assuming for simplicity that all crystal normal modes have the same frequency ω, we get instead

$$\bar{E}^{qu}_{crystal}(T) = 3N_0\,\hbar\omega\left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right] \overset{\theta_E \equiv \frac{\hbar\omega}{k_B}}{=} 3N_0 k_B \theta_E\left[\frac{1}{2} + \frac{1}{e^{\theta_E/T}-1}\right]$$

and

$$C_{qu}(T) = \frac{1}{N_0}\frac{\partial}{\partial T}\bar{E}^{qu}_{crystal}(T) = 3k_B\theta_E \frac{-\left(\frac{\theta_E}{T^2}\right)e^{\theta_E/T}}{\left(e^{\theta_E/T}-1\right)^2}$$

$$= 3k_B\left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{\left(e^{\theta_E/T}-1\right)^2} \checkmark$$

Here $\theta_E = \frac{\hbar\omega}{k_B}$ depends on the crystal frequency and varies from crystal to crystal.

Einstein's theory predict that $C(T)$ approaches zero exponentially as $T \to 0$. Experimentally one finds instead $C(T) \sim T^3$ for small T. Debye realized *) that this discrepancy was due to the (incorrect) assumption of the same ω for all normal modes; he fixed the discrepancy by including a realistic ω-spectrum for the crystal normal modes.

*) after the invention of quantum mechanics.

Problem Set #10

7.4.10

1. $$L = \frac{m}{2}(v_x^2 + v_y^2) - a(x^2+y^2)^{1/2}$$

$$x = \rho\cos\varphi \qquad y = \rho\sin\varphi$$

$$v_x = \dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi$$

$$v_y = \dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi$$

$$L = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\varphi^2) - a\rho$$

$$p_\rho = \frac{\partial L}{\partial \dot\rho} = m\dot\rho$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m\rho^2\dot\varphi$$

$$H(\rho,\varphi,p_\rho,p_\varphi) = -L + p_\varphi\dot\varphi + p_\rho\dot\rho \Big|_{\dot q = \dot q(q,p)}$$

$$= \frac{1}{2m}p_\rho^2 + \frac{1}{2m}\frac{1}{\rho^2}p_\varphi^2 + a\rho$$

ii If $\langle \rho, \varphi | P_\rho | \psi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \rho} \langle \rho, \varphi | \psi \rangle$,

and

$$\int_0^\infty d\rho \int_0^{2\pi} \rho \, d\varphi \; |\rho, \varphi\rangle\langle \rho, \varphi| = 1$$

then, with $\psi(\rho, \varphi) \equiv \langle \rho, \varphi | \psi \rangle$,

$$\langle \psi_1 | P | \psi_2 \rangle$$

$$= \int_0^\infty d\rho \int_0^{2\pi} \rho \, d\varphi \; \langle \psi_1 | \rho, \varphi \rangle \frac{\hbar}{i} \frac{\partial}{\partial \rho} \langle \rho, \varphi | \psi_2 \rangle$$

$$= \int_0^\infty d\rho \int_0^{2\pi} \rho \, d\varphi \; \psi_1^*(\rho, \varphi) \frac{\hbar}{i} \frac{\partial}{\partial \rho} \psi_2(\rho, \varphi)$$

$$= \int_0^\infty d\rho \int_0^{2\pi} \rho \, d\varphi \left[\left(\frac{\hbar}{i} \frac{\partial}{\partial \rho} \psi_1(\rho, \varphi) \right)^* \psi_2(\rho, \varphi) - \frac{\hbar}{i\rho} \psi_1(\rho, \varphi)^* \psi_2(\rho, \varphi) \right]$$

$$+ \int_0^{2\pi} d\varphi \; \frac{\hbar}{i} \Big[\rho \, \psi_1(\rho, \varphi) \psi_2(\rho, \varphi) \Big|_{\rho=0}^{\rho=\infty}$$

by integration by parts.

The surface term vanishes because the wavefunctions are finite at $\rho = 0$ and vanish as $\rho \to \infty$ faster than $\rho^{-1/2}$.

Indeed, let $\psi \sim \rho^p$ as $\rho \to \infty$.

Then $\int_0^{2\pi} d\varphi \int_0^{\infty} \rho \, d\rho \, |\psi|^2 < \infty$ implies

$p < -1$. Hence the 2nd condition is satisfied.

And let $\psi \sim \rho^q$ as $\rho \to 0$.

$\int_0^{2\pi} d\varphi \int_0^{\infty} \rho \, d\rho \, |\psi|^2 < \infty$ only implies $q > -1$.

But we also need

$$\langle \psi | H | \psi \rangle < \infty .$$

Since $$H = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial \rho^2} + \cdots$$

this requires (provided some other mild assumption concerning the behaviour of V at $\rho \to 0$ are satisfied)

$$\int_0^{2\pi} d\varphi \int_0^{\infty} d\rho\ \rho \left| \frac{\partial \psi}{\partial \rho} \right|^2 < \infty .$$

Hence $q > 0$ and the first condition is satisfied.

Thus

$$\langle \psi_1 | P | \psi_2 \rangle = \langle \psi_2 | P | \psi_1 \rangle^*$$

$$- \frac{\hbar}{i} \langle \psi_1 | \frac{1}{\rho} | \psi_2 \rangle$$

which means that $P \neq P^\dagger$.

However, if

$$\langle \rho, \varphi | P_\rho | \psi \rangle = \frac{\hbar}{i} \left(\frac{\partial}{\partial \rho} + \frac{1}{2\rho} \right) \langle \rho, \varphi | \psi \rangle$$

then

$$\langle \psi_1 | P_\rho | \psi_2 \rangle = \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\ \psi_1^* \frac{\hbar}{i} \left(\frac{\partial}{\partial \rho} + \frac{1}{2\rho} \right) \psi_2$$

$$= \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\ \psi_2 \left[\frac{\hbar}{i} \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} - \frac{1}{2\rho} \right) \psi_1 \right]^*$$

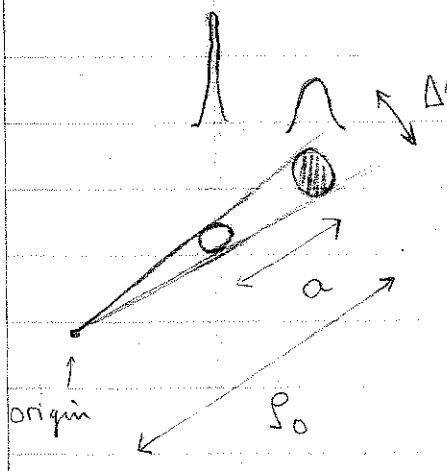
$$\langle \psi_1 | P | \psi_2 \rangle = \langle \psi_2 | P | \psi_1 \rangle^*$$

which means that $P = P^\dagger$.

iii. The situation is very different.

In Ex. 7.4.9. $f(x)$ must be real for P to be hermitean, whereas here

$$p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial \varphi} + \underbrace{\frac{\hbar}{i}\frac{1}{2\varphi}}_{\text{pure imaginary}}$$



Consider a wavefunction $\psi(\rho, \varphi)$ localized in the shaded area.

Apply the unitary operator

$$U(a) = e^{+i a/\hbar\, p_\varphi}$$

to the corresponding state.

The new state is localized in the unshaded area, which is smaller than the shaded area. The new wavefunction must therefore have larger amplitude than the old one.

$$|\psi'\rangle = U(a)\,|\psi\rangle$$

$$\psi'(\rho,\varphi) \simeq \psi(\rho+a,\varphi)\left(\frac{\rho+a}{\rho}\right)^{1/2}$$

for small a

to insure that

$$\int_0^{2\pi} d\varphi \int_0^{\rho} \rho\, d\rho\, |\psi'(\rho,\varphi)|^2 = \int_0^{2\pi} d\varphi \int_0^{\rho} \rho\, d\rho\, |\psi(\rho,\varphi)|^2 = 1$$

$$\psi'(\rho,\varphi) \simeq \left[\psi(\rho,\varphi) + a\frac{\partial}{\partial\rho}\psi(\rho,\varphi)\right]\left(1+\frac{a}{2\rho}\right)$$

$$\simeq \psi(\rho,\varphi) + a\left(\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right)\psi(\rho,\varphi)$$

Thus $$\frac{i}{\hbar}P_\rho = \frac{\partial}{\partial\rho} + \frac{1}{2\rho}$$

iv. $$H = \frac{1}{2m}\left(P_\rho^2 + \frac{1}{\rho^2}P_\varphi^2\right) + a\rho$$

$$\rightarrow \frac{-\hbar^2}{2m}\left[\left(\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right)^2 + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right] + a\rho$$

$$H \rightarrow -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}\cdot + \frac{1}{2\rho}\frac{\partial}{\partial\rho}\cdot + \frac{\partial}{\partial\rho}\left(\frac{1}{2\rho}\cdot\right) + \frac{1}{4\rho^2}\cdot + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\cdot\right) + a\rho\cdot$$

$$= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}\cdot + \frac{1}{\rho}\frac{\partial}{\partial\rho}\cdot - \frac{1}{2\rho^2}\cdot + \frac{1}{4\rho^2}\cdot + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\cdot\right) + a\rho\cdot$$

$$H \rightarrow -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{4\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right) + a\rho$$

S7.5.4

a. $$-\frac{\partial}{\partial\beta}\ln Z = \frac{-1}{Z}\frac{\partial}{\partial\beta}\left(\sum_i e^{-\beta E_i}\right)$$

$$= \sum_i E_i\, e^{-\beta E_i}\,\frac{1}{Z} = \sum_i E_i P_i = \bar{E}$$

b. $$Z_{cl} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\; e^{-\beta\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)}$$

$$= \sqrt{\pi\,\frac{2m}{\beta}}\cdot\sqrt{\pi\,\frac{2}{\beta m\omega^2}} = \frac{2\pi}{\omega\beta}$$

$$\bar{E}_{cl} = -\frac{\partial}{\partial\beta}\ln Z_{cl} = \frac{1}{\beta} = kT$$

c. $$Z_{qu} = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}$$

$$= e^{-\frac{1}{2}\beta\hbar\omega}\sum_{n=0}^{\infty}\left(e^{-\beta\hbar\omega}\right)^n$$

$$= \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$$

$$\bar{E}_{qu} = \frac{-\partial}{\partial \beta}\left[\ln\left(e^{-\frac{1}{2}\beta\hbar\omega}\right) - \ln\left(1 - e^{-\beta\hbar\omega}\right)\right]$$

$$= \hbar\omega\left[\frac{1}{2} + \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}\right]$$

$$= \hbar\omega\left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right]$$

d. For $\beta\hbar\omega \ll 1$, or $kT \gg \hbar\omega$,

$$\bar{E}_{qu} \simeq \hbar\omega\left(\frac{1}{2} + \frac{1}{\beta\hbar\omega + \frac{1}{2}(\beta\hbar\omega)^2 + O(\beta\hbar\omega)^3}\right)$$

$$\simeq \hbar\omega\left(\frac{1}{2} + \frac{1}{\beta\hbar\omega}\left(1 - \frac{1}{2}(\beta\hbar\omega) + O(\beta\hbar\omega)^2\right)\right)$$

$$\simeq kT\left(1 + O(\beta\hbar\omega)^2\right)$$

e.

$$\bar{E}_{cl\,cr} = 3N_0 \bar{E}_{cl} = 3N_0 kT$$

$$C_{cl}(T) = \frac{1}{N_0} \frac{\partial \bar{E}_{cl\,cr}}{\partial T} = 3k$$

$$\bar{E}_{qu\,cr} = 3N_0 \bar{E}_{qu} = 3N_0 \hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right)$$

$$C_{qu}(T) = \frac{1}{N_0} \frac{\partial \bar{E}_{qu\,cr}}{\partial T} = \frac{1}{N_0} \left(\frac{d\beta}{dT} \right) \frac{\partial \bar{E}_{qu\,cr}}{\partial \beta}$$

$$= \frac{1}{N_0} \left(\frac{-1}{kT^2} \right) 3N_0 \hbar\omega \frac{-\hbar\omega e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1 \right)^2}$$

$$= 3k \left(\frac{\theta_E}{T} \right)^2 \frac{e^{\theta_E/T}}{\left(e^{\theta_E/T} - 1 \right)^2} \qquad \text{with } \theta_E \equiv \frac{\hbar\omega}{k}$$

$$C_{qu}(T) \to 3k \left(\frac{\theta_E}{T} \right)^2 \frac{1}{\left(\frac{\theta_E}{T} \right)^2} = 3k \qquad \text{for } T \gg \theta_E$$

$$\to 3k \left(\frac{\theta_E}{T} \right)^2 e^{-\theta_E/T} \qquad \text{for } T \ll \theta_E$$

§8.6.1 $V(x) = -fx$

$$m \frac{d^2 x_d}{dt^2} = +f$$

$$\therefore \quad x_d(t) = x_d(0) + \dot{x}_d(0)\, t + \frac{f}{2m} t^2 \,,$$

For $x_d(t') = x'$, $x_d(t'') = x''$,

$$x_d(t) = x' + v_0 (t - t') + \frac{f}{2m} (t - t')^2$$

with

$$x'' = x' + v_0 (t'' - t') + \frac{f}{2m} (t'' - t')^2 .$$

Hence

$$v_0 = \frac{x'' - x'}{t'' - t'} - \frac{f}{2m} (t'' - t') .$$

$$S_d = \int_{t'}^{t''} dt \left[\frac{m}{2} v^2 + fx \right]$$

$$= \int_{t'}^{t''} dt \left\{ \frac{m}{2} \left[v_0 + \frac{f}{m} (t - t') \right]^2 + f \left[x' + v_0 (t - t') + \frac{f}{2m} (t - t')^2 \right] \right\}$$

$$S_{cl} = \int_{t'}^{t''} dt \left\{ \frac{m}{2} v_0^2 + f x' + 2 f v_0 (t - t') + \frac{f^2}{m} (t - t')^2 \right\}$$

$$= \left(\frac{m}{2} v_0^2 + f x' \right) (t'' - t') + f v_0 (t'' - t')^2 + \frac{f^2}{3m} (t'' - t')^3$$

$$\frac{m}{2} v_0^2 (t'' - t') + f v_0 (t'' - t')$$

$$= \frac{m}{2} \left[v_0 (t'' - t') + \frac{f}{m} (t'' - t')^2 \right]^2 \frac{1}{t'' - t'} - \frac{f^2}{2m} (t'' - t')^3$$

$$= \frac{m}{2} \left[x'' - x' + \frac{f}{2m} (t'' - t')^2 \right]^2 \frac{1}{t'' - t'} - \frac{f^2}{2m} (t'' - t')^3$$

$$= \frac{m}{2} \frac{(x'' - x')^2}{t'' - t'} + \frac{f}{2} (x'' - x')(t'' - t') - \frac{3}{8} \frac{f^2}{m} (t'' - t')^3$$

$\left(\frac{1}{8} - \frac{1}{2} = \frac{1-4}{8} = \frac{-3}{8} \right)$

$$S_d = \frac{m}{2}\frac{(x''-x')^2}{t''-t'} + \frac{f}{2}(x''+x')(t''-t') - \frac{1}{24}\frac{f^2}{m}(t''-t')^3$$

$\left(\frac{-3}{8}+\frac{1}{3} = \frac{-9+8}{24} = \frac{-1}{24}\right)$

Therefore

$$U(x,t;x',0) = A(t)\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x-x')^2}{t} + \frac{f}{2}(x+x')t - \frac{f^2}{24m}t^3\right]\right\}$$

See section 8.6.
In particular, see Eq. (8.6.11).
$A(t)$ does not depend upon f.

For $f=0$, $U(x,t;x',0) = \left(\frac{m}{2\pi\hbar i t}\right)^{3/2} e^{\frac{im}{2\hbar}\frac{(x-x')^2}{t}}$

from Eq. (5.1.10)

Therefore, for $f \neq 0$,

$$A(t) = \left(\frac{m}{2\pi\hbar i t}\right)^{3/2}$$ as well.

6 3 2 H solutions

HW #1

Ans 9.4.1 $\langle n=1 | \frac{1}{x^2} | n=1 \rangle$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar\, 2^{2(1)}}\right)^{1/4} \exp\left(\frac{-m\omega x^2}{2\hbar}\right) 2\left(\frac{m\omega}{\hbar}\right)^{1/2} x$$

$$\left\langle \frac{1}{x^2} \right\rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar\,4}\right)^{1/2} 4 \exp\left(\frac{-m\omega x^2}{\hbar}\right) \left(\frac{m\omega}{\hbar}\right) \frac{1}{x^2}\, x^2\, dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-m\omega x^2}{\hbar}\right) dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) 2\int_{0}^{\infty} \exp\left(\frac{-m\omega x^2}{\hbar}\right) dx = 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \left(\frac{\pi\hbar}{m\omega}\right)^{1/2}$$

$$= 2\left(\frac{m\omega}{\hbar}\right)$$

(7)

$$\langle n=1 | x^4 | n=1 \rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar\,4}\right)^{1/2} 4 \exp\left(\frac{-m\omega x^2}{\hbar}\right) \left(\frac{m\omega}{\hbar}\right) x^4\, dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \int_{-\infty}^{\infty} x^4 \exp\left(\frac{-m\omega x^2}{\hbar}\right) dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \sqrt{\frac{\hbar\pi}{m\omega}}\, \frac{(2(2)-1)!!}{\left(2\frac{m\omega}{\hbar}\right)^2} = \frac{4}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} \frac{3}{4\left(\frac{m\omega}{\hbar}\right)^2}$$

$$= \frac{3\hbar}{2(m\omega)} \quad \therefore \quad \frac{1}{\langle n=1 | x^4 | n=1 \rangle} = \frac{2m\omega}{3\hbar}$$

Thus,

$$\left\langle \frac{1}{x^2} \right\rangle \simeq \frac{1}{\langle x^2 \rangle} \simeq \frac{m\omega}{\hbar}$$

Prob 9.4.2. $\psi = \frac{1}{(\pi a_0^3)^{1/2}} e^{-r/a_0}$

$\langle \psi | \psi \rangle = 4\pi \int_0^\infty \frac{1}{(\pi a_0^3)} e^{-2r/a_0} r^2 dr = \frac{4}{a_0^3} \int_0^\infty e^{-2r/a_0} r^2 dr$

$= \frac{4}{a_0^3} \frac{2}{(2/a_0)^3} = \frac{8}{8} \frac{a_0^3}{a_0^3} = \boxed{1}$ $\quad \alpha = \left(\frac{2}{a_0}\right)$, $n = 2$

(a) $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$

We have spherical wavefunctions. So $\langle x \rangle = \langle y \rangle = \langle z \rangle = 0$

$\therefore (\Delta x)^2 = \langle x^2 \rangle$ $\quad \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = \langle r^2 \rangle$

↳ ∵ spherical, symmetric w.r.t x, y & z

(b) $\therefore \langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle$

$= \frac{1}{3} 4\pi \int \frac{1}{\pi a_0^3} e^{-2r/a_0} r^4 dr$

$= \frac{4}{3a_0^3} \int e^{-2r/a_0} r^4 dr = \frac{4}{3a_0^3} \frac{4!}{(2/a_0)^5} = \frac{4}{3} a_0^2 \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2}$

$\therefore (\Delta x)^2 = a_0^2 \Rightarrow \boxed{(\Delta x) = a_0 = \frac{\hbar^2}{me^2}}$

(3) $\left\langle \frac{1}{r} \right\rangle \xrightarrow{\text{spherical coordinate}} 4\pi \int_0^\infty \frac{1}{(\pi a_0^3)} e^{-2r/a_0} r \, dr$

$= \frac{4}{a_0^3} \int_0^\infty r e^{-2r/a_0} dr = \frac{4}{a_0^3} \frac{1}{\left(\frac{2}{a_0}\right)^2} = \frac{1}{a_0} = \frac{me^2}{\hbar^2}$

$\langle r \rangle = 4\pi \int_0^\infty \frac{1}{\pi a_0^3} e^{-2r/a_0} r^3 dr = \left(\frac{4\pi}{\pi a_0^3}\right)\left(\frac{3 a_0^4}{8}\right) = \frac{3}{2} a_0 = \frac{3}{2} \frac{\hbar^2}{me^2}$

$\therefore \frac{1}{\langle r \rangle} = \frac{2}{3} \frac{me^2}{\hbar^2}$ $\quad \therefore \left\langle \frac{1}{r} \right\rangle \simeq \frac{1}{\langle r \rangle} \simeq \frac{me^2}{\hbar^2}$

Ans 3.4.3. $H = \frac{p^2}{2m} - \frac{e^2}{(R^2)^{1/2}}$; $\langle H \rangle = \frac{\langle p^2 \rangle}{2m} - \frac{e^2}{\langle R^2 \rangle^{1/2}}$

$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$

$(\Delta R)^2 = \langle R^2 \rangle - \langle R \rangle^2$

We are interested in ground state $\Rightarrow \langle H \rangle$ minimum. So, work with states in which $\langle p \rangle = \langle R \rangle = 0$

(7) $\therefore \langle H \rangle \approx \frac{(\Delta p)^2}{2m} - \frac{e^2}{[(\Delta R)^2]^{1/2}}$ $\quad (\Delta p)(\Delta R) \geq \frac{\hbar}{2}$

$\therefore \langle H \rangle \geq \frac{\hbar^2}{8(\Delta R)^2 m} - \frac{e^2}{(\Delta R)}$ $\quad \therefore (\Delta p) \geq \frac{\hbar}{2(\Delta R)}$

Differentiating, $\frac{\hbar^2(-2)}{8m(\Delta R)^3} + \frac{e^2}{(\Delta R)^2} = 0$

$\therefore \frac{e^2}{(\Delta R)^2} = \frac{2\hbar^2}{8m(\Delta R)^3}$ $\quad \therefore e^2 = \frac{\hbar^2}{4m(\Delta R)}$ $\quad \therefore (\Delta R) = \frac{\hbar^2}{4me^2}$

$\therefore \langle H \rangle \geq \frac{\hbar^2}{8m\left(\frac{\hbar^2}{4me^2}\right)^2} - \frac{e^2(4me^2)}{\hbar^2}$

$\geq \frac{2\hbar^2 4m^2 e^4}{4m\hbar^4} - \frac{4e^4 m}{\hbar^2}$

$\geq \frac{2e^4 m}{\hbar^2} - \frac{4e^4 m}{\hbar^2}$ $\quad \therefore \boxed{\langle H \rangle \geq -\frac{2me^4}{\hbar^2}}$

$\therefore$ The ground state energy is of the order of $-\frac{2me^4}{\hbar^2}$

Ans 9.4.4

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\langle\psi|[\hat{\omega},\hat{\lambda}]_+|\psi\rangle^2 + \frac{1}{4}\langle\psi|\hat{\Gamma}|\psi\rangle^2$$

where, $\hat{\omega} = \hat{\Omega} - \bar{\Omega}$, $\hat{\lambda} = \hat{\Lambda} - \bar{\Lambda}$; $[\hat{\omega},\hat{\lambda}] = [\hat{\Omega},\hat{\Lambda}] = i\hat{\Gamma}$

$$[T,x] = \frac{1}{2m}[p^2,x] = p[p,x] + [p,x]p = p(-i\hbar) - i\hbar p$$

$$= \frac{-2i\hbar p}{2m}$$

$\therefore$ Here $\hat{\Gamma} = -2\hbar\hat{p}$.

$[\hat{\omega},\hat{\lambda}]_+$ is Hermitean $\therefore \langle[\hat{\omega},\hat{\lambda}]_+\rangle^2 \geq 0$

(Q)

$$\therefore (\Delta T)^2(\Delta x)^2 \geq \frac{1}{4}\frac{\langle\hat{p}\rangle^2}{4m^2}4\hbar^2 \qquad (\Delta T)(\Delta x) \geq \frac{\hbar\langle p\rangle}{2m}$$

This relation is not proper because it involves the particle momentum $\langle p\rangle$ & therefore is not universal.

Also for bound states $\langle p\rangle = 0$ $\therefore$ This relation just gives $(\Delta T)(\Delta x) \geq 0$, which is trivial information.

Homework 9

Shanshan Cao

9.4.2

(1) $\psi = \frac{1}{\sqrt{a_0^3\pi}} e^{-r/a_0}$

$$\langle\psi|\psi\rangle = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^{\infty} dr \cdot r^2 \cdot \psi^*\psi$$

$$= 4\pi \int_0^{\infty} dr \ \frac{1}{\pi a_0^3} e^{-2r/a_0} \cdot r^2$$

$$= \frac{4}{a_0^3} \int_0^{\infty} r^2 e^{-2r/a_0} dr$$

$$= \frac{4}{a_0^3} \cdot \left(-\frac{a_0}{2}\right)\left(r^2 e^{-2r/a_0}\Big|_0^{\infty} - \int_0^{\infty} 2r e^{-2r/a_0} dr\right)$$

$$= \frac{4}{a_0^3} \cdot \left(-\frac{a_0}{2}\right)^2 \cdot (-2) \cdot \left(r e^{-2r/a_0}\Big|_0^{\infty} - \int_0^{\infty} e^{-2r/a_0} dr\right)$$

$$= \frac{4}{a_0^3} \cdot \left(-\frac{a_0}{2}\right)^3 \cdot 2 \cdot e^{-2r/a_0}\Big|_0^{\infty}$$

$$= \frac{4}{a_0^3} \cdot \frac{a_0^3}{8} \cdot 2 \cdot 1$$

$$= 1.$$

(2) $(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$

$$\langle X\rangle = \int r\sin\theta\cos\varphi \cdot \psi^*\psi \cdot r^2\sin\theta \, dr d\theta d\varphi$$

The φ part: $\int_0^{2\pi} d\varphi \cos\varphi \, d\varphi = 0$.

Therefore, $\langle X\rangle = 0$.

$$\langle X^2\rangle = \int r^2\sin^2\theta\cos^2\varphi \cdot \psi^*\psi \cdot r^2\sin\theta \, dr d\theta d\varphi$$

$$= \left(\int_0^{\infty} \frac{1}{\pi a_0^3} \cdot r^4 e^{-2r/a_0} \cdot dr\right)\left(\int_0^{\pi} \sin^3\theta \, d\theta\right)\left(\int_0^{2\pi} \cos^2\varphi \, d\varphi\right)$$

$$= \frac{3}{4\pi} a_0^2 \cdot \frac{4}{3} \cdot \pi$$

$$= a_0^2$$

Thus, $(\Delta X)^2 = a_0^2 = \left(\frac{\hbar^2}{me^2}\right)^2 \Rightarrow \Delta X = \frac{\hbar^2}{me^2}$.

$r^2 = X^2 + Y^2 + Z^2$

$$\Rightarrow \langle r^2\rangle = \langle X^2\rangle + \langle Y^2\rangle + \langle Z^2\rangle \overset{\text{By symmetry}}{=} 3\langle X^2\rangle = 3(\Delta X)^2$$

$$\Rightarrow (\Delta X)^2 = \frac{1}{3}\langle r^2\rangle.$$

It might be easier to compute $\langle r^2 \rangle$ first.

$$\begin{aligned}\langle r^2 \rangle &= 4\pi \int_0^\infty r^2 \cdot \frac{1}{\pi a_0^3} e^{-2r/a_0} \cdot r^2 dr \\ &= 4\pi \cdot \frac{1}{\pi a_0^3} \int_0^\infty \left(\frac{2r}{a_0}\right)^4 e^{-2r/a_0} d\left(\frac{2r}{a_0}\right) \cdot \left(\frac{a_0}{2}\right)^5 \\ &= 4\pi \cdot \frac{1}{\pi a_0^3} \cdot \left(\frac{a_0}{2}\right)^5 \cdot \Gamma(5) \\ &= 4 \cdot a_0^2 \cdot \frac{1}{32} \cdot 4! \\ &= 3a_0^2\end{aligned}$$

$$\Rightarrow (\Delta X)^2 = \frac{1}{3}\langle r^2 \rangle = a_0^2$$

(3) $$\begin{aligned}\left\langle \frac{1}{r} \right\rangle &= 4\pi \int_0^\infty \frac{1}{r} \cdot \frac{1}{\pi a_0^3} e^{-2r/a_0} \cdot r^2 dr \\ &= 4\pi \cdot \frac{1}{\pi a_0^3} \int_0^\infty \frac{2r}{a_0} e^{-2r/a_0} d\left(\frac{2r}{a_0}\right) \cdot \left(\frac{a_0}{2}\right)^2 \\ &= 4\pi \cdot \frac{1}{\pi a_0^3} \cdot \frac{a_0^2}{4} \cdot 1! \\ &= \frac{1}{a_0}.\end{aligned}$$

$$\begin{aligned}\langle r \rangle &= 4\pi \int_0^\infty r \cdot \frac{1}{\pi a_0^3} e^{-2r/a_0} \cdot r^2 dr \\ &= 4\pi \cdot \frac{1}{\pi a_0^3} \int_0^\infty \left(\frac{2r}{a_0}\right)^3 e^{-2r/a_0} d\left(\frac{2r}{a_0}\right) \cdot \left(\frac{a_0}{2}\right)^4 \\ &= 4\pi \cdot \frac{1}{\pi a_0^3} \cdot \frac{a_0^4}{16} \cdot 3! \\ &= \frac{3}{2} a_0\end{aligned}$$

$$\Rightarrow \frac{1}{\langle r \rangle} = \frac{2}{3a_0}$$

Therefore, $\left\langle \frac{1}{r} \right\rangle \approx \frac{1}{\langle r \rangle} \approx \frac{1}{a_0} = \frac{me^2}{\hbar^2}$.

10.1.1.

(1) $$\begin{aligned}&[\Omega_1^{(1)} \otimes I^{(2)},\ I^{(1)} \otimes \Lambda_2^{(2)}]\ |\chi_1\rangle \otimes |\chi_2\rangle \\ &= (\Omega_1^{(1)} \otimes I^{(2)})(I^{(1)} \otimes \Lambda_2^{(2)})(|\chi_1\rangle \otimes |\chi_2\rangle) - (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} \otimes I^{(2)})(|\chi_1\rangle \otimes |\chi_2\rangle) \\ &= (\Omega_1^{(1)} \otimes I^{(2)})(|\chi_1\rangle \otimes |\Lambda_2^{(2)}\chi_2\rangle) - (I^{(1)} \otimes \Lambda_2^{(2)})(|\Omega_1^{(1)}\chi_1\rangle \otimes |\chi_2\rangle) \\ &= |\Omega_1^{(1)}\chi_1\rangle \otimes |\Lambda_2^{(2)}\chi_2\rangle - |\Omega_1^{(1)}\chi_1\rangle \otimes |\Lambda_2^{(2)}\chi_2\rangle \\ &= 0\end{aligned}$$

$$\Rightarrow [\Omega_1^{(1)} \otimes I^{(2)},\ I^{(1)} \otimes \Lambda_2^{(2)}] = 0.$$

(2) $(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\theta_1^{(1)} \otimes \Lambda_2^{(2)})(|\chi_1\rangle \otimes |\chi_2\rangle)$

$= (\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(|\theta_1^{(1)}\chi_1\rangle \otimes |\Lambda_2^{(2)}\chi_2\rangle)$

$= |\Omega_1^{(1)}\theta_1^{(1)}\chi_1\rangle \otimes |\Gamma_2^{(2)}\Lambda_2^{(2)}\chi_2\rangle$

$= (\Omega_1^{(1)}\theta_1^{(1)}) \otimes (\Gamma_2^{(2)}\Lambda_2^{(2)})(|\chi_1\rangle \otimes |\chi_2\rangle)$

$= (\Omega\theta)_1^{(1)} \otimes (\Gamma\Lambda)_2^{(2)}(|\chi_1\rangle \otimes |\chi_2\rangle)$

$\Rightarrow (\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\theta_1^{(1)} \otimes \Lambda_2^{(2)}) = (\Omega\theta)_1^{(1)} \otimes (\Gamma\Lambda)_2^{(2)}$.

(3) $[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}]$

$= [\Omega_1^{(1)} \otimes I^{(2)}, \Lambda_1^{(1)} \otimes I^{(2)}]$

$= (\Omega_1^{(1)} \otimes I^{(2)})(\Lambda_1^{(1)} \otimes I^{(2)}) - (\Lambda_1^{(1)} \otimes I^{(2)})(\Omega_1^{(1)} \otimes I^{(2)})$

(by part (2)) $= (\Omega_1^{(1)}\Lambda_1^{(1)}) \otimes I^{(2)} - (\Lambda_1^{(1)}\Omega_1^{(1)}) \otimes I^{(2)}$

(by linearality) $= (\Omega_1^{(1)}\Lambda_1^{(1)} - \Lambda_1^{(1)}\Omega_1^{(1)}) \otimes I^{(2)}$

$= [\Omega_1^{(1)}, \Lambda_1^{(1)}] \otimes I^{(2)}$

$= \Gamma_1^{(1)} \otimes I^{(2)}$.

Similarly, we can prove $[\Omega_2^{(1)\otimes(2)}, \Lambda_2^{(1)\otimes(2)}] = I^{(1)} \otimes \Gamma_2^{(2)}$ if $[\Omega_2^{(2)}, \Lambda_2^{(2)}] = \Gamma_2^{(2)}$.

(4) $(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)})^2$

$= (\Omega_1^{(1)} \otimes I^{(2)} + I^{(1)} \otimes \Omega_2^{(2)})^2$

$= (\Omega_1^{(1)} \otimes I^{(2)})(\Omega_1^{(1)} \otimes I^{(2)}) + (\Omega_1^{(1)} \otimes I^{(2)})(I^{(1)} \otimes \Omega_2^{(2)}) + (I^{(1)} \otimes \Omega_2^{(2)})(\Omega_1^{(1)} \otimes I^{(2)})$
$+ (I^{(1)} \otimes \Omega_2^{(2)})(I^{(1)} \otimes \Omega_2^{(2)})$

(by part (2)) $= (\Omega_1^2)^{(1)} \otimes I^{(2)} + \Omega_1^{(1)} \otimes \Omega_2^{(2)} + \Omega_1^{(1)} \otimes \Omega_2^{(2)} + I^{(1)} \otimes (\Omega_2^2)^{(2)}$

$= (\Omega_1^2)^{(1)} \otimes I^{(2)} + I^{(1)} \otimes (\Omega_2^2)^{(2)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)}$.

10.1.2.

* If $A_1^{(1)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $B_2^{(2)} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, we can prove that

$$A_1^{(1)} \otimes B_2^{(2)} = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \end{array} = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix}$$

Proof: The new matrix element

$$
\begin{aligned}
M_{i_1 i_2, j_1 j_2} &= (\langle i_1| \otimes \langle i_2|)(A_1^{(1)} \otimes B_2^{(2)})(|j_1\rangle \otimes |j_2\rangle) \\
&= (\langle i_1| \otimes \langle i_2|)(A_1^{(1)}|j_1\rangle \otimes B_2^{(2)}|j_2\rangle) \\
&= \langle i_1 | A_1^{(1)} | j_1 \rangle \langle i_2 | B_2^{(2)} | j_2 \rangle \\
&= A_1^{(1)}{}_{i_1 j_1} \cdot B_2^{(2)}{}_{i_2 j_2} \qquad\qquad i_1, i_2, j_1, j_2 = 1 \text{ or } 2 .
\end{aligned}
$$

This is equivalent to the matrix form above.

(1)

$$
\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes I^{(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}
$$

$$
= \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}
$$

* We set $1 \leftrightarrow$ "+", $2 \leftrightarrow$ "−", and use the statement above.

(2)

$$
\sigma_2^{(1)\otimes(2)} = I^{(1)} \otimes \sigma_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix}
$$

$$
= \begin{pmatrix} 1\begin{pmatrix} e & f \\ g & h \end{pmatrix} & 0\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ 0\begin{pmatrix} e & f \\ g & h \end{pmatrix} & 1\begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{pmatrix}
$$

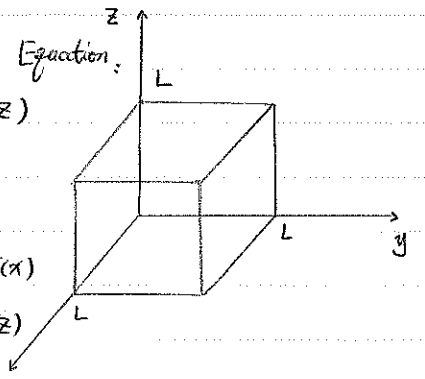
$$
= \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix}
$$

(3) $(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes \sigma_2^{(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$
= \begin{pmatrix} a\begin{pmatrix} e & f \\ g & h \end{pmatrix} & b\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ c\begin{pmatrix} e & f \\ g & h \end{pmatrix} & d\begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{pmatrix}
$$

$$= \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

10.2.1.



Inside the box, $V = 0$. The Schrödinger Equation:

$$E\psi(x,y,z) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x,y,z)$$

Seperation of variables:

$$\psi(x,y,z) = X(x)Y(y)Z(z)$$

$$\Rightarrow E X(x)Y(y)Z(z) = -\frac{\hbar^2}{2m}\cdot Y(y)Z(z)\frac{\partial^2}{\partial x^2}X(x) - \frac{\hbar^2}{2m}X(x)Z(z)\frac{\partial^2}{\partial y^2}Y(y) - \frac{\hbar^2}{2m}X(x)Y(y)\frac{\partial^2}{\partial z^2}Z(z)$$

Divide both sides by $X(x)Y(y)Z(z)$

$$\Rightarrow E = -\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2}{\partial x^2}X - \frac{\hbar^2}{2m}\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y - \frac{\hbar^2}{2m}\frac{1}{Z}\frac{\partial^2}{\partial z^2}Z$$

The only way to satisfy this equation is to set

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2}{\partial x^2}X = E_x \\ -\frac{\hbar^2}{2m}\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = E_y \\ -\frac{\hbar^2}{2m}\frac{1}{Z}\frac{\partial^2}{\partial z^2}Z = E_z \end{cases} \qquad E = E_x + E_y + E_z .$$

We have got 3 one-dimensional Schrödinger Equation. The solutions are:

$$\begin{cases} X(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi x}{L}\right), & E_x = \frac{\hbar^2\pi^2}{2ML^2}n_x^2 \\ Y(y) = \sqrt{\frac{2}{L}}\sin\left(\frac{n_y\pi y}{L}\right), & E_y = \frac{\hbar^2\pi^2}{2ML^2}n_y^2 \\ Z(z) = \sqrt{\frac{2}{L}}\sin\left(\frac{n_z\pi z}{L}\right), & E_z = \frac{\hbar^2\pi^2}{2ML^2}n_z^2 . \end{cases}$$

$$n_x, n_y, n_z = 1, 2, 3, \cdots$$

Therefore,

$$\psi_E(x,y,z) = X_E(x)Y_E(y)Z_E(z) = \sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n_y\pi y}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n_z\pi z}{L}\right),$$

$$E = E_x + E_y + E_z = \frac{\hbar^2\pi^2}{2ML^2}(n_x^2 + n_y^2 + n_z^2).$$

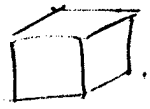
Total points : 35

PHY 632H

Solutions : HW #2

Ans 10.2. Let $|E\rangle$ represent the state vector

$$H|E\rangle = E|E\rangle$$



(7) $$\therefore \left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right)|E\rangle = E|E\rangle$$

In x representation,

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\right)\psi_E(x,y,z) = E\,\psi_E(x,y,z)$$

$$\psi_E(x,y,z) = \psi_{E_1}(x)\,\psi_{E_2}(y)\,\psi_{E_3}(z) \quad \text{.... Ansatz}$$

$$\therefore \psi_{E_2}(y)\,\psi_{E_3}(z)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_{E_1}(x)\right) - \frac{\hbar^2}{2m}\psi_{E_1}(x)\,\psi_{E_3}(z)\frac{\partial^2\psi_{E_2}(y)}{\partial y^2}$$

$$-\frac{\hbar^2}{2m}\psi_{E_1}(x)\,\psi_{E_2}(y)\frac{\partial^2\psi_{E_3}(z)}{\partial z^2} = E\,\psi_{E_1}(x)\,\psi_{E_2}(y)\,\psi_{E_3}(z)$$

$$\therefore \frac{1}{\psi_{E_1}(x)}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi_{E_1}}{\partial x^2}\right) + \frac{1}{\psi_{E_2}(y)}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi_{E_2}}{\partial y^2}\right) + \frac{1}{\psi_{E_3}(z)}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi_{E_3}}{\partial z^2}\right) = E$$

depends only on x | depends only on y | depends only on z.

$\therefore$ Each term is separately equal to a constant

$$\therefore -\frac{\hbar^2}{2m}\frac{\partial^2\psi_{E_1}}{\partial x^2} = E_1\,\psi_{E_1}(x) \;;\; -\frac{\hbar^2}{2m}\frac{\partial^2\psi_{E_2}(y)}{\partial y^2} = E_2\,\psi_{E_2}(y) \ldots$$

$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_{E_1}}{\partial x^2} = E_1 \psi_{E_1}(x)$ & $\psi_{E_1}(x) = 0$ if $x < c$ or $x > L$

↳ we know the solutions for this is

$$\psi_{E_1}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_x \pi x}{L}\right) \quad \& \quad E_1 = \frac{n_x^2 \pi^2 \hbar^2}{2m L^2}$$

III^y $\psi_{E_2}(y) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_y \pi y}{L}\right)$; $E_2 = \frac{n_y^2 \hbar^2 \pi^2}{2m L^2}$

& ~~E~~ $\psi_{E_3}(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_z \pi z}{L}\right)$, $E_3 = \frac{n_z^2 \hbar^2 \pi^2}{2m L^2}$

By our ansatz, $\psi_E(x,y,z) = \psi_{E_1}(x)\, \psi_{E_2}(y)\, \psi_{E_3}(z)$

$$\therefore \psi_E(x,y,z) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_x \pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n_y \pi y}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n_z \pi z}{L}\right)$$

$$n_x, n_y, n_z = 1, 2, 3, \ldots$$

& $E = E_1 + E_2 + E_3$

$$= \frac{\hbar^2 \pi^2}{2m L^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

Ans 10.2-2 $\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \omega_x^2 x^2 + \frac{1}{2} m \omega_y^2 y^2$

$$H|\psi\rangle = E|\psi\rangle$$

In x representation,

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2} m \omega^2 y^2\right) \psi(x,y) = E \psi(x,y)$$

Ansatz : $\psi(x,y) = \psi_x(x)\, \psi_y(y)$

$$\therefore \frac{1}{\psi_x(x)}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega_x^2x^2\right)\psi_x + \frac{1}{\psi_y(y)}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}+\frac{1}{2}m\omega_y^2y^2\right)\psi_y = E$$

E_x $\qquad$ E_y

$(\because E_x + E_y = E)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_x}{\partial x^2}+\frac{1}{2}m\omega_x^2x^2\psi_x = E_x\psi_x$$

$$\& \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi_y}{\partial y^2}+\frac{1}{2}m\omega_y^2y^2\psi_y = E_y\psi_y.$$

We know from Sec 7.3, Shankar that equations as above will admit a solution only if,

$$E_x = (n_x+\tfrac{1}{2})\hbar\omega \quad , n_x = 0,1,2\ldots \qquad (9)$$

$$\& \quad E_y = (n_y+\tfrac{1}{2})\hbar\omega \quad , n_y = 0,1,2,\ldots$$

$$\therefore E = (n_x+\tfrac{1}{2})\hbar\omega + (n_y+\tfrac{1}{2})\hbar\omega \quad ; n_x, n_y = 0,1,2,\ldots$$

(2) $\psi_E(x,y) = \psi_E(x)\psi_y(y)$

$$= \left(\frac{m\omega_x}{\pi\hbar\, 2^{2n_x}(n_x!)^2}\right)^{1/4}\left(\frac{m\omega_y}{\pi\hbar\, 2^{2n_y}(n_y!)^2}\right)^{1/4}\exp\left(-\frac{m\omega_x x^2}{2\hbar}\right)\exp\left(-\frac{m\omega_y y^2}{2\hbar}\right)$$

$$H_{n_x}\left[\left(\frac{m\omega_x}{\hbar}\right)^{1/2}x\right]\, H_{n_y}\left[\left(\frac{m\omega_y}{\hbar}\right)^{1/2}y\right]$$

$$H_{n_x}\left[\left(\frac{m\omega_x}{\hbar}\right)^{1/2}(-x)\right] = (-1)^{n_x}\, H_{n_x}\left[\left(\frac{m\omega_x}{\hbar}\right)^{1/2}x\right]$$

$$H_{n_y}\left[\left(\frac{m\omega_y}{\hbar}\right)^{1/2}(-y)\right] = (-1)^{n_y}\, H_{n_y}\left[\left(\frac{m\omega_y}{\hbar}\right)^{1/2}y\right]$$

Thus $\psi_{E_n}(-x,-y) = (-1)^{n_x+n_y}\ \psi_{E_n}(x,y)$

$= -\psi_{E_n}(x,y)$ if n odd

$= \psi_{E_n}(x,y)$ if n even.

(3) $\psi_{n=0}(x,y) = \left(\frac{m\omega_x}{\pi\hbar 1}\right)^{1/4}\left(\frac{m\omega_y}{\pi\hbar}\right)^{1/4} \exp\left(\frac{-m\omega x^2}{2\hbar}\right)\exp\left(\frac{-m\omega y^2}{2\hbar}\right)\ 1\ 1$ $(\omega_x = \omega_y = \omega)$

$\therefore \psi_{n=0}(x,y) = \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}(x^2+y^2)\right)$

$\therefore \psi_{n=0}(\rho,\phi) = \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}\rho^2\right)$

$\psi_{\substack{n=1\\(n_x,n_y)}}(\rho,\phi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega}{\pi\hbar 2^2}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}(x^2+y^2)\right)\ 1\ (x+y)\left(\frac{m\omega}{\hbar}\right)^{1/2}$

$\psi_{\substack{n=1\\(0,1)}}(\rho,\phi) = \sqrt{\frac{m\omega}{2\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}\rho^2\right)\ 2\rho\sin\phi\ \sqrt{\frac{m\omega}{\hbar}}$

$\psi_{\substack{n=1\\(n_x=1,n_y=0)}}(x,y) = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/2}\exp\left(-\frac{m\omega}{2\hbar}(x^2+y^2)\right)\ 2x\ \left(\frac{m\omega}{\hbar}\right)^{1/2}$

$\therefore \psi_{n=1}(\rho,\phi) = \frac{1}{\sqrt{2\pi}}\left(\frac{m\omega}{\hbar}\right)\exp\left(-\frac{m\omega}{2\hbar}\rho^2\right)\ 2\rho\cos\phi$

$E = (n+1)\hbar\omega = (n_x+\tfrac{1}{2}+n_y+\tfrac{1}{2})\hbar\omega \quad ; n_x+n_y = n$

	n_x	n_y
Total possible ways $(n+1)$	0	n
	1	$n-1$
	2	$n-2$
	⋮	
	n	0

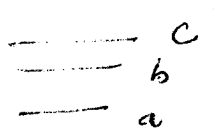
degeneracy is $(n+1)$

Ans 10.3.2 $|\Psi\rangle = N\left[|3,3,4\rangle + |3,4,3\rangle + |4,3,3\rangle\right]$

Using $\langle\Psi|\Psi\rangle = N \;;\; N = \frac{1}{\sqrt{3}}$ (3)

$\therefore |\Psi\rangle = \frac{1}{\sqrt{3}}\left[|3,3,4\rangle + |3,4,3\rangle + |4,3,3\rangle\right]$

Ans 10.3.3



particles are labelled

→ allowed distinct configurations

the possible choices for each particle are 3.

∴ Total number of possible configurations = 3×3×3 = 27

(2) Bosons are indistinguishable

possible distinct configurations are 10

(6)

1) a a a
2) a a b
3) a a c
4) b b b
5) b b a
6) b b c
7) c c a
8) c c b
9) c c c
10) a b c

(3) Fermions are indistinguishable & no two fermions can occupy the same energy level. So there is only one possible configuration, abc.

Ans 10.3.4 For a particle in box, $E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$

$E_{system} = \frac{\pi^2\hbar^2}{mL^2} \Rightarrow E_1 = \frac{\pi^2\hbar^2}{2mL^2}$ & $E_2 = \frac{\pi^2\hbar^2}{2mL^2}$

∴ both particles have same energy, they have to be bosons.

$\therefore |\psi\rangle = |E_1\rangle |E_1\rangle$

$E_{system} = \frac{5\hbar^2\pi^2}{2mL^2} = \frac{4\pi^2\hbar^2}{2mL^2} + \frac{\hbar^2\pi^2}{2mL^2}$

$\therefore$ Possible states, $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|E_1\rangle|E_2\rangle + |E_2\rangle|E_1\rangle\right]$ ... symmetric

or $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|E_1\rangle|E_2\rangle - |E_2\rangle|E_1\rangle\right]$ ... antisymmetric.

Ans 10.3.1 $|S\rangle = N\left[|\phi,\psi\rangle + |\psi,\phi\rangle\right] = N\left[|\phi\rangle\otimes|\psi\rangle + |\psi\rangle\otimes|\phi\rangle\right]$

(3) $\therefore N^2\left(\langle\phi|\otimes\langle\psi| + \langle\psi|\otimes\langle\phi|\right)\left(|\phi\rangle\otimes|\psi\rangle + |\psi\rangle\otimes|\phi\rangle\right) = 1$

$\frac{1}{N^2} = \langle\phi|\phi\rangle\langle\psi|\psi\rangle + \langle\phi|\psi\rangle\langle\psi|\phi\rangle + \langle\psi|\phi\rangle\langle\phi|\psi\rangle + \langle\psi|\psi\rangle\langle\phi|\phi\rangle$

$= 2\langle\phi|\phi\rangle\langle\psi|\psi\rangle + 2\langle\psi|\phi\rangle\langle\phi|\psi\rangle$

$= 2 + 2|\langle\phi|\psi\rangle|^2$ ... (assuming $|\phi\rangle$ & $|\psi\rangle$ are normalized)

$\therefore |S\rangle = \frac{1}{\left[2 + 2|\langle\phi|\psi\rangle|^2\right]^{1/2}}\left[|\phi\rangle\otimes|\psi\rangle + |\psi\rangle\otimes|\phi\rangle\right]$

Ans 10.3.6 Fermions have an half integer spin $n\frac{\hbar}{2}$, n odd
& Bosons have n integral spins; $n\hbar$.

If we have even number of fermions, the total spin will add up to an integer & thus the particle will behave as a boson.

If we have odd number of fermions, the total spin will add up to an half integer & the particle will behave as a fermion.

H atom has a proton (fermion) & an electron (also a fermion). So it is <u>boson</u>.

Physics 828: Homework Set No. 3

Due date: Friday, January 27, 2011, 1:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 60 points + 10 bonus points

Problem 1 (20 pts.): Exercise 10.3.5 (Shankar, p. 278)

Problem 2 (10 pts.): Exercise 10.3.6 (Shankar, p. 278)

Problem 3 (5 pts.): Exercise 11.2.2 (Shankar, p. 283)

Problem 4 (5 pts.): Exercise 11.4.1 (Shankar, p. 300)

Problem 5 (5 pts.): Exercise 11.4.2 (Shankar, p. 300). If you correctly derive in closed form the explicit expression for $[\hat{P}, \hat{H}]$ you receive **10 bonus points**.

Problem 6 (10 pts.): Exercise 11.4.3 (Shankar, p. 300)

Problem 7 (5 pts.): Exercise 11.4.4 (Shankar, p. 300)

Exercise 10.3.5 (p. 278)

- $\hat{P}_{12}|x_1 x_2\rangle = |x_2 x_1\rangle$ defn. of exchange operator

(1) $\hat{P}_{12}^2 |x_1 x_2\rangle = \hat{P}_{12}|x_2 x_1\rangle = |x_1 x_2\rangle \quad \forall x_1, x_2 \Rightarrow \hat{P}_{12}^2 = \mathbb{1}$

$\Rightarrow P_{12} = (P_{12})^{-1}$ ✓

since $\langle x_1' x_2' | \hat{P}_{12}^+ \hat{P}_{12} | x_1 x_2\rangle = \left(\hat{P}_{12}|x_1' x_2'\rangle\right)^+ \hat{P}_{12}|x_1 x_2\rangle$

$= \langle x_2' x_1' | x_2 x_1\rangle = \delta(x_2 - x_2')\,\delta(x_1 - x_1') \quad \forall x_1, x_2$,

we see that $\hat{P}_{12}^+ \hat{P}_{12} = \hat{\mathbb{1}} \Rightarrow \hat{P}_{12}^+ = (\hat{P}_{12})^{-1}$ (unitary) ✓

$\Rightarrow \hat{P}_{12}^+ \hat{P}_{12} = \hat{P}_{12}^2$ and hence $\hat{P}_{12}^+ = \hat{P}_{12}$ (Hermitean) ✓

and

since $\hat{P}_{12}$ is unitary, its eigenvalues have the form

$\lambda = e^{i\alpha}$ $(\alpha \in \mathbb{R})$

since $\hat{P}_{12}$ is Hermitean, its eigenvalues are real.

since $\hat{P}_{12}^2 = \hat{\mathbb{1}}$, all eigenvalues obey $\lambda^2 = 1$

$\Rightarrow P_{12}$ has eigenvalues $\boxed{\lambda_\pm = \pm 1}$ ✓

(2) $\hat{P}_{12}|\omega_1 \omega_2\rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \left(\hat{P}_{12}|x_1 x_2\rangle\right) \langle x_1|\omega_1\rangle \langle x_2|\omega_2\rangle$

$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, |x_2 x_1\rangle \underbrace{\langle x_2|\omega_2\rangle \langle x_1|\omega_1\rangle}_{= \langle x_2 x_1 | \omega_2 \omega_1\rangle} = |\omega_2 \omega_1\rangle$ ✓

$= \hat{\mathbb{1}}$

$\mathbb{V}_S = \{ |\omega_1 \omega_2, S\rangle \mid \omega_1, \omega_2 \in \{\text{eigenvalues of } \hat{S}^2\}\}$

$|\psi_S\rangle \in \mathbb{V}_S$ can be written as

$$|\psi_S\rangle = \sum_{\omega_1 \le \omega_2} c_{\omega_1\omega_2} |\omega_1\omega_2, S\rangle$$

$$\hat{P}_{12}|\psi_S\rangle = \sum_{\omega_1 \le \omega_2} c_{\omega_1\omega_2} \hat{P}_{12}|\omega_1\omega_2, S\rangle$$

$$= \sum_{\omega_1 \le \omega_2} c_{\omega_1\omega_2} (+1) |\omega_1\omega_2, S\rangle$$

$$= |\psi_S\rangle \checkmark$$

← for $\omega_1 = \omega_2$, $\hat{P}_{12}|\omega\omega, S\rangle = \hat{P}_{12}|\omega\omega\rangle = |\omega\omega\rangle = (+1)|\omega\omega\rangle$

for $\omega_1 \neq \omega_2$, $\hat{P}_{12}|\omega_1\omega_2, S\rangle = \frac{1}{\sqrt{2}}\hat{P}_{12}(|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle) = \frac{1}{\sqrt{2}}(|\omega_2\omega_1\rangle + |\omega_1\omega_2\rangle) = (+1)|\omega_1\omega_2, S\rangle$

$|\psi_A\rangle \in \mathbb{V}_A$ can be written as

$$|\psi_A\rangle = \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} |\omega_1\omega_2, A\rangle$$

$$\hat{P}_{12}|\psi_A\rangle = \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} \hat{P}_{12}|\omega_1\omega_2, A\rangle = \frac{1}{\sqrt{2}} \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} \hat{P}_{12}(|\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle)$$

$$= \frac{1}{\sqrt{2}} \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} (|\omega_2\omega_1\rangle - |\omega_1\omega_2\rangle) = \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} (-1) \frac{1}{\sqrt{2}} (|\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle)$$

$$= -|\psi_A\rangle$$

$\Rightarrow \mathbb{V}_S \subset \mathbb{V}_{1\otimes 2}$ is the eigenspace of $\hat{P}_{12}$ with eigenvalue $+1$

$\mathbb{V}_A \subset \mathbb{V}_{1\otimes 2}$ " " " " $\hat{P}_{12}$ " " -1 ✓

(4) For two identical particles

$$\hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2) = \hat{H}(\hat{X}_2, \hat{P}_2; \hat{X}_1, \hat{P}_1)$$ (switching the two particles does not change their total energy)

$$= \hat{P}_{12}\, \hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2)\, \hat{P}_{12}$$

$$\Rightarrow \quad \hat{P}_{12}\, \hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2) = \hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2)\, \hat{P}_{12}$$

$$\Rightarrow \underline{[\hat{P}_{12}, \hat{H}] = 0} \quad \checkmark$$

This holds irrespective of whether or not $\hat{H}$ depends on time t.

Since $\hat{U}(t) = \hat{T} e^{-\frac{i}{\hbar}\int_0^t dt' \hat{H}(t')}$ (time-ordered product)

and $[\hat{P}_{12}, \hat{H}(t')] = 0 \quad \forall\, t'$

$$\Rightarrow \underline{[\hat{P}_{12}, \hat{U}(t)] = 0} \quad \checkmark \qquad (*)$$

If $\hat{P}_{12}|\psi(0)\rangle = \lambda|\psi(0)\rangle$, $\lambda = \pm 1$, at $t = 0$,

then

$$\underline{\hat{P}_{12}|\psi(t)\rangle} = \hat{P}_{12}\hat{U}(t)|\psi(0)\rangle \overset{(*)}{=} \hat{U}(t)\hat{P}_{12}|\psi(0)\rangle$$

$$= \hat{U}(t)\lambda|\psi(0)\rangle = \underline{\lambda|\psi(t)\rangle}$$

i.e. $|\psi(t)\rangle$ is an eigenstate of $\hat{P}_{12}$ with eigenvalue λ at at all times.

$\Rightarrow$ Symmetric states always remain in $\mathcal{H}_S$,
antisymmetric " " " $\mathcal{H}_A$ ✓

(3) For a general observable $\hat{\Omega}$ with eigenstates $|\omega_i\rangle$ and eigenvalues ω_i we have

$$\hat{P}_{12}\hat{\Omega}_1^{1\otimes 2}\hat{P}_{12}|\omega_1\omega_2\rangle = \hat{P}_{12}(\hat{\Omega}_1\times\hat{\mathbb{1}}_2)|\omega_2\omega_1\rangle = \hat{P}_{12}(\omega_2|\omega_2\omega_1\rangle$$

$$= \omega_2\hat{P}_{12}|\omega_2\omega_1\rangle = \omega_2|\omega_1\omega_2\rangle = \Omega_2^{1\otimes 2}|\omega_1\omega_2\rangle \quad \forall\,\omega_1,\omega_2$$

$$\Rightarrow \underline{\hat{P}_{12}\hat{\Omega}_1^{1\otimes 2}\hat{P}_{12} = \hat{\Omega}_2^{1\otimes 2}}$$

Similarly

$$\hat{P}_{12}\hat{\Omega}_2^{1\otimes 2}\hat{P}_{12}|\omega_1\omega_2\rangle = \hat{P}_{12}(\mathbb{1}_1\otimes\hat{\Omega}_2)|\omega_2\omega_1\rangle = \omega_1\hat{P}_{12}|\omega_2\omega_1\rangle$$

$$= \omega_1|\omega_1\omega_2\rangle = \hat{\Omega}_1^{1\otimes 2}|\omega_1\omega_2\rangle \quad \forall\,\omega_1,\omega_2$$

$$\Rightarrow \underline{\hat{P}_{12}\hat{\Omega}_2^{1\otimes 2}\hat{P}_{12} = \hat{\Omega}_1^{1\otimes 2}}$$

These hold in particular for $\hat{\Omega}=\hat{x}$ and $\hat{\Omega}=\hat{P}$ ✓

Now consider

$$\underline{\hat{P}_{12}\hat{\Omega}(\hat{X}_1,\hat{P}_1;\hat{X}_2,\hat{P}_2)\hat{P}_{12}} = \hat{P}_{12}\Big(\sum_{n_1,\dots n_4=0}^{\infty} C_{n_1n_2n_3n_4}\hat{X}_1^{n_1}\hat{P}_1^{n_2}\hat{X}_2^{n_3}\hat{P}_2^{n_4}\Big)\hat{P}_{12}$$

$$= \sum_{n_1\dots n_4=0}^{\infty} C_{n_1n_2n_3n_4}\,\hat{P}_{12}\hat{X}_1^{n_1}\underbrace{P_{12}P_{12}}_{\hat{\mathbb{1}}}\hat{P}_1^{n_2}\hat{P}_{12}\underbrace{\hat{P}_{12}\hat{X}_2^{n_3}\hat{P}_{12}}_{=\hat{X}_1^{n_3}}\underbrace{\hat{P}_{12}\hat{P}_2^{n_4}\hat{P}_{12}}_{=\hat{P}_1^{n_4}}$$

$$= \underbrace{\hat{P}_{12}\hat{X}_1\hat{P}_{12}}_{=\hat{P}_2^{n_2}}\underbrace{\hat{P}_{12}\hat{X}_1P_{12}}\cdots\underbrace{\hat{P}_{12}\hat{X}_1\hat{P}_{12}}_{=\hat{X}_2^{n_1}}$$

(n_1 factors)

$$= \sum_{n_1\dots n_4=0}^{\infty} C_{n_1n_2n_3n_4}\hat{X}_2^{n_1}\hat{P}_2^{n_2}\hat{X}_1^{n_3}\hat{P}_1^{n_4} = \underline{\hat{\Omega}(\hat{X}_2,\hat{P}_2;\hat{X}_1,\hat{P}_1)}$$ ✓

Exercise 10.3.6 (p. 278)

Consider a state consisting of two identical particles P positioned at x_1^P and x_2^P:

$$|\psi\rangle = |x_1^P x_2^P\rangle$$

If each particle consist of N fermions, the position eigenstates for the particle P can be expanded into a complete set of position eigenstates for each fermion:

$$|x^P\rangle = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \, C(y_1, \ldots, y_N) \, |y_1 y_2 \cdots y_N\rangle$$

We don't know to what extent the states $|y_1, y_2, \ldots, y_N\rangle$ must be antisymmetrized, since we are not told whether the N fermions are all distinguishable species or some of them are identical. But our answer will not depend on this.

With this expansion the state $|\psi\rangle$ can be written as

$$|\psi\rangle = \int_{-\infty}^{\infty} dy_1 \, dy_2 \cdots dy_N \int_{-\infty}^{\infty} dz_1 \, dz_2 \cdots dz_N \, C_1(y_1 \cdots y_N) \, C_2(z_1 \cdots z_N) \cdot$$

$$\cdot \, |y_1 y_2 \cdots y_N\rangle \otimes |z_1 z_2 \cdots z_N\rangle$$

$$= \left(|y_1\rangle \otimes |y_2\rangle \otimes \cdots \otimes |y_n\rangle\right) \otimes \left(|z_1\rangle \otimes |z_2\rangle \otimes \cdots \otimes |z_N\rangle\right)$$

$|y_1\rangle \in \mathbb{V}_1$ for fermion 1; $|y_n\rangle \in \mathbb{V}_1$ for fermion N; $|z_1\rangle \in \mathbb{V}_2$ for fermion 1; $|z_N\rangle \in \mathbb{V}_2$ for fermion N

Let us reorder the factors on the right hand side as

$$\left(|y_1\rangle \otimes |z_1\rangle\right) \otimes \left(|y_2\rangle \otimes |z_2\rangle\right) \otimes \ldots \otimes \left(|y_N\rangle \otimes |z_N\rangle\right) \equiv$$
$$\equiv |y_1 z_1\rangle |y_2 z_2\rangle \ldots |y_N z_N\rangle$$

and write the expansion in the form

$$|\psi\rangle = \int_{-\infty}^{\infty} dy_1 dz_1 \int_{-\infty}^{\infty} dy_2 dz_2 \int \ldots \int_{-\infty}^{\infty} dy_N dz_N \; \tilde{C}(y_1 z_1; y_2 z_2; \ldots; y_N z_N) \cdot$$
$$\cdot |y_1 z_1\rangle |y_2 z_2\rangle \ldots |y_N z_N\rangle \qquad (*)$$

Under interchange of the two composite particles the states $|y_1 z_1\rangle |y_2 z_2\rangle \ldots |y_N z_N\rangle$ go into $|z_1 y_1\rangle |z_2 y_2\rangle \ldots |z_N y_N\rangle$ since all of the constituent fermions get interchanged. Since all constituents are fermions, each of the states $|y_k z_k\rangle$ $(k = 1, \ldots, N)$ must really be antisymmetrized. For this to be true, the expansion coefficients $\tilde{C}(y_1 z_1; \ldots; y_N z_N)$ must have appropriate symmetries under the exchange $y_k \leftrightarrow z_k$ such that the expansion $(*)$ takes the form

$$|\psi\rangle = \int_{y_1 < z_1} dy_1 dz_1 \int_{y_2 < z_2} dy_2 dz_2 \ldots \int_{y_N < z_N} dy_N dz_N \; \bar{C}(y_1 z_1; y_2 z_2; \ldots; y_N z_N) \cdot$$
$$\cdot |y_1 z_1, A\rangle |y_2 z_2, A\rangle \ldots |y_N z_N, A\rangle$$

What matters is the antisymmetry between the fermions in particle 1 and their identical partners in particle 2, not the antisymmetry between possibly

identical constituents inside each of the composite particles.

The state in which the two composite particles are interchanged is given by ($\hat{P}_{12}^{(k)}$ = exchange operator for two particles of type k)

$$\hat{P}_{12}|\psi\rangle = \int_{y_1<z_1} dy_1 dz_1 \cdots \int_{y_N<z_N} dy_N dz_N \, \bar{C}(y_1 z_1; \cdots; y_N, z_N) \cdot$$

$$\cdot \underbrace{\left(\hat{P}_{12}^{(1)}|y_1 z_1, A\rangle\right)}_{= |z_1 y_1, A\rangle = -|y_1 z_1; A\rangle} \left(\hat{P}_{12}^{(2)}|y_2 z_2, A\rangle\right) \cdots \underbrace{\left(\hat{P}_{12}^{(N)}|y_N z_N, A\rangle\right)}_{= -|y_N z_N, A\rangle}$$

$$= (-1)^N |\psi\rangle$$

So $|\psi\rangle$ is an eigenstate of $\hat{P}_{12}$ with eigenvalue

$$(-1)^N = \begin{cases} +1 & N \text{ even}, \\ -1 & N \text{ odd}. \end{cases}$$

Hence a composite particle consisting of an even/odd number of fermions behaves like a boson/fermion ✓

In particular, hydrogen atoms behave like bosons, deuterium behaves like a fermion, tritium like a boson but ^{3}He like a fermion. ^{6}Li atoms are fermions.

Exercise 11.2.2 (p.283)

$$\hat{T}^{+}(\varepsilon)\hat{T}(\varepsilon) = \left(\mathbb{1} + \frac{i\varepsilon}{\hbar}\hat{G}^{+} + O(\varepsilon^2)\right)\left(\mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{G} + O(\varepsilon^2)\right)$$

$$= \mathbb{1} + \frac{i\varepsilon}{\hbar}\left(\hat{G}^{+} - \hat{G}\right) + O(\varepsilon^2)$$

Since $\hat{T}(\varepsilon)|x\rangle = |x+\varepsilon\rangle$ is normalized,

$$\langle x'|x\rangle = \delta(x-x') = \langle x'+\varepsilon|x+\varepsilon\rangle = \langle x'|\hat{T}^{+}(\varepsilon)\hat{T}(\varepsilon)|x\rangle \quad \forall\, x, x',$$

we have $\hat{T}(\varepsilon)\hat{T}(\varepsilon) = \hat{\mathbb{1}}$ exactly $(\forall \varepsilon)$.

$$\Rightarrow \hat{G}^{+} - \hat{G} = 0 \quad \Rightarrow \boxed{\hat{G}^{+} = \hat{G}}$$

Exercise 11.4.1 (p. 300)

At time $t=0$ we have by assumption

$$\hat{\pi}|\psi(0)\rangle = \lambda|\psi(0)\rangle \qquad \text{with } \lambda = \pm 1.$$

$$\Rightarrow \hat{\pi}|\psi(t)\rangle = \hat{\pi}\,\hat{u}(t)|\psi(0)\rangle$$

Since $[\hat{\pi}, \hat{H}] = 0 \;\; \forall t$ by assumption,

$$[\hat{\pi}, \hat{u}(t)] = [\hat{\pi}, \hat{T}e^{-i/\hbar \int_0^t dt' \hat{H}(t')}] = 0$$

$$\Rightarrow \hat{\pi}\,\hat{u}(t) = \hat{u}(t)\hat{\pi} \quad \text{and}$$

$$\hat{\pi}|\psi(t)\rangle = \hat{u}(t)\,\hat{\pi}|\psi(0)\rangle = \hat{u}(t)\left(\lambda|\psi(0)\rangle\right)$$

$$= \lambda|\psi(t)\rangle$$

$\Rightarrow |\psi(t)\rangle$ remains a parity eigenstate with the same eigenvalue at all times.

Exercise 11.4.2 (p. 300)

$$\hat{H} = \frac{\hat{P}^2}{2m} + V_0 \sin\left(\frac{2\pi}{a}\hat{X}\right).$$

For momentum conservation to hold,

$$\frac{d}{dt}\langle \hat{P} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{P}] \rangle = 0 \qquad \forall \text{ states}$$

we need $[\hat{H}, \hat{P}] = 0$.

However, here

$$[\hat{H}, \hat{P}] = \left[\frac{\hat{P}^2}{2m} + V_0 \sin\left(\frac{2\pi}{a}\hat{X}\right), \hat{P}\right] = V_0 \left[\sin\left(\frac{2\pi}{a}\hat{X}\right), \hat{P}\right] \neq 0$$

hence momentum is not conserved.

(
$$\sin\left(\frac{2\pi}{a}\hat{X}\right) = \frac{2\pi}{a}\hat{X} - \frac{1}{3!}\left(\frac{2\pi}{a}\right)^3 \hat{X}^3 + \frac{1}{5!}\left(\frac{2\pi}{a}\right)^5 \hat{X}^5$$

$$[\hat{X}^n, \hat{P}] = \hat{X}[\hat{X}^{n-1}, \hat{P}] + [\hat{X}, \hat{P}]\hat{X}^{n-1} = \hat{X}[\hat{X}^{n-1}, \hat{P}] + i\hbar\hat{X}^{n-1}$$

$$[\hat{X}^2, \hat{P}] = 2i\hbar\hat{X}$$

$$[\hat{X}^3, \hat{P}] = \hat{X}(2i\hbar\hat{X}) + i\hbar\hat{X}^2 = 3i\hbar\hat{X}^2$$

$$\Rightarrow [\hat{X}^n, \hat{P}] = \hat{X}(n-1)i\hbar X^{n-2} + i\hbar\hat{X}^{n-1} = ni\hbar X^{n-1}$$

$$\Rightarrow \left[\sin\left(\frac{2\pi}{a}\hat{X}\right), \hat{P}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{2\pi}{a}\right)^{2n+1} [\hat{X}^{2n+1}, \hat{P}]$$

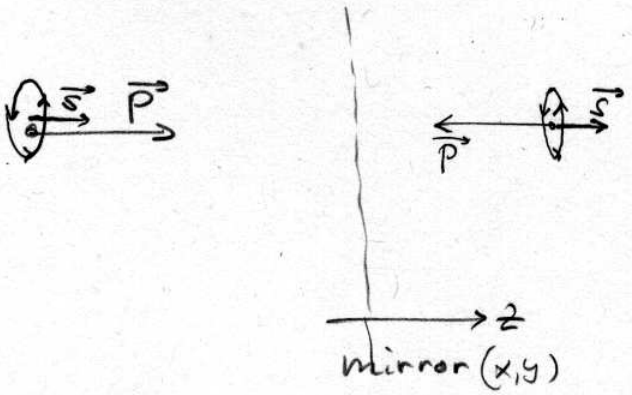
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{2\pi}{a}\right)^{2n+1} (2n+1) i\hbar \hat{X}^{2n}$$

$$= i\hbar \frac{2\pi}{a} \sum_{n=0}^{\infty} \frac{(-)^n}{2n!}\left(\frac{2\pi\hat{X}}{a}\right)^n = i\hbar \frac{2\pi}{a}\cos\left(\frac{2\pi}{a}\hat{X}\right)$$

$$\Rightarrow [\hat{P}, \hat{H}] = -[\hat{H}, \hat{P}] = -i\hbar \frac{2\pi V_0}{a}\cos\left(\frac{2\pi}{a}\hat{X}\right) = -i\hbar \frac{\partial \hat{V}(\hat{X})}{\partial \hat{X}}.$$
)

Exercise 11.4.3 (p. 300)

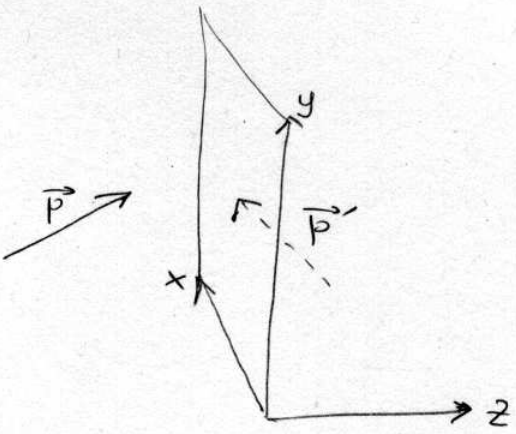
We are told that in a certain reaction the electron always comes out with its spin pointing in its direction of motion, $\vec{s} \parallel \vec{p}$. Let us imagine putting a mirror in the plane $\perp \vec{p}$:



If we imagine the spin of the electron (which is associated with a magnetic moment $\vec{\mu} \parallel \vec{s}$) as being caused by some imaginary electric ring current, we see that in the mirror that current remains unchanged, so the spin of the electron in the mirror still points in the $+z$ direction although it moves in the $-z$ direction. So the electron in the mirror represents an impossible outcome of the reaction under consideration, so the mechanism causing this reaction is not reflection symmetric. It therefore violates parity.

Exercise 11.4.4 (p. 300)

Verify that in 3-d reflection on a mirror in the x-y plane equals a parity transformation followed by a rotation by π about the z-axis.



Consider a polar vector $\vec{p} = (p_x, p_y, p_z)$

Under mirror reflection it turns into

$$\vec{p}\,' = (p_x, p_y, -p_z)$$

Under a parity transformation $\vec{p}$ turns into

$$\hat{\pi}\,\vec{p} = -\vec{p} = (-p_x, -p_y, -p_z)$$

A rotation R_z^z by an angle φ around the z-axis turns the complex number $p_\perp = p_x + i p_y$ into

$$e^{i\varphi} p_\perp = (\cos\varphi + i\sin\varphi)(p_x + i p_y) = (p_x\cos\varphi - p_y\sin\varphi) + i(p_x\sin\varphi + p_y\cos\varphi)$$

For $\varphi = \pi$ this means $p_\perp = (p_x, p_y) \longrightarrow e^{i\pi} p_\perp = -p_\perp = -(p_x, p_y)$

Hence $\vec{p}\,' = (p_x, p_y, -p_z) \xrightarrow{R_\pi^z} (-p_x, -p_y, -p_z) = -\vec{p} = \hat{\pi}\,\vec{p}$ ✓

P4410 PROBLEM SET 5 SOLUTIONS

(1) 11.2.1

$$\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle$$

$$= \int dx' \int dx'' \, \langle \psi | x' \rangle \langle x' | \hat{p} | x'' \rangle \langle x'' | \psi \rangle$$

$$= \int dx' \int dx'' \, \psi^*(x') (-i\hbar) \delta(x'-x'') \frac{d}{dx''} \psi(x'') = -i\hbar \int dx' \psi^*(x') \frac{d}{dx'} \psi(x')$$

$$\langle \psi | \hat{T}^\dagger \hat{p} \hat{T} | \psi \rangle = \int dx' \int dx'' \langle \psi | x' \rangle \langle x' | \hat{T}^\dagger \hat{p} \hat{T}^\dagger | x'' \rangle \langle x'' | \psi \rangle$$

$$= \int dx' \int dx'' \, \psi^*(x') \langle x'+\varepsilon | e^{-i\varepsilon g(x')/\hbar} \hat{p} e^{i\varepsilon g(x'')/\hbar} | x''+\varepsilon \rangle \psi(x'')$$

$$= \int dx' \int dx'' \, \psi^*(x') e^{-i\varepsilon g(x')/\hbar} (-i\hbar) \delta((x'+\varepsilon)-(x''+\varepsilon)) \frac{d}{dx''} e^{i\varepsilon g(x'')/\hbar} \psi(x'')$$

$$= \int dx' \, \psi^*(x') e^{-i\varepsilon g(x')/\hbar} (-i\hbar) \frac{d}{dx'} e^{i\varepsilon g(x')/\hbar} \psi(x')$$

$$= \int dx' \, \psi^*(x') e^{-i\varepsilon g(x')/\hbar} (-i\hbar) \left\{ \left[\frac{i\varepsilon}{\hbar} \frac{dg(x')}{dx'} e^{i\varepsilon g(x')/\hbar} \psi(x') \right] + \left[e^{i\varepsilon g(x')/\hbar} \frac{d\psi(x')}{dx'} \right] \right\}$$

$$= \left(-i\hbar \int dx' \, \psi^*(x') \frac{i\varepsilon}{\hbar} f(x') \psi(x') \right) + \left(i\hbar \int dx' \, \psi^*(x') \frac{d}{dx'} \psi(x') \right)$$

$$= \varepsilon \langle f(x) \rangle + \langle p \rangle \qquad \text{QED}$$

(2) 11.4.1 $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$

Recall that $i\hbar\frac{d\hat{U}}{dt} = \hat{H}\hat{U} \Rightarrow [\hat{P}, \hat{U}(t)] = 0$ if $[\hat{P}, \hat{H}] = 0$

so, if $\hat{P}|\psi(0)\rangle = \pm|\psi(0)\rangle$ then

$$\hat{P}|\psi(t)\rangle = \hat{P}\hat{U}|\psi(0)\rangle$$
$$= \hat{U}\hat{P}|\psi(0)\rangle$$
$$= \pm\hat{U}|\psi(0)\rangle$$
$$= \pm|\psi(t)\rangle$$

QED $\Rightarrow$ Parity of state is unchanged.

11.4.2) Particle in potential $V(x) = V_0 \sin\left(\frac{2\pi x}{a}\right)$

Is momentum conserved?

$$i\hbar \frac{d}{dt}\langle p \rangle = \langle [p, H] \rangle$$

$$= \left\langle \left[p, \frac{p^2}{2m} + V(x)\right] \right\rangle$$

$$= \left\langle \left[p, \frac{p^2}{2m}\right] + [p, V(x)] \right\rangle \quad \text{let } |\omega\rangle \text{ be test state}$$

$$= \left\langle 0 + \frac{\hbar}{i}\frac{d}{dx}\left(V_0 \sin\frac{2\pi x}{a}\right)|\omega\rangle - \left(V_0 \sin\frac{2\pi x}{a}\right)\frac{\hbar}{i}\frac{d}{dx}|\omega\rangle \right\rangle$$

$$= \frac{\hbar}{i} V_0 \left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi x}{a}\right)|\omega\rangle + \frac{\hbar}{i} V_0 \sin\left(\frac{2\pi x}{a}\right)\frac{d}{dx}|\omega\rangle - V_0 \sin\frac{2\pi x}{a}\frac{\hbar}{i}\frac{d}{dx}|\omega\rangle \rangle$$

$$= \frac{\hbar}{i} V_0 \frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right) \neq 0$$

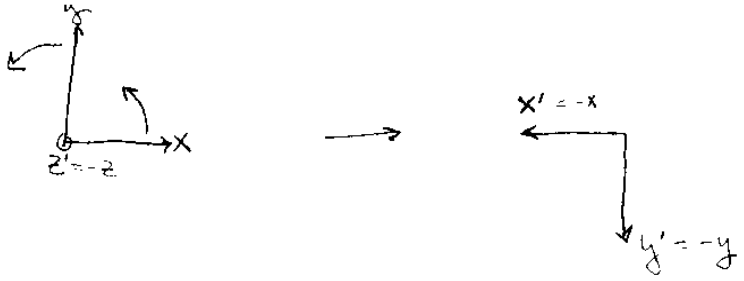
Not conserved.

11.4.3 In a certain reaction an electron comes out with its spin always parallel to its momentum. If we look at this reaction under parity the spin changes direction but the direction of the emitted particle does not. That is, the spin and the emitted particle are antiparallel under parity. But we are told this process never happens. Thus parity is violated.

(5) 11.4.4

First, reflect about x-y plane: $(x,y,z) \to (x,y,-z)$

Now rotate by π about z axis:



so $(x,y,-z) \to (-x,-y,-z)$ i.e. the same as $\hat{P}(x,y,z)$.

(6) 12.3.4 Express azimuthal part of wave function as:

$$\left(\frac{\rho}{\Delta}\cos\phi + \sin\phi\right) = \left[\frac{\rho}{2\Delta}\left(e^{i\phi}+e^{-i\phi}\right) + \frac{1}{2i}\left(e^{i\phi}-e^{-i\phi}\right)\right]$$

$$= e^{i\phi}\underbrace{\left(\frac{\rho}{2\Delta}-\frac{i}{2}\right)}_{=C^*} + e^{-i\phi}\underbrace{\left(\frac{\rho}{2\Delta}+\frac{i}{2}\right)}_{=C}$$

$$\langle\phi|\psi\rangle = C^*\langle\phi|m=1\rangle + C\langle\phi|m=-1\rangle$$

$$\Rightarrow |\psi\rangle = C^*|1\rangle + C|-1\rangle$$

So $P(m=1) = \frac{|C^*|^2}{|C^*|^2+|C|^2} = \frac{1}{2}$

$$P(m=-1) = \frac{|C|^2}{|C^*|^2+|C|^2} = \frac{1}{2}$$

12.2.3) $L_z \xrightarrow[\text{basis.}]{\text{coordinate}} x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)$

Transforming to polar coordinates:

$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad \tan\phi = \frac{y}{x}$

$\frac{\partial}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial y} = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}$

So, $\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\tan\phi = \frac{\partial}{\partial x}\frac{y}{x}$ and $\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\tan\phi = \frac{\partial}{\partial y}\frac{y}{x}$

$\frac{\partial\phi}{\partial x}\sec^2\phi = \frac{y}{-x^2}$ so $\frac{\partial\phi}{\partial y}\sec^2\phi = \frac{1}{x}$

$\frac{\partial\phi}{\partial x} = \frac{y\cos^2}{-x^2}$ $\qquad$ $\frac{\partial\phi}{\partial y} = \frac{\cos^2\phi}{x}$

Now plugging in,

$L_z \to x\left(-i\hbar\frac{\cos^2\phi}{x}\frac{\partial}{\partial\phi}\right) - y\left(+i\hbar\frac{y\cos^2\phi}{x^2}\frac{\partial}{\partial\phi}\right)$

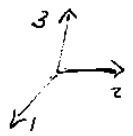
$= \cos^2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right) - \frac{y^2}{x^2}\cos^2\left(+i\hbar\frac{\partial}{\partial\phi}\right)$

$= \cos^2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right) + \sin^2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right)$

$= \left(-i\hbar\frac{\partial}{\partial\phi}\right)$ ✓

T2.4.1 $\vec{c} = \vec{a} \times \vec{b}$, Eq (12.4.8)

$$\vec{c} = \begin{vmatrix} 1 & 2 & 3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



$$\boxed{\vec{c} = (a_2 b_3 - a_3 b_2)1 + (a_3 b_1 - a_1 b_3)2 + (a_1 b_2 - a_2 b_1)3}$$

And,

$$c_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k, \quad \text{Eq (12.4.9)}$$

$$= \sum_{j=1}^{3} (\varepsilon_{ij1} a_j b_1 + \varepsilon_{ij2} a_j b_2 + \varepsilon_{ij3} a_j b_3)$$

$$= \varepsilon_{i11} a_1 b_1 + \underset{-1}{\varepsilon_{i21}} a_2 b_1 + \underset{+}{\varepsilon_{i31}} a_3 b_1 + \underset{+}{\varepsilon_{i12}} a_1 b_2 + \varepsilon_{i22} a_2 b_2$$

$$+ \underset{-1}{\varepsilon_{i32}} a_3 b_2 + \underset{-1}{\varepsilon_{i13}} a_1 b_3 + \underset{+}{\varepsilon_{i23}} a_2 b_3 + \varepsilon_{i33} a_3 b_3$$

$\Rightarrow$ $c_1 = a_2 b_3 - a_3 b_2$, $c_2 = a_3 b_1 - a_1 b_3$, $c_3 = a_1 b_2 - a_2 b_1$

$$\vec{c} = c_1 1 + c_2 2 + c_3 3$$

$$\boxed{\vec{c} = (a_2 b_3 - a_3 b_2)1 + (a_3 b_1 - a_1 b_3)2 + (a_1 b_2 - a_2 b_1)3}$$

Thus Eq (12.4.8) is equivalent to Eq (12.4.9)

Homework 11

Shanshan Cao

11.2.2

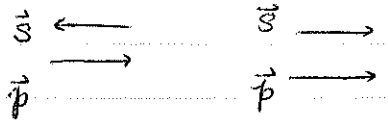
$$I = T^+(\varepsilon)T(\varepsilon) = (I + \tfrac{i\varepsilon}{\hbar}G^+)(I - \tfrac{i\varepsilon}{\hbar}G)$$
$$= I + \tfrac{i\varepsilon}{\hbar}(G^+ - G) + O(\varepsilon^2)$$
$$\Rightarrow \tfrac{i\varepsilon}{\hbar}(G^+ - G) = 0$$
$$\Rightarrow G^+ = G.$$

11.4.1

$$\psi(t) = U(t)\psi(0)$$
$$U(t) = e^{-i\frac{H}{\hbar}t} = \sum_{n=0}^{\infty} \frac{1}{n!}(-i\tfrac{H}{\hbar}t)^n$$
Since $[\Pi, H] = 0$, $[\Pi, U(t)] = 0$.
Therefore, $\Pi\psi(t) = \Pi U(t)\psi(0) = U(t)\Pi\psi(0)$.
If $\Pi\psi(0) = \lambda\psi(0)$, $\lambda = \pm 1$, we have
$$\Pi\psi(t) = U(t)[\lambda\psi(0)] = \lambda U(t)\psi(0) = \lambda\psi(t)$$
$\Rightarrow$ A system that starts out in a state of even / odd parity maintains its parity.

11.4.3

$$\Pi\vec{S} = \vec{S}, \quad \Pi\vec{p} = -\vec{p}$$
$$\Rightarrow \Pi(\vec{S}\cdot\vec{p}) = -\vec{S}\cdot\vec{p}$$
Therefore, we should have $-\vec{S}\cdot\vec{p}$ and $\vec{S}\cdot\vec{p}$ with equal rate if



parity is a good symmetry. If $\vec{p} \parallel \vec{S}$, $\vec{S}\cdot\vec{p}$ is positive only, parity must be violated.

12.2.1.

$$U[R] = I - \frac{i\epsilon_z L_z}{\hbar} \quad \text{when } \epsilon_z \text{ is small.}$$

$$\langle x,y|U[R]|\psi\rangle = \iint dx'dy' \langle x,y|U[R]|x',y'\rangle\langle x',y'|\psi\rangle$$

$$= \iint dx'dy' \langle x,y|x'-y'\epsilon_z, x'\epsilon_z+y'\rangle\langle x',y'|\psi\rangle \quad (*)$$

Change of variables:

$$\begin{cases} r = x' - y'\epsilon_z \\ s = x'\epsilon_z + y' \end{cases} \Rightarrow \begin{cases} x' = \frac{1}{1+\epsilon_z^2}(r+s\epsilon_z) \approx r+s\epsilon_z \\ y' = \frac{1}{1+\epsilon_z^2}(s-r\epsilon_z) \approx s-r\epsilon_z \end{cases}$$

Jacobian: $\left|\frac{\partial(x',y')}{\partial(r,s)}\right| = 1+\epsilon_z^2 \approx 1$

Therefore,

$$(*) = \iint dr ds \langle x,y|r,s\rangle\langle r+s\epsilon_z, s-r\epsilon_z|\psi\rangle$$

$$= \iint dr ds\, \delta(x-r)\delta(y-s)\langle r+s\epsilon_z, s-r\epsilon_z|\psi\rangle$$

$$= \langle x+y\epsilon_z, y-x\epsilon_z|\psi\rangle$$

$$= \psi(x+y\epsilon_z, y-x\epsilon_z).$$

12.2.2.

First check the consistency.

$$[x, L_z] = [x, xp_y - yp_x] = -y[x,p_x] = -i\hbar y$$

$$[y, L_z] = [y, xp_y - yp_x] = x[y,p_y] = i\hbar x$$

$$[p_x, L_z] = [p_x, xp_y - yp_x] = [p_x,x]p_y = -i\hbar p_y$$

$$[p_y, L_z] = [p_y, xp_y - yp_x] = -[p_y,y]p_x = i\hbar p_x$$

Let's derive the form of L_z from the commutation relations.

$$[x, L_z] = i\hbar\frac{\partial L_z}{\partial p_x} = -i\hbar y \Rightarrow \frac{\partial L_z}{\partial p_x} = -y \quad ①$$

$$[y, L_z] = i\hbar\frac{\partial L_z}{\partial p_y} = i\hbar x \Rightarrow \frac{\partial L_z}{\partial p_y} = x \quad ②$$

$$[p_x, L_z] = -i\hbar\frac{\partial L_z}{\partial x} = -i\hbar p_y \Rightarrow \frac{\partial L_z}{\partial x} = p_y \quad ③$$

$$[p_y, L_z] = -i\hbar\frac{\partial L_z}{\partial y} = i\hbar p_x \Rightarrow \frac{\partial L_z}{\partial y} = -p_x \quad ④$$

Set $L_z = L_z(x,y,p_x,p_y)$

① $\Rightarrow L_z = -yp_x + f_1(x,y,p_y)$

② $\Rightarrow \frac{\partial L_z}{\partial p_y} = \frac{\partial f_1}{\partial p_y} = x \Rightarrow f_1 = xp_y + f_2(x,y)$

$\Rightarrow L_z = -yp_x + xp_y + f_2(x,y)$

③ $\Rightarrow \frac{\partial L_z}{\partial x} = p_y + \frac{\partial f_2}{\partial x} = p_y \Rightarrow$ f_2 is not a function of x

④ $\Rightarrow \frac{\partial L_z}{\partial y} = -p_x + \frac{\partial f_2}{\partial y} = -p_x \Rightarrow$ f_2 is not a function of y

Therefore, f_2 can only be a const.

Hence, $L_z = -yp_x + xp_y + \text{Const.}$

12.2.3

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases}$$

$$\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} = -\rho\sin\phi\frac{\partial}{\partial x} + \rho\cos\phi\frac{\partial}{\partial y}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Since $L_z = x(-i\hbar\frac{\partial}{\partial y}) - y(-i\hbar\frac{\partial}{\partial x})$

$= -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$,

we have

$$L_z = -i\hbar\frac{\partial}{\partial\phi}.$$

Or directly,

$$(1 - i\frac{\delta\phi}{\hbar}L_z)\psi(\rho,\phi) = \psi(\rho,\phi-\delta\phi)$$

$$= \psi(\rho,\phi) + \frac{\partial\psi}{\partial\phi}(-\delta\phi) + O[(\delta\phi)^2]$$

$$\Rightarrow -i\frac{\delta\phi}{\hbar}L_z\psi(\rho,\phi) = \frac{\partial\psi}{\partial\phi}(-\delta\phi)$$

$$\Rightarrow L_z\psi(\rho,\phi) = -i\hbar\frac{\partial\psi}{\partial\phi}$$

$$\Rightarrow L_z = -i\hbar\frac{\partial}{\partial\phi}.$$

12.3.5

$$[-\frac{\hbar^2}{2\mu}(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}) + V(\rho)]R_{Em}(\rho) = ER_{Em}(\rho)$$

↑ generated by angular momentum

$$V_{\text{arg.}} = \frac{\hbar^2m^2}{2\mu\rho^2}$$

$$\vec{F} = -\nabla V_{\text{arg.}} = \frac{\hbar^2m^2}{\mu\rho^3}\hat{\rho} \quad \text{(repulsive force)}$$

Since $L_z = m\hbar$, we have $F = \frac{L_z^2}{\mu\rho^3}$.

12.6.1

(1) $\psi_E(r,\theta,\phi) = Ae^{-r/a_0}$ does not depend on θ, ϕ. Therefore its angular momentum is zero.

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$
$$\hat{L}^2\psi_E = 0 \quad \Rightarrow \quad \ell = 0.$$

(2)

$$\hat{H}\psi_E = \left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r)\right]\psi_E = E\psi_E \quad \text{(no angular part)}$$

Substitute $\psi_E = Ae^{-r/a_0}$, we have

$$EAe^{-r/a_0} = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\frac{A}{a_0}e^{-r/a_0}\right) + V(r)Ae^{-r/a_0}$$
$$\Rightarrow \quad Ee^{-r/a_0} = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\left(-2r\frac{1}{a_0}e^{-r/a_0} + \frac{r^2}{a_0^2}e^{-r/a_0}\right) + V(r)e^{-r/a_0}$$
$$\Rightarrow \quad E = \frac{\hbar^2}{a_0\mu r} - \frac{\hbar^2}{2\mu a_0^2} + V(r) \qquad (*)$$

In the limit of $r\to\infty$, $\frac{\hbar^2}{a_0\mu r}\to 0$, $V(r)\to 0$, we have

$$E = -\frac{\hbar^2}{2\mu a_0^2}.$$

(3) If r is finite, substitute $E = -\frac{\hbar^2}{2\mu a_0^2}$ into $(*)$, we have

$$\frac{\hbar^2}{a_0\mu r} + V(r) = 0$$
$$\Rightarrow \quad V(r) = -\frac{\hbar^2}{a_0\mu r}.$$

12.6.4

(1) $\int f(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')\,d\vec{r}' = f(\vec{r})$

In Cartesian Coordinate, we have,

$$\int f(x',y',z')\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')\,dx'dy'dz' = f(x,y,z) = f(\vec{r}).$$

In Spherical Coordinate, we have,

$$\int f(r',\theta',\phi')\frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')\,r'^2\sin\theta'\,dr'd\theta'd\phi'$$
$$= \frac{1}{r^2\sin\theta}f(r,\theta,\phi)\,r^2\sin\theta$$
$$= f(r,\theta,\phi)$$
$$= f(\vec{r})$$

Therefore, $\delta^3(\vec{r}-\vec{r}') \equiv \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$

(2) If $r \neq 0$, we have

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\cdot\frac{1}{r}\right)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}(-1)$$
$$= 0.$$

For $r \to 0$, consider a small sphere centered at the origin.

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = \int_V \nabla\cdot\left(\nabla\frac{1}{r}\right) dV$$
$$= \int_S \left(\nabla\frac{1}{r}\right)\cdot d\vec{S} \qquad \text{(Gauss's law)}$$
$$= \int \left(\nabla\frac{1}{r}\right)\cdot\left(r^2\hat{r}\,d\Omega\right)$$
$$= \int \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right]\cdot r^2\, d\Omega$$
$$= \int(-1)\, d\Omega$$
$$= -4\pi.$$

Meanwhile,

$$\int_V -4\pi\,\delta^3(\vec{r})\, dV = -4\pi.$$

Therefore,

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(\vec{r}).$$

QM Honors HW#3 Bixo Huang

12.3.3/ $\psi(\rho,\varphi) = A e^{-\frac{\rho^2}{2\Delta^2}} \cos^2\varphi \equiv R(\rho)\,Q(\varphi)$, where

$R(\rho)$, $Q(\varphi)$ normalize respectively. Since we only care about the angular momentum part, we can integrate the $R(\rho)$ part out, leaving only $Q(\varphi) = B\cos^2\varphi$.

Now $Q(\varphi) = B\,\frac{1+\cos 2\varphi}{2} = \tilde{B}\left[1 + \frac{e^{i2\varphi} + e^{i(-2\varphi)}}{2}\right]$

$= \sqrt{2\pi}\tilde{B}\left[\Phi_0 + \frac{1}{2}\Phi_2 + \frac{1}{2}\Phi_{-2}\right]$.

(7)

$\Rightarrow P(l_z = m\hbar) = \dfrac{\left|\int_0^{2\pi} d\varphi\, \Phi_m^* Q(\varphi)\right|^2}{\underbrace{\int_0^{2\pi} d\varphi\, |Q(\varphi)|^2}_{=1,\ \text{just to make calculation simpler}}}$, Use $\int \Phi_{m'}^* \Phi_m\, d\varphi = \delta_{mm'} \Rightarrow$

$\Rightarrow P(l_z = 0) = \dfrac{|\sqrt{2\pi}\tilde{B}|^2}{|\sqrt{2\pi}\tilde{B}|^2}\dfrac{1^2}{1^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{2}{3}$

$P(l_z = 2\hbar) = P(l_z = -2\hbar) = \dfrac{|\sqrt{2\pi}\tilde{B}|^2}{|\sqrt{2\pi}\tilde{B}|^2}\dfrac{\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{6}$

* Or from the expression of $Q(\varphi)$ we know

$\dfrac{P(l_z=0)}{P(l_z=2\hbar)} = \dfrac{P(l_z=0)}{P(l_z=-2\hbar)} = \dfrac{1}{\left(\frac{1}{2}\right)^2} = 4$, and $P(l_z=0)+P(l_z=2\hbar)+P(l_z=-2\hbar) = 1$

$\Rightarrow$ The same result.

12.3.6/ $H = \frac{p^2}{2\mu} = -\frac{\hbar^2}{2\mu}\vec{\nabla}^2 = -\frac{\hbar^2}{2\mu}\left(\vec{e}_\varphi \frac{1}{a}\frac{\partial}{\partial\varphi}\right)^2 = -\frac{\hbar^2}{2\mu a^2}\frac{\partial^2}{\partial\varphi^2}$.

Here we used $\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\varphi \frac{1}{a}\frac{\partial}{\partial\varphi}$ in polar coordinates. But because of the constraint, φ is the only coordinate any wave functions would depend on, so $\vec{\nabla} = \vec{e}_\varphi \frac{1}{a}\frac{\partial}{\partial\varphi}$.

On the other hand $L_z^2 = -\hbar^2\frac{\partial^2}{\partial\varphi^2}$. It is then clear

(5) $$H = \frac{L_z^2}{2\mu a^2}$$

We can use the eigenfunction of L_z:

$$H\Phi_m(\varphi) = \frac{L_z^2}{2\mu a^2}\Phi_m(\varphi) = \frac{m^2\hbar^2}{2\mu a^2}\Phi_m(\varphi).$$

$$\Rightarrow E_m = \frac{\hbar^2}{2\mu a^2}m^2, \quad \text{degeneracy} = 2$$

$\sim \pm m$

12.3.8/ (1) $\vec{A} = \frac{B}{2}(-y\vec{i} + x\vec{j})$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{B}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \frac{B}{2}\cdot[\vec{i}(-\partial_z x) - \vec{j}(\partial_z y) + \vec{k}(\partial_x x + \partial_y y)]$$

(15)

$$= \frac{B}{2}\cdot\vec{k}\cdot 2 = B\vec{k}$$

(2) For a classical particle,

$$\mu\frac{v^2}{r} = qvB \Rightarrow v = \frac{qBr}{\mu}$$

$$\omega_0 = \frac{v}{2\pi r} = \frac{qB}{2\pi\mu}$$

If we change the unit $B \to B\frac{2\pi}{c}$, $\omega_0 = \frac{qB}{\mu c}$

(3) $$H = \frac{(P_x + \frac{qB}{2c}Y)^2}{2\mu} + \frac{(P_y - \frac{qB}{2c}X)^2}{2\mu}$$

$$Q = (cp_x + \frac{qB}{2}Y)/qB, \quad P = (P_y - \frac{qB}{2c}X)$$

Clearly $[Q, Q] = [P, P] = 0$ because there is no non-vanishing commutators $[X, P_x] = [Y, P_y] = i\hbar$.

$$[Q, P] = \left[\frac{cP_x + \frac{qB}{2}Y}{qB}, P_y - \frac{qB}{2c}X\right]$$

$$= \frac{c}{qB}\cdot\frac{qB}{2c}[P_x, -X] + \frac{1}{2}[Y, P_y]$$

$$= i\hbar \Rightarrow Q, P \text{ are canonical.}$$

4

$$H = \frac{\left(\frac{qB}{c}Q\right)^2}{2\mu} + \frac{P^2}{2\mu} = \frac{q^2B^2}{2\mu c^2}Q^2 + \frac{P^2}{2\mu}$$

Let $\omega_0 = \frac{qB}{\mu c}$, $H = \frac{P^2}{2\mu} + \frac{1}{2}\mu\omega_0^2 Q^2$.

This is the Hamiltonian for 1-D harmonic oscillator

with energy levels $E_n = (n + \frac{1}{2})\hbar\omega_0$

* These levels are called Landau Levels. (quantization) And it shows that in QM, energy levels get quantized even in some cases where particles are free to travel into infinity, eg. $z \to \infty$ in this case.

(4) $$H = \frac{1}{2\mu}\left(P_x^2 + \left(\frac{qB}{2c}\right)^2 Y^2 + \frac{qB}{c}YP_x + P_y^2 + \left(\frac{qB}{2c}\right)^2 X^2 - \frac{qB}{c}XP_y\right)$$

$$= \frac{P_x^2 + P_y^2}{2\mu} + \frac{\mu\omega_0^2}{8}(X^2 + Y^2) - \frac{\omega_0}{2}L_z$$

$L_z = (XP_y - YP_x)$

$\omega_0 = \frac{qB}{\mu c}$

$$= H\left(\frac{\omega_0}{2}, \mu\right) - \frac{\omega_0}{2}L_z$$

where $H(\frac{\omega_0}{2}, \mu)$ is the Hamiltonian for isotropic 2-D oscillator with frequency $\frac{\omega_0}{2}$

$$[HC(\tfrac{\omega_c}{2}, \mu), L_z] = [HC(\tfrac{\omega_0}{2}, \mu), XP_y - YP_x]$$

$$= \frac{1}{2\mu}\left([P_x^2, XP_y] + [P_y^2, -YP_x] \right)$$

$$+ \frac{\mu\omega_0^2}{8}\left([X^2, -YP_x] + [Y^2, XP_y] \right)$$

$$= \frac{1}{2\mu}\left(2P_xP_y(-i\hbar) + 2(i\hbar)P_yP_x \right)$$

$$+ \frac{\mu\omega_0^2}{8}\left(2YX(-i\hbar) + 2XY(i\hbar) \right)$$

$$= 0 \qquad ([P_x, P_y] = [X, Y] = 0)$$

$\Rightarrow$ For $H(\frac{\omega_0}{2}, \mu)|n\rangle = \hbar(\frac{\omega_0}{2})(n+1)|n\rangle$, where $|n\rangle$ is the basis diagonizing $HC(\frac{\omega_0}{2}, \mu)$,

$$HC(\tfrac{\omega_0}{2}, \mu)(L_z|n\rangle) = L_z HC(\tfrac{\omega_0}{2}, \mu)|n\rangle = \hbar(\tfrac{\omega_0}{2})(n+1)(L_z|n\rangle)$$

$\Rightarrow$ $L_z|n\rangle$ is also an eigenfunction of $HC(\frac{\omega_0}{2}, \mu)$ with quantum number n.

$\Rightarrow$ $L_z|n\rangle = \tilde{k}|n\rangle$. # This is just the eigenfunction for L_z, so we know $\tilde{k} \equiv l_z = m\hbar$, where m is another quantum number specifying the degeneracy in $|n\rangle$ (Recall $n = n_x + n_y$)

See exercise 10.2.2

6

$$\Rightarrow \quad H|n\rangle = \left(HC\frac{\omega_0}{2}, \mu\right) - \frac{\omega_0}{2} L_3 \Big) |n\rangle$$

$$= \left(\hbar \frac{\omega_0}{2}(n+1) - \frac{\omega_0}{2} m\hbar \right) |n\rangle$$

So $|n\rangle$ also diagonizes H. The energy levels are

$$E_{n,m} = \frac{\hbar\omega_0}{2}(n+1) - \frac{\hbar\omega_0}{2} m$$

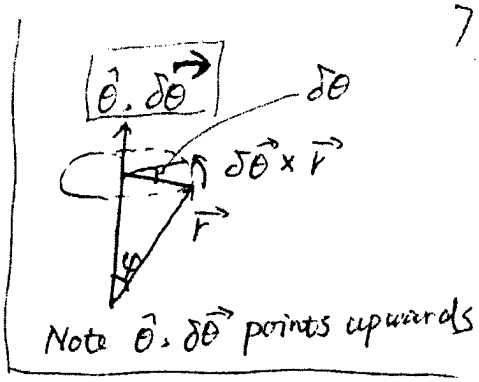
Take $n \equiv 2k + |m|$, $\quad E = \hbar\omega_0\left(k + \frac{|m|}{2} - \frac{m}{2} + \frac{1}{2}\right)$

For $m \geq 0$, $\quad E = \hbar\omega_0\left(k + \frac{1}{2}\right)$

$m < 0$, $\quad E = \hbar\omega_0\left(k - m + \frac{1}{2}\right)$

Thus, levels are the same as $E = \hbar\omega_0\left(n + \frac{1}{2}\right)$

12.4.3/(1) According to the picture, when $\vec{r}$ is rotated by an angle $\delta\theta$ along $\hat{\theta}$ (denote $\delta\vec{\theta} = \delta\theta\,\hat{\theta}$), it moves the axis ~~to~~ $\vec{r}$ the distance of $r\sin\varphi\,\delta\theta$ along the direction ~~$\hat{\theta}$~~ $\hat{\theta}\times\vec{r}$. Thus, we can summarized it as



$$\vec{r} \to \vec{r} + \delta\vec{\theta}\times\vec{r}$$

(2) $U[R(\delta\vec{\theta})]\,\psi(\vec{r}) = (I - \frac{i}{\hbar}\delta\theta\, L_{\hat{\theta}})\,\psi(\vec{r})$

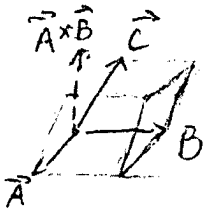
$$= \psi(\vec{r}) - \frac{i}{\hbar}\delta\theta\, L_{\hat{\theta}}\,\psi(\vec{r}) \qquad (1)$$

8

Besides, $U[R(\delta\vec{\theta})]\,\psi(\vec{r}) = \psi(\vec{r} - \delta\vec{\theta}\times\vec{r})$

$$= \psi(\vec{r}) - (\delta\vec{\theta}\times\vec{r})\cdot\vec{\nabla}\psi(\vec{r}).$$

Use the identity: $(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{B}\times\vec{C})$ ≠



(This represents the volume of a "leaning" cube with $\vec{A}$, $\vec{B}$, $\vec{C}$ as three edges)

$$\Rightarrow U[R(\delta\vec{\theta})]\,\psi(\vec{r}) = \psi(\vec{r}) - \delta\vec{\theta}\cdot(\vec{r}\times\vec{\nabla})\,\psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{\hbar}\delta\theta\,(\hat{\theta}\cdot(\vec{r}\times\vec{p}))\,\psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{\hbar}\delta\theta\,(\hat{\theta}\cdot\vec{L})\,\psi(\vec{r}) \qquad (2)$$

(1)(2) $\Rightarrow$ $L_{\hat{\theta}} = \hat{\theta}\cdot\vec{L}$

8

12.5.3/ ~~##~~ $J_\pm = J_x \pm i J_y$

$$\Rightarrow \begin{cases} J_x = \dfrac{J_+ + J_-}{2} & J_x^2 = \dfrac{J_+^2 + J_-^2 + J_+J_- + J_-J_+}{4} \\ J_y = \dfrac{J_+ - J_-}{2i} & J_y^2 = \dfrac{-J_+^2 - J_-^2 + J_+J_- + J_-J_+}{4} \end{cases}$$

(1) Since $J_\pm |j, m\rangle \rightarrow |j, m\pm 1\rangle$, we have

$\langle J_\pm \rangle \equiv \langle jm | J_\pm | jm \rangle = 0$, and therefore

15 $\langle J_x \rangle = \langle J_y \rangle = 0$

(2) $\langle J_x^2 \rangle = \frac{1}{4} \langle jm | (J_+J_- + J_-J_+) | jm \rangle$

Here we have $J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 + i(J_yJ_x - J_xJ_y)$

$= J^2 - J_z^2 + \hbar J_z$

$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J^2 - J_z^2 - \hbar J_z$

$$\Rightarrow \langle J_x^2 \rangle = \frac{1}{4} \langle jm | 2J^2 - 2J_z^2 | jm \rangle = \frac{1}{2}[j(j+1)\hbar^2 - m^2\hbar^2]$$

$$= \frac{\hbar^2}{2}(j(j+1) - m^2)$$

$$\langle J_y^2 \rangle = \frac{1}{4} \langle jm | (J_+J_- + J_-J_+) | jm \rangle = \langle J_x^2 \rangle .$$

(3) $\Delta J_x = \Delta J_y = \sqrt{\langle J_x^2 \rangle} = \hbar\sqrt{\frac{1}{2}[j(j+1) - m^2]}$

$$\Delta J_x \cdot \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - m^2]$$

From (9.2.9), $(\Delta J_x)^2(\Delta J_y)^2 \geq |\langle jm | J_x J_y | jm \rangle|^2$

$$= \left|\langle jm| \frac{J_+J_+ - J_+J_+}{4i} |jm\rangle\right|^2$$

$$\Rightarrow \Delta J_x \cdot \Delta J_y \geq \left|\langle jm| \frac{2\hbar J_z}{4} |jm\rangle\right| = \frac{\hbar}{2}|m|\hbar$$

$$= \frac{|m|}{2}\hbar^2$$

We can see that $[j(j+1) - |m|^2] - |m| = j(j+1) - |m|(|m|+1)$

Since $|m| \leq j$, $j(j+1) - |m|(|m|+1) \geq 0$

$\Rightarrow \Delta J_x \cdot \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - m^2] \geq \frac{|m|}{2}\hbar^2$ is satisfied

(4) When $m = \pm j \Rightarrow$

$\Delta J_x \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - j^2] = \frac{\hbar^2}{2}j = \frac{\hbar^2}{2}|m|$, the inequality is saturated.

12.5.13/(1) $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$

$= \mp\sqrt{\frac{3}{8\pi}}\frac{1}{r}[r\sin\theta\cos\varphi \pm i r\sin\theta\sin\varphi]$

$= \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}$

* $\begin{cases} Y_1^1 = \sqrt{\frac{3}{8\pi}}\frac{-x-iy}{r} \\ Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\frac{x-iy}{r} \end{cases}$

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$

(2) $\psi = N(x+y+2z)e^{-ar}$

Here $\begin{cases} x = (Y_1^{-1} - Y_1^1)\sqrt{\frac{2\pi}{3}}r \\ y = i(Y_1^{-1} + Y_1^1)\sqrt{\frac{2\pi}{3}}r \end{cases}$ $\quad z = Y_0^1\sqrt{\frac{4\pi}{3}}r$

$\Rightarrow \psi = Ne^{-ar}\left[\sqrt{\frac{2\pi}{3}}r\left(Y_1^{-1}(1+i) - Y_1^1(1-i)\right) + 2\sqrt{\frac{4\pi}{3}}r\,Y_1^0\right]$

$\Rightarrow P(l_z=0) = \frac{\left|\int d\Omega\, Y_1^{0*}\psi\right|^2}{\int|\psi|^2 d\Omega} = \frac{(2\sqrt{2})^2}{(2\sqrt{2})^2 + |1+i|^2 + |1-i|^2} = \frac{2}{3}$

* Here we drop the integration over r because this variable can be separated in ψ and this part is integrated into unity in both numeritor & denominator.

Similarly $P(l_z=\pm\hbar) = \frac{|\int d\Omega\, Y_1^{\pm 1*}\psi|^2}{\int|\psi|^2 d\Omega} = \frac{|1\pm i|^2}{(2\sqrt{2})^2 + |1+i|^2 + |1-i|^2}$

$= \frac{1}{6}$

* $\psi = Ne^{-ar}\sqrt{\frac{2\pi}{3}}r\left[Y_1^{-1}(1+i) - Y_1^1(1-i) + 2\sqrt{2}\,Y_1^0\right]$

Physics 828 Sketch of Solution to Set 2

Shankar 12.3.7

(1) The two-dimensional harmonic oscillator is obviously invariant under rotations about the z-axis: the magnitude of the position and momenta are unaltered by rotations around the z-axis. Therefore, the Hamiltonian commutes with the generator of rotations about the z-axis, L_z.

(2) So we write $\psi(\rho,\phi) = e^{im\phi} R_{Em}(\rho)$ where m is an integer, positive or negative. In two dimensions since we know the Laplacian we have

$$-\frac{\hbar^2}{2\mu}\left(R''_{Em} + \frac{1}{\rho}R'_{Em}\right) + \left(\frac{\hbar^2 m^2}{2\mu\rho^2} + \frac{1}{2}\mu\omega^2\rho^2\right) R_{Em}(\rho) = E R_{Em}\,.$$

For small ρ assuming that $R_{Em}(\rho) \propto \rho^k$ (derivatives decrease the power by unity increasing its importance for small ρ etc.) we can neglect the potential energy term and the constant term on the right-hand side. Thus we have

$$R''_{Em} + \frac{R'_{Em}}{\rho} \propto \frac{m^2}{\rho^2} R_{Em} \;\Rightarrow\; [k(k-1)+k] = m^2 \;\Rightarrow\; k^2 = m^2\,.$$

For the wave function to be normalizable, for $\int_0^\infty d\rho\,\rho\, R^2_{Em}(\rho)$ to be finite at the lower limit $k \geq 0$. Thus we have $R_{Em}(\rho) \overset{\rho\to 0}{\longrightarrow} \rho^{|m|}$.

(3) For large ρ, terms with inverse powers of ρ including the centrifugal term and also the constant term on the right-hand side can be neglected. So we have

$$R''_{Em} = \frac{\mu^2\omega^2\rho^2}{\hbar^2} R_{Em}\,;$$

this is identical to the one-dimensional oscillator equation. See the careful analysis on page 191. Up to powers of ρ the solution is[1]

$$R_{Em}(\rho) \overset{\rho\to\infty}{\longrightarrow} e^{-\frac{\mu\omega}{2\hbar}\rho^2}\,.$$

So we write as instructed

[1] $$R'_{Em} = -\frac{\mu\omega}{\hbar}\,\rho\, e^{-\frac{\mu\omega}{2\hbar}\rho^2} \text{ and } R''_{Em} = \frac{\mu\omega}{\hbar}\left(-1 + \frac{\mu\omega}{\hbar}\rho^2\right) e^{-\frac{\mu\omega}{2\hbar}\rho^2}\,.$$
Thus we can neglect the constant term in R'' and recover the solution. One can use this to check that terms we have neglected are indeed small compared to the terms we have retained. When you neglect terms it is a good idea to substitute the solution you have obtained and check that the terms you have neglected are indeed smaller.

$$R_{Em}(\rho) \,=\, U_m(\rho)\,\rho^{|m|}\,e^{-\frac{\mu\omega}{2\hbar}\rho^2}\,.$$

(4) Use the dimensionless variable $\epsilon \,=\, \frac{E}{\hbar\omega}$ and $y^2 \,=\, \frac{\mu\omega}{\hbar}\rho^2$. Dividing the radial equation by $\hbar\omega$ we have dropping the annoying subscripts

$$\left[-\frac{1}{2}\left(\frac{d^2}{dy^2} \,+\, \frac{1}{y}\frac{d}{dy} \,-\, \frac{m^2}{y^2}\right) \,+\, \frac{1}{2}y^2\right]\,R(y) \,=\, \epsilon\,R(y)\,.$$

(5) We do the substitution and do elementary calculus and obtain the result given:

$$U'' \,+\, \left[\left(\frac{2|m|+1}{y}\right) \,-\, 2y\right]\,U' \,+\, (2\epsilon - 2|m| - 2)\,U \,=\, 0\,.$$

(6) We substitute $U(y) \,=\, \sum_{r=0}^{\infty} C_r y^r$ and collect the coefficient of y^r. The second derivative reduces the power by two and so we use the C_{r+2} term etc.

$$(r+2)(r+1)C_{r+2} \,+\, (2|m|+1)(r+2)C_{r+2} - 2rC_r \,+\, (2\epsilon - 2|m| - 2)C_r \,=\, 0$$

yielding a two-term recursion relation.

(7) We write this as

$$\frac{C_{r+2}}{C_r} \,=\, -\frac{(2(\epsilon - |m| - r - 1)}{(r+2)(2|m|+r+2)}\,.$$

First if C_0 is given C_2 and the other even terms can be computed. As $r \to \infty$ we have

$$\frac{C_{r+2}}{C_r} \,\to\, \frac{2}{r}\,.$$

This implies that $U(y)$ grows as e^{y^2} which overwhelms the $e^{-y^2/2}$ in R pushing it out of the Hilbert space. So the series must terminate. Thus the boundary condition at infinity leads as usual to energy quantization.

What about the odd terms? A series only with odd terms (set $C_0 = 0$ so that all even terms vanish) is inconsistent since then $U(y) \,\sim\, y$ for small y and thus $R(\rho) \,\sim\, \rho^{|m|+1}$ inconsistent with our earlier result in (2). This appears to be suggested as an argument. What if one starts with C_0 and C_1 non-zero? Substituting into the equation for U we find that the $(1/y)(dU/dy)$ leads to the term C_1/y and there is no other source of y^{-1} terms. Thus $C_1 \,=\, 0$ and therefore, all odd terms vanish.

Therefore, we have $r = 2k$ and the termination of the series condition yields

$$\epsilon \,=\, 1 \,+\, r \,+\, |m| \,=\, |m| \,+\, 2k \,+\, 1 \,\equiv\, (n+1)$$

with $k = 0, 1, , 2, \cdots$.

(8) Since $|m| \,=\, n \,-\, 2k$ for a given n (i.e., for a given energy) the maximum value of m is n which occurs for $k = 0$, The azimuthal quantum number m decreases by steps of 2 until m reaches the value $-n$. It is easy to se that there are $n+1$ allowed values of m yielding a degeneracy of $n+1$. In Cartesian coordinates the energy is $n_x + n_y + 1$ in units of $\hbar\omega$. So the degeneracy corresponds to the number of ways in which we can choose two non-negative integers to add up to n. We can choose n_x to be any integer from 0 to n and $n_y = n - n_x$. This yields the same degeneracy.

Shankar 12.6.1

(1) Since there is no angular dependence $\ell = 0$.

(b) We have $R(r) \,\propto\, e^{-r/a_0}$ and for large r (retaining only the dominant terms)

$$-\frac{\hbar^2}{2m} R'' \;=\; E R(r)\,.$$

Substituting the given form we obtain $\boxed{E \,=\, -\frac{\hbar^2}{2ma_0^2}}$.

(c) Clearly the R'' term and the energy term cancel for all r. If the equation is valid for all r we must have

$$-\frac{\hbar^2}{2m}\frac{2}{r}\, R' \,+\, V(r)\, R(r) \,=\, 0\,.$$

Substituting $R(r) \,=\, e^{-r/a_0}$ we obtain

$$\frac{\hbar^2}{ma_0 r} \,+\, V(r) \,=\, 0 \;\;\Rightarrow\;\; V(r) \,=\, -\frac{\hbar^2}{ma_0 r}\,.$$

Shankar 13.1.1 and 13.1.3 You should be able to fill in the steps. Here are some steps dropping some subscripts for notational simplicty.

$$v \,=\, \sum_{k=0}^{\infty} C_k\, \rho^{k+\ell+1}\,.$$

Substituting into

$$v'' \,-\, 2v' \,+\, \left(\frac{e^2\lambda}{\rho} \,-\, \frac{\ell(\ell+1)}{\rho^2}\right) v \;=\; 0 \tag{1}$$

we extract the coefficient of ρ^{k+l} carefully. Since two derivatives reduce the power of ρ by two we should start from the term with $\rho^{k+\ell+2}$ with coefficient C_{k+1} for the first term and similarly for the last term. For the second and third terms which reduce the power of ρ by unity we can start with the $\rho^{k+\ell+1}$ term with coefficient C_k. Thus we have

$$(k+\ell+2)(k+\ell+1)C_{k+1} \,-\, 2(k+\ell+1)C_k \,+\, e^2\lambda C_k \,-\, \ell(\ell+1)C_{k+1} \;=\; 0 \tag{2}$$

which yields

$$\frac{C_{k+1}}{C_k} \;=\; \frac{-q^2\lambda \,+\, 2(k+\ell+1)}{(k+\ell+2)(k+\ell+1) \,-\, \ell(\ell+1)} \tag{3}$$

Since $\lambda^2 \,=\, \frac{2m}{\hbar^2 W}$ from 13.1.9. For the numerator to vanish we have

$$e^4\lambda^2 \;=\; -\frac{2me^2}{\hbar^2 E} \;=\; 4(k+\ell+1)^2\,.$$

This yields 13.1.14.

Mathematical aside: Here is a different representation using classical mathematical physics. Note that the function $L(\rho) \,\equiv\, \sum_{k=0}^{\infty} C_k\,\rho^k$ obeys the equation

$$\rho L''(\rho) \,+\, [\,2(\ell+1) \,-\, 2\rho\,]L'(\rho) \,+\, (q^2\lambda \,-\, 2(\ell \,+\, 1))L(\rho) \;=\; 0 \quad .$$

Substituting $q^2\lambda \,=\, 2n$ (where n is an integer eventually) we have

$$\frac{1}{2}\rho L'' \,+\, [\,(\ell+1) \,-\, \rho\,]L' \,+\, (n \,-\, (\ell \,+\, 1))L \;=\; 0 \quad .$$

Let $z = 2\rho$ we have

$$z\frac{d^2L}{dz^2} \,+\, [2(\ell+1) \,-\, z]\frac{dL}{dz} \,-\, (\ell+1-n)L \;=\; 0\,.$$

We know (with a good mathematical methods course) that the general solution to

$$zw'' \,+\, (c-z)\,w' \,-\, aw \;=\; 0$$

is given by the confluent hypergeometric function $w \,=\, {}_1F_1(a;c;z)$. Thus we find that the solution $L(\rho)$ is ${}_1F_1(\ell+1-n;\, 2\ell+2;\, 2\rho)$. The particular terminating (for integer n)

confluent hypergeometric function can be related to the *associated Laguerre polynomial.*

Shankar 13.3

We are considering the case $n = 2$ and $\ell = 1$ and thus $k = 0$ in the notation of the text. So we have $W = \frac{me^4}{8\hbar^2}$ from 13.1.4 and form 13.1.6

$$\rho = \sqrt{\frac{2mW}{\hbar^2}}\, r = \frac{me^2}{2\hbar^2}\, r = \frac{r}{2a_0}$$

using 13.1.24. From 13.1.10 since $k = 0$ and $\ell = 1$ we have $v = C_0\rho^2$ and thus

$$R(r) = \frac{U(r)}{r} = C\, e^{-\frac{r}{2a_0}}\, r$$

where C is an overall constant. We know that $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$ from 12.5.39 which is normalized. So all we need is that $Ce^{-r/(2a_0)}r$ is notmalized when integrated over the radial coordinate. We have

$$C^2 \int_0^\infty dr\, r^2\, r^2\, e^{-\frac{r}{a_0}} = C^2\, 24\, a_0^5\,.$$

Thus $C = \sqrt{\frac{1}{24a_0^3}} \times 1/a_0$ and including the normalization from the spherical harmonic yields the quoted answer.

Shankar 13.5

Since we are asked to compute $\langle\Omega\rangle$ for stationary states $|n\ell m\rangle$ its time derivative vanishes. Thus we have $\langle[\Omega,\, H]\rangle = 0$ by Ehrenfests' theorem. So we compute the commutator for $\Omega = \vec{R}\cdot\vec{P}$ as ordered. We calculate (using the summation convention)

$$\left[R_jP_j, \frac{P_iP_i}{2m}\right] = \left[R_j, \frac{P_iP_i}{2m}\right] P_j = 2i\hbar\, \frac{\vec{P}\cdot\vec{P}}{2m} = 2i\hbar\, T\,.$$

We have used

$$[R_j,\, P_iP_i] = P_i[R_j,\, P_i] + [R_j,\, P_i]P_i = 2i\hbar\, P_j\,.$$

We consider the potential energy term next:

$$[R_jP_j,\, V(R)] = R_j[P_j,\, V(R)] = R_j\left(-i\hbar\frac{d}{dR_j}V(R)\right)\,.$$

Note that $[P_i,\, V(R)]$ can be evaluated in the coordinate representation (in Cartesian coordinates) by acting on a function $f(r)$:

$$[P_i,\, V(r)]f(r) \,=\, -i\hbar\left(\frac{\partial}{\partial r_i}V(r)f(r) \,-\, V(r)\,\frac{\partial}{\partial r_i}V(r)f(r)\right] \,=\, -i\hbar\,\frac{\partial}{\partial r_i}V(r) \,=\, -i\hbar\vec{R}\cdot\vec{\nabla}V.$$

Thus we can write formally $[P_i,\, V(R)] \,=\, -i\hbar\partial V(R)/\partial R_i$. So we need to evaluate $\vec{R}\cdot\vec{\nabla}V(R)$. In the coordinate representation using spherical coordinates for central potentials this is just $rV'(r)$. For the Coulomb potential we obtain $-V(r)$ and including the factor of $-i\hbar$ we obtain $i\hbar\,V(r)$. Substituting into the basic relation we have

$$\langle 2T \,+\, V\rangle \,=\, 0$$

as asserted. If $V(R) \,\propto\, R^n$, $rV'(r) \,=\, nV$ and thus we obtain $\langle T\rangle \,=\, \frac{n}{2}\,\langle V\rangle$.

Physics 710 March 12, 2010

Problem Set 15

Problem 12.6.1: $\psi_E = Ae^{-r/a_0}$.

(1) No (θ,ϕ)-dependence implies $\psi_E \propto Y_0^0$, so we must have $\ell = 0$ and $m = 0$.

(2) Therefore $\psi_E = R_{E,\ell=0} = \frac{1}{r}U_{E,0}$ which satisfies eqn. (12.6.5) with $\ell = 0$:

$$(r\psi_E)'' + \frac{2\mu}{\hbar^2}[E - V(r)](r\psi_E) = 0. \tag{1}$$

As $r \to \infty$, $V(r) \to 0$, which implies in this limit $(r\psi_E)' = A(1 - \frac{r}{a_0})e^{-r/a_0} \approx -(A/a_0)re^{-r/a_0}$, so $(r\psi_E)'' \approx (A/a_0^2)re^{-r/a_0}$. Therefore, eqn. (1) reads in this limit

$$\frac{A}{a_0^2}re^{-r/a_0} = -\frac{2\mu E}{\hbar^2}Are^{-r/a_0},$$

from which it follows that

$$E = -\frac{\hbar^2}{2\mu a_0^2}.$$

(3) Now plug E into (1) and use $(r\psi_E)'' = A(-\frac{2}{a_0} + \frac{r}{a_0^2})e^{-r/a_0}$ to get

$$A\left(-\frac{2}{a_0} + \frac{r}{a_0^2}\right)e^{-r/a_0} + \frac{2\mu}{\hbar^2}\left(-\frac{\hbar^2}{2\mu a_0^2} - V(r)\right)Are^{-r/a_0} = 0,$$

which gives

$$V(r) = -\frac{\hbar^2}{\mu a_0 r}.$$

Problem 12.6.4:

(1) $\delta^3(\mathbf{r} - \mathbf{r}')$ is defined by the property that $\int d^3r\delta^3(\mathbf{r} - \mathbf{r}')f(\mathbf{r}) = f(\mathbf{r}')$. So simply check:

$$\int r^2 dr \sin\theta d\theta d\phi \frac{1}{r^2\sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')f(r,\theta,\phi)$$
$$= \int drd\theta d\phi\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')f(r,\theta,\phi) = f(r',\theta',\phi').$$

(2) If $r \neq 0$ then $\nabla^2(\frac{1}{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}(\frac{1}{r}))$+(angular parts) $= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(\frac{-1}{r^2})) = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0$. When $r = 0$ the above calculation breaks down since terms are singular there. So consider an arbitrary continuous function $f(\mathbf{r})$ and the integral $\int d^3x\nabla^2(\frac{1}{r})f(\mathbf{r}) = \lim_{\epsilon\to 0}\int_0^\epsilon r^2dr\int d\Omega\nabla^2(\frac{1}{r})f(\mathbf{r})$, since $\nabla^2(\frac{1}{r}) = 0$ for $r > 0$. Then, integrating by parts we get

$$\int d^3x\nabla^2\left(\frac{1}{r}\right)f(\mathbf{r}) = \lim_{\epsilon\to 0}\int_0^\epsilon r^2dr\int d\Omega\left\{\vec{\nabla}\cdot\left(f(\mathbf{r})\vec{\nabla}\left(\frac{1}{r}\right)\right) - \left(\vec{\nabla}f(\mathbf{r})\right)\cdot\vec{\nabla}\left(\frac{1}{r}\right)\right\}.$$

Now recall that in spherical coordinates $\vec{\nabla} = \widehat{r}\frac{\partial}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \widehat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$, implying that $\vec{\nabla}(1/r) = \widehat{r}\frac{\partial}{\partial r}(1/r) = -\widehat{r}/r^2$, so

$$\int d^3x \nabla^2 \left(\frac{1}{r}\right) f = \lim_{\epsilon\to 0}\int_0^\epsilon r^2 dr \int d\Omega \left\{\vec{\nabla}\cdot\left(-\widehat{r}\frac{f}{r^2}\right) + \left(\widehat{r}\cdot\vec{\nabla}f\right)\frac{1}{r^2}\right\}.$$

The second term on the right side vanishes, because as $\epsilon \to 0$, $\widehat{r}\cdot\vec{\nabla}f = (\partial f/\partial r)|_{r=0} =$ const., so:

$$\lim_{\epsilon\to 0}\int_0^\epsilon r^2 dr \int d\Omega \left(\widehat{r}\cdot\vec{\nabla}f\right)\frac{1}{r^2} = \text{const.}\cdot\lim_{\epsilon\to 0}\int_0^\epsilon r^2 dr\int d\Omega\frac{1}{r^2} = \text{const.}\cdot\lim_{\epsilon\to 0} 4\pi\epsilon = 0.$$

Therefore,

$$\begin{aligned}\int d^3x\nabla^2\left(\frac{1}{r}\right) f &= \lim_{\epsilon\to 0}\int_0^\epsilon r^2 dr\int d\Omega\vec{\nabla}\cdot\left(-\widehat{r}\frac{f}{r^2}\right) = \lim_{\epsilon\to 0}\int d\Omega r^2\widehat{r}\cdot\left(-\widehat{r}\frac{f}{r^2}\right)\Big|_{r=\epsilon}\\ &= -\lim_{\epsilon\to 0}\int d\Omega\, f|_{r=\epsilon} = -f(0)\lim_{\epsilon\to 0}\left(\int d\Omega\right) = -f(0)\lim_{\epsilon\to 0} 4\pi\\ &= -4\pi f(0).\end{aligned}$$

In the second step I used the divergence theorem which states $\int_R d^3x\vec{\nabla}\cdot\vec{g} = \int_{\partial R} d^2a\,\widehat{n}\cdot\vec{g}$ where R is any region, ∂R is its boundary, $\widehat{n}$ is the normal unit vector to ∂R (pointing out of R) and d^2a is the surface area element. In our case $R = \{r < \epsilon\}$, $d^2a = d\Omega$, $\widehat{n} = \widehat{r}$, and $\vec{g} = -\widehat{r}(f/r^2)$. Thus we have shown that $\nabla^2(1/r) = 0$ for $r \neq 0$ and for any $f(\vec{r})$ that $\int d^3x\nabla^2(1/r)f = -4\pi f(0)$. This is the *definition* of the delta function, so

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}).$$

Problem 12.6.9: Since $\ell = 0$, $\psi = R(r)Y_0^0(\theta,\phi) = R(r)$. So the radial equation becomes, with $\psi(r) = (1/r)U(r)$,

$$\begin{aligned}\left(\frac{d^2}{dr^2} + k^2\right)U_{\text{in}} = 0 \qquad & r \le r_0, \qquad k \equiv \sqrt{\frac{2\mu(E+V_0)}{\hbar^2}},\\ \left(\frac{d^2}{dr^2} - \kappa^2\right)U_{\text{out}} = 0 \qquad & r \ge r_0, \qquad \kappa \equiv \sqrt{\frac{-2\mu E}{\hbar^2}},\end{aligned}$$

where k and κ are defined to be the positive root. The solutions of these equations are

$$\begin{aligned}U_{\text{in}} &= A\sin kr + \widetilde{A}\cos kr,\\ U_{\text{out}} &= Be^{-\kappa r} + \widetilde{B}e^{+\kappa r}.\end{aligned}$$

The boundary conditions are that $U_{\rm in} \to 0$ at $r = 0$, implying $\widetilde{A} = 0$, and $U_{\rm out} \to 0$ at $r = \infty$, implying that $\widetilde{B} = 0$. The matching conditions at $r = r_0$ are the continuity of ψ and its first derivative:

$$\psi_{\rm in} = \psi_{\rm out} \qquad \Rightarrow \qquad U_{\rm in} = U_{\rm out}|_{r=r_0}, \tag{2}$$

$$\psi'_{\rm in} = \psi'_{\rm out} \qquad \Rightarrow \qquad (U_{\rm in}/r)' = (U_{\rm out}/r)'|_{r=r_0}. \tag{3}$$

Eqn. (2) implies $A \sin kr_0 = Be^{-\kappa r_0}$, or,

$$B = A \sin kr_0 \, e^{\kappa r_0}. \tag{4}$$

Eqn. (3) implies $\frac{d}{dr}[(A/r) \sin kr - (B/r)e^{-\kappa r}]_{r=r_0} = 0$, which gives

$$-\frac{A}{r_0^2} \sin kr_0 + \frac{Ak}{r_0} \cos kr_0 + \frac{B}{r_0^2} e^{-\kappa r_0} + \frac{B\kappa}{r_0} e^{-\kappa r_0} = 0.$$

Plugging (4) into this gives $(Ak/r_0) \cos kr_0 + (A\kappa/r_0) \sin kr_0 = 0$, or

$$-\tan kr_0 = \frac{k}{\kappa}, \tag{5}$$

which is what we wanted to show.

If $V_0 < \pi^2\hbar^2/(8\mu r_0^2)$ and $-V_0 < E < 0$ (for a bound state), then $k^2 = \frac{2\mu}{\hbar^2}(E + V_0) < \frac{2\mu}{\hbar^2}\pi^2\hbar^2/(8\mu r_0^2) = \pi^2/(4r_0^2)$. Thus $0 < kr_0 < (\pi/2)$, which implies that $-\tan kr_0 < 0$. On the other hand, by definition, $k/\kappa > 0$. So there is no solution to eqn. (5).

QM Honors HW#5 Biao Huang

TOTAL=35

(10) 13.1.5/ $0 = \frac{\partial}{\partial t}\langle \vec{R}\cdot\vec{P}\rangle = \langle [\vec{R}\cdot\vec{P}, \vec{H}] \rangle$

$= \langle [xP_x + yP_y + zP_z, T+V] \rangle$

Here $\langle [xP_x, \frac{P_x^2}{2m}] \rangle = \frac{1}{2m}\langle [x, P_x^2] P_x \rangle$

$= i\hbar \langle \frac{P_x^2}{m} \rangle = 2i\hbar \langle T_x \rangle$

$\Rightarrow \langle [\vec{R}\cdot\vec{P}, T] \rangle = 2i\hbar \langle T \rangle$

For the potential part, we work in the coordinate representation:

$[xP_x, \frac{-e^2}{r}]\psi(x) = -e^2\left(x(-i\hbar\frac{\partial}{\partial x})\frac{\psi}{r} - \frac{x}{r}(-i\hbar\frac{\partial \psi}{\partial x})\right)$

$= i\hbar e^2 x\left(\psi \frac{\partial}{\partial x}\frac{1}{r}\right)$

$= -i\hbar e^2 x \psi \frac{x}{r^3} = -i\hbar \frac{x^2}{r^2}\left(\frac{e^2}{r}\psi\right)$

$\Rightarrow [\vec{R}\cdot\vec{P}, V] = -i\hbar \frac{e^2}{r} = i\hbar V(r)$

$\Rightarrow \langle [\vec{R}\cdot\vec{P}, H] \rangle = (2\langle T\rangle + \langle V\rangle) i\hbar = 0$

$\Rightarrow \langle T \rangle = -\frac{1}{2}\langle V \rangle$

(5) 13.3.1/ The range corresponds to Compton wavelength

$$\lambda_\pi = \frac{\hbar}{m_\pi c} \Rightarrow m_\pi c^2 = \frac{\hbar c}{\lambda e} = \frac{10^{-34} \times 3 \times 10^8}{10^{-15} \times 1.6 \times 10^{-19}} eV$$

$$\cong 200 \text{ MeV}$$

(5) 13.3.3/ $-(E_2 - E_1) = -13.6\left(1 - \frac{1}{4}\right) eV = -10.2 eV$

For $T = 300K$, $kT = \frac{1}{40} eV$

$\Rightarrow T = 6000K$. $kT = \frac{1}{40} \times 20 eV = \frac{1}{2} eV$

$$\Rightarrow \frac{P(n=2)}{P(n=1)} \cong 4e^{-20} \rightarrow 0$$

$T \sim 10^5 K$, $kT \cong \frac{1}{40} \times 400 eV = 10 eV$ $(T = 1.2 \times 10^5 K)$

Then kT is comparable to $10.2 eV$, and

$\frac{P(n=2)}{P(n=1)}$ starts to become significant.

(5)

13.4.1/ If we neglect interelectron interaction and change the potential to $\frac{Ze}{r}$, we change $V \to \frac{Ze^2}{r}$. Thus, we should replace $e^2 \to Ze^2$ in all equations. Then

$$E_n = -\frac{me^4}{2\hbar^2}\frac{1}{n^2} \to -\frac{m(Ze^2)^2}{2\hbar^2}\frac{1}{n^2} = Z^2\left[-\frac{me^4}{2\hbar^2}\frac{1}{n^2}\right]$$

This is because 1) the interaction is stronger: $\phi = \frac{e^2}{r} \to \frac{Ze^2}{r}$.

2) Distance is decreased: $a_0 = \frac{\hbar^2}{me^2} \to \frac{\hbar^2}{mZe^2}$

$\Rightarrow$ The total effect is E_n increases by Z^2

(5)

13.4.2/ For uranium, $Z = 92$. $a_0 = \frac{a_0}{Z}$

If electrons fill in the order of increasing n, we know the degeneracy for each n is $2\times\sum_{l=0}^{n-1}(2l+1) = 2n^2$. Then from

$\begin{cases} \sum_{n=1}^{5} 2n^2 = 110, \\ \sum_{n=1}^{4} 2n^2 = 60 \end{cases}$

we can take the radius of uranium atom to be $\langle r\rangle = n^2\frac{a_0}{Z} = \frac{5^2}{92}a_0 = \frac{25}{92}a_0$, the average radius of the 5th level

4

(5)

13.4.3/ For visible light,

$$\omega_{nn'} = \frac{2\pi}{T} = \frac{2\pi c}{\lambda} \approx 3.6 \times 10^{15} \cdot s^{-1}$$

Then,

$$\omega_{nn'} = \frac{Ry}{\hbar}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right) \approx 20 \times 10^{15}\, s^{-1}$$

Lyman: $\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = 0.75$

Balmer: $\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{5}{36} \sim \frac{1}{7}$

Paschen: $\left(\frac{1}{3^2} - \frac{1}{4^2}\right) = \frac{7}{144} \sim \frac{1}{20}$

Thus, Balmer series is most likely the first series discovered

Problem Set 16

Problem 13.3.1: To say the pion has a range of $\lambda \simeq 10^{-5}$ angstroms is to say that a single pion can be localized on this scale: $\Delta X \sim \lambda$. From the uncertainty principle $\Delta X\, \Delta P \gtrsim \hbar$, we then deduce $\Delta P \gtrsim \hbar/\lambda$. Since we assume λ is the smallest scale on which we can localize the pion, it is plausible that the inequality is saturated, so $P \sim \Delta P \sim \hbar/\lambda$. The relation between the energy and momentum is (from special relativity) $E^2 = m^2c^4 + c^2P^2 \sim m^2c^4 + c^2\hbar^2/\lambda^2$. Now, for the notion of a "single pion" to exist, we must have $E \lesssim 2mc^2$. (See discussion in text on p. 363). So $4m^2c^4 \gtrsim m^2c^4 + c^2\hbar^2/\lambda^2$, or $mc^2 \gtrsim c\hbar/(\sqrt{3}\lambda)$. Again, since λ is the smallest scale, it is plausible that the inequality is saturated, giving

$$mc^2 \sim \frac{c\hbar}{\sqrt{3}\lambda} \sim \frac{2000\,\mathrm{eV}\ \mathring{A}}{1.7\times 10^{-5}\ \mathring{A}} \sim 100\,\mathrm{MeV}.$$

(In reality $m_\pi c^2 = 140$ MeV.)

Problem 13.3.2: Since the kinetic energy is $T = 200$ eV $\ll 0.5$ MeV $\simeq mc^2$, the electron is non-relativistic, so we can use $T = p^2/(2m) = (p^2c^2)/(2mc^2)$ which implies $pc = \sqrt{2mc^2T}$. Then

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\hbar c}{\sqrt{2mc^2T}} \simeq \frac{\sqrt{2}\pi(2000\,\mathrm{eV}\ \mathring{A})}{\sqrt{(0.5\times 10^6\mathrm{eV})(200\,\mathrm{eV})}} = \frac{\sqrt{2}\pi}{10}\ \mathring{A} \simeq 1\ \mathring{A}\,.$$

Problem 13.3.3: Recalling $E_n = -\mathrm{Ry}/n^2 \simeq -13\mathrm{eV}/n^2$, we have

$$\frac{P(n=2)}{P(n=1)} = 4e^{-(E_2-E_1)/kT} = 4e^{-[(-1/4)-(-1)]13\,\mathrm{eV}/(k_BT)} \simeq 4e^{-10^5/(T/K)}$$

where I used $(k_B/\mathrm{eV}) \sim 9\times 10^{-5}\mathrm{K}^{-1}$. So it is clear that we need $T \gtrsim 10^5$ K so that the exponent is not very small. For example, if $T = 6000$ K, then

$$\frac{P(n=2)}{P(n=1)} \simeq 4e^{-10^5/(6000)} \simeq 4e^{-16} \sim 2\times 10^{-7} \ll 1.$$

PHY 615 – 2012
PROBLEM SET #1

Problem 1 (14.3.1)

Consider spinor $\begin{pmatrix} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{pmatrix}$

Normalization $\Rightarrow \quad \rho_1^2 + \rho_2^2 = 0$

Parametrize $\quad \rho_1 = \cos\frac{\theta}{2}$

$\rho_2 = \sin\frac{\theta}{2}$

$$\chi = e^{i\frac{\phi_1+\phi_2}{2}} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi_2-\phi_1}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi_2-\phi_1}{2}} \end{pmatrix}$$

$\Rightarrow$ n is along direction

$\theta, \phi \equiv \theta, \frac{\phi_2-\phi_1}{2}$.

Problem 2 (14.3.2)

(1) $\hat{n}\cdot\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$

$$(\hat{n}\cdot\vec{\sigma})|\hat{n}+\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2}\left(\cos\theta\cos\theta/2 + \sin\theta\sin\frac{\theta}{2}\right) \\ e^{i\phi/2}\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix} = |\hat{n}+\rangle$$

$$(\hat{n}\cdot\vec{\sigma})|\hat{n}-\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\theta/2\, e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2}\left(-\sin\frac{\theta}{2}\cos\theta + \sin\theta\cos\frac{\theta}{2}\right) \\ e^{i\phi/2}\left(-\sin\theta\sin\theta - \cos\theta\cos\theta/2\right) \end{pmatrix}$$

$$= -|\hat{n},-\rangle$$

(2) $\langle n+|\sigma_1|n+\rangle$

$$= \left(\cos\frac{\theta}{2}e^{+i\phi/2}, \sin\frac{\theta}{2}e^{-i\phi/2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}e^{-i} \\ \sin\frac{\theta}{2}e^{i} \end{pmatrix}$$

$$= \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi}\right)$$

$$= 2\cos\phi\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\cos\phi$$

$$= n_x$$

Thus $\langle n+ | S_1 | n+ \rangle = \frac{\hbar}{2} n_x$

The other relations can be verified similarly

Problem 3 (14.3.3)

From (14.3.32)

$$\mathrm{Tr}\, \sigma_i \cdot \sigma_j = -\mathrm{Tr}(\sigma_j \cdot \sigma_i) = -\mathrm{Tr}(\sigma_i \cdot \sigma_j)$$

(Since $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$)

$$\Rightarrow \mathrm{Tr}\, \sigma_i \cdot \sigma_j = 0 \quad (i \neq j)$$

From (14.3.33)

$$\sigma_i \cdot \sigma_j = i \epsilon_{ijk} \sigma_k .$$

Thus we have

$$i \epsilon_{ijk}\, \mathrm{Tr}\, \sigma_k = 0$$

Since ϵ_{ijk} is totally antisymmetric this implies

$$\mathrm{Tr}\, \sigma_k = 0 \quad \forall k .$$

eg for $i=1, j=2$

$$i \epsilon_{12k}\, \mathrm{Tr}\, \sigma_k = i\, \mathrm{Tr}\, \sigma_3 = 0$$

Problem 4 (4.3.4)

(1) $(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = A_i B_j \sigma_i \sigma_j$

$\sigma_i \sigma_j = \frac{1}{2}[\sigma_i, \sigma_j] + \frac{1}{2}\{\sigma_i, \sigma_j\}$

$= i\epsilon_{ijk}\sigma_k + \delta_{ij}$

$\Rightarrow$

$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = A_i B_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k)$

$= \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}$

(2) Suppose

$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = m_0 + \sum_{i=1}^{3} m_i \sigma_i$

Then

$m_0 = \frac{1}{2}\mathrm{Tr}[(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma})]$

$= \frac{1}{2} A_i B_j \mathrm{Tr}(\sigma_i \sigma_j)$

$= \frac{1}{2} A_i B_j 2\delta_{ij} = \vec{A}\cdot\vec{B}$

$m_1 = \frac{1}{2} A_i B_j \mathrm{Tr}(\sigma_i \sigma_j \sigma_1)$

The trace is nonzero and equal to $2i$ only if $i=2, j=3$ or $i=3, j=2$

$\Rightarrow m_1 = i(A_2 B_3 - A_3 B_2)$

$= i(\vec{A}\times\vec{B})_1$

Similarly for m_2

Problem 5 (14.3.5)

Suppose a 2×2 matrix M is expanded as follows

$$M = \sum_{\mu=0}^{3} m_\mu \sigma_\mu \qquad \sigma_0 \equiv \mathbb{I}_{2\times 2}$$

Then $\mathrm{Tr}\, M = 2m_0$

since $\mathrm{Tr}\,\sigma_i = 0 \qquad i = 1, 2, 3 \ldots$

$$\mathrm{Tr}(M\sigma_i) = m_0 \mathrm{Tr}\,\sigma_i + \sum_j m_j \mathrm{Tr}(\sigma_i \sigma_j)$$

Now use

$$\mathrm{Tr}\,\sigma_i \sigma_j = 2\delta_{ij}$$

and $\mathrm{Tr}\,\sigma_i = 0$

$\Rightarrow$

$$\mathrm{Tr}(M\sigma_i) = 2m_i$$

Thus

$$m_0 = \frac{1}{2}\mathrm{Tr}\, M$$

$$m_i = \frac{1}{2}\mathrm{Tr}(M\sigma_i)$$

Direct evaluation

$\Rightarrow$

$$m_0 = \frac{1}{2}(\alpha + \delta)$$

$$m_1 = \frac{1}{2}(\beta + \gamma)$$

$$m_2 = \frac{1}{2} i(\beta - \gamma)$$

$$m_3 = \frac{1}{2}(\alpha - \delta)$$

which may be solved for $\alpha, \beta, \gamma, \delta$

6

Problem 6 (14.3.7)

(1) $I + i\sigma_x = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sigma_x\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sigma_x \sin\frac{\pi}{4}\right)$

$= \sqrt{2}\, e^{i\sigma_x \pi/4}$

Thus

$(I + i\sigma_x)^{1/2} = 2^{1/4} e^{i\sigma_x \pi/8}$

$= 2^{1/4}\left[\cos\frac{\pi}{8} + i\sigma_x \sin\frac{\pi}{8}\right]$

(2) $(2I + \sigma_x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$(2I + \sigma_x)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$= \frac{1}{3}(2I - \sigma_x)$

(3) $\sigma_x^{-1} = \sigma_x$ since $\sigma_x^2 = 1$

7 # #

Problem 5 (14.3.8)

Consider a general matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Then, as in Problem 4

$$M = \sum_{\mu=0}^{3} m_\mu \sigma_\mu$$

(1) $$[M, \sigma_i] = \sum_{\mu=0}^{3} m_\mu [\sigma_\mu, \sigma_i]$$

$$= \sum_{k=1}^{3} m_k [\sigma_k, \sigma_i]$$

$$= 2 \sum_{k=1}^{3} m_k \, i \epsilon_{kij} \sigma_j$$

Now, $\sum_{i=1}^{3} n_i \sigma_i = 0 \Rightarrow n_i = 0 \ \forall i$

Thus

$$\sum_k \epsilon_{jki} m_k = 0 \quad \forall (i,j)$$

ϵ_{jki} (2 3)

$[M, \sigma_1] = 0 \Rightarrow m_2 = m_3 = 0$

$[M, \sigma_2] = 0 \Rightarrow m_3 = m_1 = 0$

Thus only $m_0 \neq 0$ possible

ie M must be a multiple of the unit matrix

(2) For the anticommutator

$$\{M, \sigma_i\} = 2m_0\sigma_i + \sum_j m_j \{\sigma_j, \sigma_i\}$$

However

$$\{\sigma_j, \sigma_i\} = 2\delta_{ji}$$

Thus

$$\{M, \sigma_i\} = 2(m_0\sigma_i + m_i)$$

Thus $\{M, \sigma_i\} = 0 \Rightarrow m_0, m_i = 0$

e.g.

$$0 = \{M, \sigma_3\} = 2(m_0\sigma_3 + m_3)$$

$$= 2\begin{pmatrix} m_0+m_3 & 0 \\ 0 & m_0-m_3 \end{pmatrix}$$

$$\Rightarrow m_0 = m_3 = 0$$

$$\{M, \sigma_1\} = 0 = 2(m_0\sigma_1 + m_1)$$

$$\Rightarrow m_1 = 0$$
$$m_0 = 0$$

$\{M, \sigma_2\} = 0 \Rightarrow m_2 = 0$ as well

Thus such a M must vanish

① 14.33

know $\sigma_i \sigma_j = -\sigma_j \sigma_i \quad i \neq j \quad (1)$

$\sigma_i \sigma_j = i\sigma_k$ for $\varepsilon_{ijk}=1 \quad (2)$

Using (1), can turn (2) into $\sigma_j \sigma_i = -i\sigma_k$ for $\varepsilon_{ijk}=1$

So $\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k$

Now $Tr(\sigma_i \sigma_j) = Tr(\sigma_j \sigma_i)$ (see Ex. 17.1)

But if $\sigma_i \sigma_j = -\sigma_j \sigma_i$ then

$$Tr(\sigma_i \sigma_j) = -Tr(\sigma_j \sigma_i)$$

so $Tr(\sigma_i \sigma_j) = 0$ for $i \neq j$

Now

$$0 = Tr(i\varepsilon_{ijk}\sigma_k) = i\varepsilon_{ijk} Tr(\sigma_k)$$

$$\rightarrow Tr(\sigma_k) = 0 \text{ for all } k.$$

(2) 14.3.4 Show $(\hat{\vec{\sigma}}\cdot\vec{A})(\hat{\vec{\sigma}}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\hat{\vec{\sigma}}\cdot\vec{A}\times\vec{B}$

$$\sigma_i\sigma_j = \frac{1}{2}\left([\hat{\sigma}_i,\hat{\sigma}_j]_+ + [\hat{\sigma}_i,\hat{\sigma}_j]\right) = \delta_{ij}\mathbb{1} + i\sum_k \varepsilon_{ijk}\hat{\sigma}_k$$

see p. 384

(1) $$(\hat{\vec{\sigma}}\cdot\vec{A})(\hat{\vec{\sigma}}\cdot\vec{B}) = \left(\sum_i \hat{\sigma}_i A_i\right)\left(\sum_j \hat{\sigma}_j B_j\right) = \sum_{i,j}\hat{\sigma}_i\hat{\sigma}_j A_i B_j$$

$$= \sum_{i,j}\left[\delta_{ij}\mathbb{1} + i\sum_k \varepsilon_{ijk}\hat{\sigma}_k\right]A_iB_j$$

$$= \sum_i A_iB_i\mathbb{1} + i\sum_{ijk}\varepsilon_{ijk}A_iB_j\hat{\sigma}_k$$

$$= A\cdot B + i\hat{\vec{\sigma}}\cdot(\vec{A}\times\vec{B}) \quad \checkmark$$

(2) Let $\hat{M} = (\hat{\vec{\sigma}}\cdot\vec{A})(\hat{\vec{\sigma}}\cdot\vec{B}) = \sum_{ij}\hat{\sigma}_i\hat{\sigma}_j A_iB_j$ as above.

if $\hat{M} = \sum_i m_i\sigma_i$, $m_i = \frac{1}{2}\mathrm{Tr}(\hat{M}\hat{\sigma}_i)$

$$\hat{M} = \sum_{ij}\left(\delta_{ij}\mathbb{1} + \sum_k i\varepsilon_{ijk}\sigma_k\right)A_iB_j$$

$$\Rightarrow M_0 = \frac{1}{2}\mathrm{Tr}(\hat{M}\mathbb{1}) = \frac{1}{2}\mathrm{Tr}(\hat{M}) = \sum_{ij}\delta_{ij}A_iB_j = \vec{A}\cdot\vec{B}$$

$$M_x = \frac{1}{2}\mathrm{Tr}(\hat{M}\sigma_x) = i\Sigma_{ij}\varepsilon_{ijx}A_iB_i = i[A_yB_z - A_zB_y]$$

+ cyc. perm.

$$\Rightarrow M = \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\hat{\vec{\sigma}} \quad \checkmark$$

(3) 14.3.7

(1) Recall rotations effectively add:

$$\hat{R}(a)\hat{R}(b) = \hat{R}(a+b)$$

so $$\left[\hat{R}_{\vec{n}}(a)\right]^{\frac{1}{2}} = R_{\vec{n}}\left(\frac{a}{2}\right)$$

Now $\sqrt{2}\left(\mathbb{1} + i\hat{\sigma}_x\right) = \hat{R}_x\left(\frac{-\pi}{2}\right)$

$$\longrightarrow \quad \left(\mathbb{1} + i\sigma_x\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}\hat{R}_x\left(\frac{-\pi}{4}\right)$$

$$= 2^{\frac{1}{4}}\left[\mathbb{1}\cos\frac{\pi}{8} + i\hat{\sigma}_x \sin\frac{\pi}{8}\right]$$

(2) $$\left(2+\hat{\sigma}_x\right)^{-1} = \frac{\mathbb{1}}{\left(2+\hat{\sigma}_x\right)}\,\frac{2-\hat{\sigma}_x}{2-\hat{\sigma}_x} = \frac{2-\hat{\sigma}_x}{4-\hat{\sigma}_x^2} = \frac{1}{3}\left(2-\hat{\sigma}_x\right)$$

$\hat{\sigma}_x^2 = 1$

Or you could do it with $\hat{M}\left(2+\hat{\sigma}_x\right) = \mathbb{1}$ and solve for components of $\hat{M}$.

(3) $\hat{\sigma}_x^2 = \mathbb{1}$, so $\hat{\sigma}_x^{-1} = \hat{\sigma}_x$.

14.3.8

$\vec{\sigma} = \sigma_x + \sigma_y + \sigma_z$

①

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (1-i) \\ (1+i) & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & (1-i) \\ (1+i) & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & (1-i) \\ (1+i) & -1 \end{pmatrix}$$

$$\begin{pmatrix} a + (1-i)c & b + (1-i)d \\ (1+i)a - c & (1+i)b - d \end{pmatrix} - \begin{pmatrix} a + (1+i)b & a(1-i) - b \\ c + (1+i)d & c(1-i) - d \end{pmatrix}$$

$$= \begin{pmatrix} [c(1-i) - b(1+i)] & [d(1-i) - a(1-i)] \\ [(1+i)a - (1+i)d] & [(1+i)b - (1-i)c] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$

$$c(1-i) - b(1+i) = 0$$
$$-c(1-i) + b(1+i) = 0 \quad \Big\} \text{ same eqn}$$

and

$$-a(1-i) + d(1-i) = 0$$
$$a(1+i) - (1+i)d = 0 \quad \Big\} \Rightarrow a = d$$

$$c - ic - b + ib = 0$$

$$(c-b) + i(-c-b) = 0 \Rightarrow c = b = 0$$

thus $a = d = s$, $c = b = 0$ $\Rightarrow$ $\boxed{s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

14.4.1 let $H = -\gamma \vec{L} \cdot \vec{B} = \sum_j -\gamma L_j B_j$

$i\hbar \frac{d}{dt}\langle \vec{L} \rangle = \langle [\vec{L}, H] \rangle$

$[\vec{L}, H] = -\gamma [\vec{L}, \vec{L} \cdot \vec{B}] = -\gamma \left[\vec{L}, \sum_j L_j B_j\right] = -\gamma \sum_{ij} [L_i, L_j B_j]$

Now $-[L_j B_j, L_i] = -\{L_j [B_j, L_i] + [L_j, L_i] B_j\}$

$= -[L_j, L_i] B_j$; But $[L_j, L_i] = -i\hbar \sum_k \epsilon_{ijk} L_k$

Thus $[L_i, L_j B_j] = i\hbar \sum_k \epsilon_{ijk} L_k B_j =$ [illegible]

So $[\vec{L}, H] = -\gamma i\hbar \sum_i \sum_j \sum_k \epsilon_{ijk} L_k B_j$

But $\sum_j \sum_k \epsilon_{ijk} L_k B_j = (\vec{L} \times \vec{B})_i$

$\therefore [\vec{L}, H] = -\gamma i\hbar \sum_i (\vec{L} \times \vec{B})_i = -i\hbar \gamma (\vec{L} \times \vec{B})$

So $\frac{d}{dt}\langle \vec{L} \rangle = \langle -\gamma \vec{L} \times \vec{B} \rangle = -\gamma \langle \vec{L} \rangle \times \vec{B}$

(6) 14.4.3 Let $\hat{R}_z(\omega t) = e^{\frac{-i\omega t \hat{S}_z}{\hbar}} \Rightarrow |\psi_r\rangle = \hat{R}_z(\omega t)|\psi\rangle$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\Rightarrow i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}\hat{R}_z^{\dagger}(\omega t)|\psi_r(t)\rangle$$

$$i\hbar \frac{d}{dt}\hat{R}_z^{\dagger}(\omega t)|\psi_r(t)\rangle = \hat{H}\hat{R}_z^{\dagger}(\omega t)|\psi_r(t)\rangle$$

$$i\hbar\left(\frac{d\hat{R}_z^{\dagger}(\omega t)}{dt}\right)|\psi_r(t)\rangle + i\hbar \hat{R}_z^{\dagger}(\omega t)\frac{d}{dt}|\psi_r(t)\rangle = \hat{H}\hat{R}_z^{\dagger}(\omega t)|\psi_r(t)\rangle$$

$$\Rightarrow i\hbar \frac{d}{dt}|\psi_r(t)\rangle = -i\hbar \hat{R}_z(\omega t)\left(\frac{d}{dt}\hat{R}_z^{\dagger}(\omega t)\right)|\psi_r\rangle + \hat{R}_z(\omega t)\hat{H}\hat{R}_z^{\dagger}(\omega t)|\psi_r(t)\rangle$$

$$= \hat{H}_r|\psi_r(t)\rangle$$

Now $\hat{R}_z(\omega t) = e^{\frac{-i\omega t}{\hbar}\hat{S}_z} = e^{\frac{-i\omega t}{2}\hat{\sigma}_z}$

$$= \cos\frac{\omega t}{2} - i\hat{S}_z \sin\frac{\omega t}{2}$$

so $-i\hbar \hat{R}_z(\omega t)\frac{d}{dt}\hat{R}_z^{\dagger}(\omega t) = -i\hbar \frac{i\omega}{2} e^{\frac{i\omega t}{2}\hat{\sigma}_z} \hat{\sigma}_z e^{\frac{-i\omega t}{2}\hat{\sigma}_z}$

$$= \frac{\hbar\omega}{2}\hat{\sigma}_z$$

$$\Rightarrow \hat{H}_r = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hat{R}_z(\omega t)\hat{H}\hat{R}_z^{\dagger}(\omega t)$$

$$\hat{R}_z(\omega t)\hat{H}\hat{R}_z^{\dagger}(\omega t) = \frac{-\hbar}{2}\left(\cos\frac{\omega t}{2} - i\hat{\sigma}_z \sin\frac{\omega t}{2}\right)\left[\gamma B_0 \hat{\sigma}_z + \gamma B\vec{\sigma}\cdot\left(\vec{e}_x \cos\omega t - \vec{e}_y \sin\omega t\right)\right]\cdot$$

$$\cdot\left(\cos\frac{\omega t}{2} + i\hat{\sigma}_z \sin\frac{\omega t}{2}\right)$$

Terms proportional to $B_0\gamma$ is $\frac{\hbar B_0\gamma}{2}\hat{\sigma}_z$

" " " " $-\frac{\hbar\gamma B}{2}\left(\cos\frac{\omega t}{2} - i\hat{\sigma}_z \sin\frac{\omega t}{2}\right)\left(\sigma_x\cos\omega t - \sigma_y \sin\omega t\right)\cdot$

$\cdot\left(\frac{\cos\omega t}{2} + i\hat{\sigma}_z\sin\frac{\omega t}{2}\right)$

Now, use $\sigma_i\sigma_j = -\sigma_j\sigma_i$:

$$= -\frac{\hbar\gamma B}{2}\left[\cos^2\frac{\omega t}{2}\left(\hat{\sigma}_x\cos\omega t - \hat{\sigma}_y\sin\omega t\right) + \sin^2\frac{\omega t}{2}\hat{\sigma}_z\left(\hat{\sigma}_x\cos\omega t - \hat{\sigma}_y\sin\omega t\right)\hat{\sigma}_z \right.$$

$$\left. * -2i\frac{\cos\omega t}{2}\sin\frac{\omega t}{2}\hat{\sigma}_z\left(\hat{\sigma}_x\cos\omega t - \hat{\sigma}_y\sin\omega t\right)\right]$$

(Now $\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z = -\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x = -\hat{\sigma}_x$

$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z = -\hat{\sigma}_y$

$\sigma_z\sigma_x = i\sigma_y$ + cyc. perm.)

so $$= -\frac{\hbar\gamma B}{2}\left[\cos\omega t\left(\hat{\sigma}_x\cos\omega t - \hat{\sigma}_y\sin\omega t\right) + \sin\omega t\left(\sigma_y\cos\omega t + \sigma_x\sin\omega t\right)\right]$$

$$= -\frac{\hbar\gamma B}{2}\left[\hat{\sigma}_x + \sigma_y(\sin\omega t\cos\omega t - \sin\omega t\cos\omega t)\right]$$

$$= -\frac{\hbar\gamma B}{2}\hat{\sigma}_x$$

So $\hat{H}_r = \frac{\hbar\omega}{2}\hat{\sigma}_z - \frac{\hbar B_0\gamma}{2}\hat{\sigma}_z - \frac{\hbar B\gamma}{2}\hat{\sigma}_x$ = time-independent.

$$\hat{U}_r(t) = \exp\left(\frac{-i\hat{H}_r t}{\hbar}\right) = \exp\left(-it\left[\frac{\omega - B_0\gamma}{2}\hat{\sigma}_z - \frac{B\gamma}{2}\hat{\sigma}_x\right]\right)$$

$$|\psi_r\rangle = \hat{U}_r(t)|\psi_r(0)\rangle = \exp\left[-it\left(\frac{\omega - B_0\gamma}{2}\hat{\sigma}_z - \frac{B\gamma}{2}\hat{\sigma}_x\right)\right]|\psi_r(0)\rangle$$

Start with $|\psi_r(0)\rangle = |+_z\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ so $\hat{\sigma}_z|\psi_r(0)\rangle = |\psi_r(0)\rangle$

$$|\psi_r(t)\rangle = e^{-it\left(\frac{\omega - B_0\gamma}{2}\right)\hat{\sigma}_z} e^{-\frac{itB\gamma}{2}\hat{\sigma}_x}\begin{pmatrix}1\\0\end{pmatrix}$$

$$|\psi(t)\rangle = \hat{R}^\dagger|\psi_r(t)\rangle = \hat{R}^\dagger\hat{U}_r|\psi_r(t)\rangle$$

$$= \left(\cos\frac{\omega t}{2} - i\hat{\sigma}_z\sin\frac{\omega t}{2}\right)\left(\cos\frac{\omega_r t}{2} + i\frac{\vec{\sigma}\cdot\vec{\omega}_r}{|\vec{\omega}_r|}\sin\frac{\omega_r t}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

where $\vec{\omega}_r = -(\omega - B_0\gamma)\vec{e}_z + B\gamma\vec{e}_x$

In matrix form:

$$\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = \begin{pmatrix}e^{i\frac{\omega t}{2}} & 0\\ 0 & e^{-i\frac{\omega t}{2}}\end{pmatrix}\cdot\left[\begin{pmatrix}\cos\frac{\omega_r t}{2} & 0\\ 0 & \cos\frac{\omega_r t}{2}\end{pmatrix} + \frac{-i(\omega - B_0\gamma)}{\omega_r}\begin{pmatrix}\sin\frac{\omega_r t}{2} & 0\\ 0 & -\sin\frac{\omega_r t}{2}\end{pmatrix} + \frac{i\gamma B}{\omega_r}\begin{pmatrix}0 & \sin\frac{\omega_r t}{2}\\ \sin\frac{\omega_r t}{2} & 0\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\left[\cos\frac{\omega_r t}{2} - i\frac{(\omega - B_0\gamma)}{\omega_r}\sin\frac{\omega_r t}{2}\right]e^{i\omega t/2}\\ \frac{i\gamma B}{\omega_r}\sin\frac{\omega_r t}{2}\, e^{-i\omega t/2}\end{pmatrix}$$

$$\langle M_z(t)\rangle = \mu\langle\hat{S}_z(t)\rangle = \frac{\mu\hbar}{2}\langle\hat{\sigma}_z(t)\rangle$$

$$= \left(|\psi_+(t)|^2 - |\psi_-(t)|^2\right)\frac{\mu\hbar}{2} = \cos^2\frac{\omega_r t}{2} + \frac{(\omega - B_0\gamma)^2}{\omega_r^2}\sin^2\frac{\omega_r t}{2} - \left(\frac{\gamma B}{\omega_r}\right)^2\sin^2\frac{\omega_r t}{2}$$

$$= \frac{(\omega - B_0\gamma)^2 + \gamma^2B^2\cos\omega_r t}{(\omega - B_0\gamma)^2 + \gamma^2B^2}\langle M_z(t=0)\rangle$$

15.1.1 Derive $S^2 \xrightarrow[\text{Prod. basis}]{} \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

So $S^2 = S_1^2 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes S_2^2 + 2\{S_{1x} \otimes S_{2x} + S_{1y} \otimes S_{2y} + S_{1z} \otimes S_{2z}\}$

Now act on the following basis to get columns of S^2 $\Rightarrow \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$

$S^2|++\rangle = \frac{3}{4}\hbar^2|++\rangle + \frac{3}{4}\hbar^2|++\rangle + \frac{2\hbar^2}{4}\{|--\rangle - |--\rangle + |++\rangle\}$

$= \frac{3}{2}\hbar^2|++\rangle + \frac{\hbar^2}{2}|++\rangle = 2\hbar^2|++\rangle \Rightarrow \hbar^2\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$S^2|+-\rangle = \frac{3}{4}\hbar^2|+-\rangle + \frac{3}{4}\hbar^2|+-\rangle + \frac{2\hbar^2}{4}\{|-+\rangle + |-+\rangle - |+-\rangle\}$

$= \frac{3}{2}\hbar^2|+-\rangle - \frac{1}{2}\hbar^2|+-\rangle + \hbar^2|-+\rangle$

$= \hbar^2|+-\rangle + \hbar^2|-+\rangle \Rightarrow \hbar^2\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Carrying out the same calculation w/ the last two you get

$$S^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

15.1.2

$$H_{HF} = A\vec{S}_1 \cdot \vec{S}_2 \qquad \vec{S} \equiv \vec{S}_1 + \vec{S}_2$$

$$S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$$

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S_{\pm}^2 - S_1^2 - S_2^2\right)$$

$$\langle E_{hf} \rangle = \langle A\vec{S}_1 \cdot \vec{S}_2 \rangle = A\langle \vec{S}_1 \cdot \vec{S}_2 \rangle$$

$$= \frac{A}{2}\langle S^2 - S_1^2 - S_2^2 \rangle$$

$$= \frac{A}{2}\left[\langle S^2 \rangle - \langle S_1^2 \rangle - \langle S_2^2 \rangle\right]$$

But $\langle S_1^2 \rangle = \langle S_2^2 \rangle = \frac{3}{4}\hbar^2$

And if we are in singlet state

$$\langle S^2 \rangle = 0$$

And in triplet state

$$\langle S^2 \rangle = 2\hbar^2$$

Thus we have

$$E_T = -13.6\,eV + \langle E_{HF} \rangle$$

$$= -13.6\,eV + \frac{A\hbar^2}{4} \quad (triplet)$$

$$= -13.6\,eV - A\hbar^2\frac{3}{4} \quad (singlet)$$

(2) Assume $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(i) $G_x A + A G_x = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} + \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

$$= \begin{pmatrix} [c+b] & [d+a] \\ [a+d] & [c+b] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(ii) $G_y A + A G_y = 0 \Rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -ic & -id \\ ia & ib \end{pmatrix} + \begin{pmatrix} ib & -ia \\ id & -ic \end{pmatrix}$

$$= \begin{pmatrix} i(b-c) & -i(a+d) \\ i(a+d) & i(b-c) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(iii) $G_z A + A G_z = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} + \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$

$$= \begin{pmatrix} 2a & 0 \\ 0 & -2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad a = d = 0$ from (iii)

(ii) $\Rightarrow \quad b = c$

(i) $\Rightarrow \quad b = -c$

$\Rightarrow \quad b = c = 0$

This $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(9) Math is done exactly as in lecture. Answers are on p. 667 in Shankar.

(1) Two State Problem

Let us write the states as $|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then
$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \frac{E_1+E_2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{E_1-E_2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Similarly
$$H_\phi = H_0 + \gamma \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

$$H = \frac{E_1+E_2}{2}\mathbb{1} + \gamma \cos\omega t\, \sigma_1 - \gamma \sin\omega t\, \sigma_2 + \frac{E_1-E_2}{2}\sigma_3$$

$$H = e^{\frac{i\omega t}{2}\sigma_3}\left[\frac{E_1+E_2}{2} + \gamma\sigma_1 + \frac{E_1-E_2}{2}\sigma_3\right] e^{-\frac{i\omega t}{2}\sigma_3}$$

We used the "trick" discussed in the class. If

$|\psi(t)\rangle$ is the state that satisfies the equation

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle.$$

We have shown $|\psi'(t)\rangle = e^{-\frac{i\omega t}{2}\sigma_3}|\psi(t)\rangle$ satisfies

the equation
$$i\hbar \frac{d}{dt}|\psi'(t)\rangle = H'\,|\psi'(t)\rangle$$

where
$$H' = \frac{E_1+E_2}{2} + \gamma\sigma_1 + \frac{E_1-E_2}{2}\sigma_3 + \frac{\hbar\omega}{2}\sigma_3$$

$$H_\Lambda = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}_\Lambda\right)^2 + q\phi_\Lambda$$

$$= e^{\frac{-iq}{\hbar c}\Lambda}\left[\frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + q\phi + \frac{q}{c}\frac{\partial \Lambda}{\partial t}\right]e^{\frac{iq}{\hbar c}\Lambda}$$

$$= e^{\frac{-iq}{\hbar c}\Lambda}\left[H + \frac{q}{c}\frac{\partial \Lambda}{\partial t}\right]e^{\frac{iq}{\hbar c}\Lambda}$$

(iii) If $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$

and $i\hbar \frac{d}{dt}|\psi(t)\rangle_\Lambda = \hat{H}_\Lambda\,|\psi(t)\rangle_\Lambda$

$$\Rightarrow \quad |\psi(t)\rangle_\Lambda = e^{\frac{-iq}{\hbar c}\Lambda}\,|\psi(t)\rangle$$

Substituting we see

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_\Lambda = \frac{q}{c}\frac{\partial \Lambda}{\partial t}\,|\psi(t)\rangle_\Lambda + e^{\frac{-iq}{\hbar c}\Lambda}\hat{H}\,e^{\frac{iq}{\hbar c}\Lambda}|\psi$$

$$= \hat{H}_\Lambda\,|\psi(t)\rangle_\Lambda$$

Let $E_1 - E_2 = \omega_{12}\hbar$, $\gamma = \omega_\gamma \hbar$, $\frac{E_1+E_2}{2} = \omega_0 \hbar$

Then we see that

$$H' = \hbar\left(\omega_0 + \omega_\gamma \sigma_1 + \frac{1}{2}(\omega_{12}+\omega)\sigma_3\right)$$

Let us define $\Omega = \left[\omega_\gamma^2 + (\omega_{12}+\omega)^2/4\right]^{1/2}$

$$n_1 = \frac{\omega_\gamma}{\Omega} \quad n_2 = 0 \quad n_3 = \frac{1}{2}\frac{(\omega_{12}+\omega)}{\Omega}$$

Then $$H' = \hbar\left(\omega_0 + \Omega \hat{n}\cdot\vec{\sigma}\right).$$

Now it is easy to solve for $|\psi'(t)\rangle$, we have

$$|\psi'(t)\rangle = \underbrace{\exp\left(\frac{-i}{\hbar}H't\right)}|\psi'(0)\rangle$$

$$e^{-i\omega_0 t}\left[\cos\Omega t - i\hat{n}\cdot\vec{\sigma}\sin\Omega t\right]$$

Initial state $|\psi'(0)\rangle = |\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

Final state at time t $|\psi(t)\rangle = \exp\left(\frac{i\omega t}{2}\sigma_3\right)|\psi'(t)\rangle$

$$|\psi(t)\rangle = e^{\frac{i\omega t}{2}\sigma_3} e^{-i\omega_0 t}\left[\cos\Omega t - i\hat{n}\cdot\vec{\sigma}\sin\Omega t\right]|\psi(0)\rangle$$

Amplitude to find the state in $|2\rangle$ is $\langle 2|\psi(t)\rangle$ so

$$A = \langle 2|\psi(t)\rangle = (0\ 1)\, e^{\frac{i\omega t}{2}\sigma_3} e^{-i\omega_0 t}(\cos\Omega t - i\hat{n}\cdot\vec{\sigma}\sin\Omega t)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= e^{\frac{-i\omega t}{2} - i\omega_0 t}(-in_1 + n_2)\sin\Omega t$$

Thus the probability

$$P_{|1\rangle\to|2\rangle}(t) = |A|^2 = n_1^2 \sin^2\Omega t$$

$$= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega+\omega_{12})^2/4}\sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega+\omega_{12})^2}{4}\right]^{1/2} t\right\}$$

Problem 14.4.4

The Hamiltonian of an electron in a magnetic field

$$H = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = -\frac{e}{2mc}\cdot 2\vec{S}$$

$$H = -\frac{e\hbar}{mc}\vec{\sigma}\cdot\vec{B}$$

(2 ↑ : g-factor of an electron.)

Now $\vec{B} = B\hat{i}$, which means $H = -\frac{e\hbar}{mc}B\sigma_1$.

Thus

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\frac{e\hbar B}{mc}t\,\sigma_1\right]|\psi(0)\rangle$$

Let $\omega = \frac{eB}{mc}$, then the state at time t

$$|\psi(t)\rangle = (\cos\omega t - i\sigma_1 \sin\omega t)\,|\psi(0)\rangle$$

We are given $|\psi(0)\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}$, we want $|\psi(t)\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}e^{i\alpha}$

We see that $\omega t = \pi \Rightarrow t = \frac{\pi}{\omega}$.

$$t = \frac{\pi mc}{eB}$$

we know $\frac{e\hbar}{2mc} \simeq 0.6\times10^{-8}\,eV/G$.

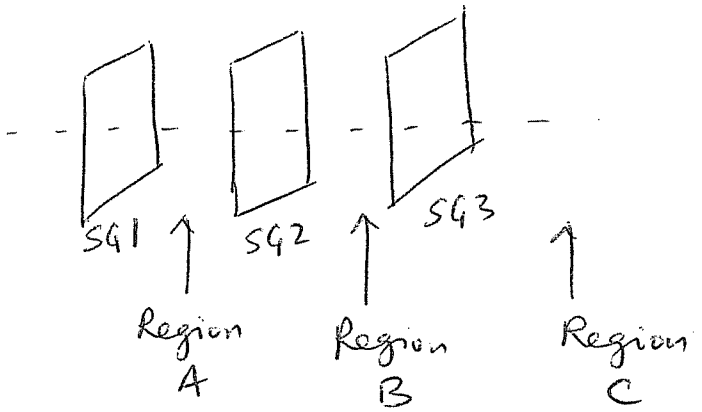
$B \simeq 100\,G$.

$\hbar = 6.6\times10^{-16}\,eV\cdot s$

$$t = \frac{\pi}{2\times 0.6\times10^{-8}}\cdot\frac{6.6\times10^{-16}}{100}$$

$$t = \frac{\pi\hbar}{1.2\times10^{-6}}\,sec \simeq 1.73\times10^{-9}\,sec$$

Problem 14.5.3



Region A has particles in state $|\hat{z},+\rangle_1$.

$$|\hat{z},+\rangle = \frac{1}{\sqrt{2}}\left[|\hat{x},+\rangle + |\hat{x},-\rangle\right]$$

(i) If the second filter allows only $|\hat{x},+\rangle$ states to pass. Then 50% of particles in region A will come out of region B.

(ii) If the 3rd filter allows only in $|\hat{z},+\rangle$ then

Since $$|\hat{x},+\rangle = \frac{1}{\sqrt{2}}\left[|\hat{z},+\rangle + |\hat{z},-\rangle\right]$$

only 50% of region B particles come into region C.

ie... overall 25% of region A particles come to region C.

(iii) If the 2nd filter allows both spins up/down (no blocking) then Region B contains $|\hat{z},+\rangle$ and then filter 3 transmits only $|\hat{z},-\rangle$ then <u>all</u> particles will be blocked and 0% will reach region C.

Problem 15.1.2

If we take into account the electron and the proton spin, the Coulomb Hamiltonian ground state $E_0 = -Ry$ is four-fold degenerate!

(1) A Hyper fine interaction

$$H_{hf} = A\,\vec{S}_1 \cdot \vec{S}_2$$

$$= \frac{A}{2}\left[\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right]$$

splits these levels. Since $\vec{S}_1$ and $\vec{S}_2$ are both spin-$\frac{1}{2}$, $\vec{S}$ can be spin-1 or spin 0.

$$E_{hf} = \frac{A}{2}\left[2\hbar^2 - \frac{3}{2}\hbar^2\right] = \frac{\hbar^2 A}{4} \quad \text{triplet}$$

$$= \frac{A}{2}\left[0 - \frac{3}{2}\hbar^2\right] = \frac{-3\hbar^2 A}{4} \quad \text{singlet.}$$

$$\therefore \quad E_+ = -Ry + \frac{\hbar^2 A}{4} \quad \text{triplet}$$

$$E_- = -Ry - \frac{3\hbar^2 A}{4} \quad \text{singlet.}$$

(2) To estimate the frequency emitted we equate

$$h\nu = \Delta E = A\hbar^2$$

In order to express it as wave length we use

$$\frac{hc}{\lambda} = \Delta E$$

$$\Rightarrow \quad \lambda = \frac{hc}{\Delta E}$$

Now $\Delta E = A\hbar^2 = \frac{1}{2}\frac{e^2\hbar^2}{a_0^3}\cdot\frac{5.6}{Mmc^2}$

$a_0 = \frac{\hbar^2}{me^2}$ and $\frac{me^4}{2\hbar^2} \simeq 13.3\,eV$

$$\Delta E = \frac{5.6}{2}\frac{m^2e^6}{\hbar^4}\frac{e^2\hbar^2}{Mmc^2}$$

$$= \frac{5.6}{2}\left(\frac{me^4}{2\hbar^2}\right)^2\cdot 4^2\cdot\frac{1}{Mc^2}$$

$$\simeq 2\cdot 5.6\cdot\frac{(13.3)^2 eV^2}{1\,GeV} = 1.98\times 10^{-6}\,eV.$$

$$\lambda = \frac{hc}{\Delta E} \simeq 62\,cms.$$

(3) The probability ratio $P(\text{triplet})/P(\text{singlet}) = 3\,\frac{\exp(-E_{\text{triplet}}/kT)}{\exp(-E_{\text{singlet}}/kT)}$ (3 ← degeneracy)

$$\simeq 3e^{-\Delta E/kT}$$

$kT \simeq \frac{1}{40}\,eV$ at $T = 300k$ (roomtemperature).

Since $\Delta E \ll kT$ we see that

$P(\text{triplet})/P(\text{singlet}) \simeq 3$.

Exercise 15.2.2 .

(1) $j_1 = 1 \quad j_2 = \frac{1}{2} \Rightarrow j = \frac{3}{2}$ or $\frac{1}{2}$

m ↓ / j →	$j = \frac{3}{2}$	$j = \frac{1}{2}$
	$\left\|\frac{3}{2}\ \frac{3}{2}\right\rangle$	
	$\left\|\frac{3}{2}\ \frac{1}{2}\right\rangle$	$\left\|\frac{1}{2}\ \frac{1}{2}\right\rangle$
	$\left\|\frac{3}{2}\ -\frac{1}{2}\right\rangle$	$\left\|\frac{1}{2}\ -\frac{1}{2}\right\rangle$
	$\left\|\frac{3}{2}\ -\frac{3}{2}\right\rangle$	

$$\rightarrow \quad \left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = |1\ 1\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle .$$

Applying J_- on both sides we get-

$$\sqrt{3}\hbar \left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{2}\hbar\, |1\,0\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \hbar\, |1\ 1\rangle \otimes \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$\rightarrow \quad \left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, \left|1\,0, \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \left|1\,1, \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$\rightarrow \quad \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, \left|1\,0, \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \left|1\,-1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$\rightarrow \quad \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \left|1\,-1, \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle .$$

$$\rightarrow \quad \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, \left|1\,1, \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\, \left|1\,0, \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

$$\rightarrow \quad \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\, \left|1\,0, \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\, \left|1\,-1, \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

One can read off the clebsch-Gordan coefficients .

(2) $j_1 = 1 \quad j_2 = 1 \Rightarrow j = 2, 1, 0$

m ↓ / j →	$j=2$	$j=1$	$j=0$
	$\|2,2\rangle$		
	$\|2,1\rangle$	$\|1,1\rangle$	
	$\|2,0\rangle$	$\|1,0\rangle$	$\|0,0\rangle$
	$\|2,-1\rangle$	$\|1,-1\rangle$	
	$\|2,-2\rangle$		

$\rightarrow \quad |2\,2\rangle = |11,11\rangle$

Applying J_- on both sides:

$$2\hbar\,|2\,1\rangle = \hbar\sqrt{2}\,|10,11\rangle + \hbar\sqrt{2}\,|11,10\rangle$$

$$\rightarrow \quad |2\,1\rangle = \frac{1}{\sqrt{2}}|10,11\rangle + \frac{1}{\sqrt{2}}|11,10\rangle$$

$$\sqrt{3}\cdot\sqrt{2}\hbar\,|2\,0\rangle = \frac{\hbar\sqrt{2}}{\sqrt{2}}|1\text{-}1,11\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|11,1\text{-}1\rangle$$

$$\rightarrow \quad |2\,0\rangle = \frac{1}{\sqrt{6}}|1\text{-}1,11\rangle + \sqrt{\frac{2}{3}}|10,10\rangle + \frac{1}{\sqrt{6}}|11,1\text{-}1\rangle$$

$$\rightarrow \quad |2\,\text{-}1\rangle = \frac{1}{\sqrt{2}}|10,1\text{-}1\rangle + \frac{1}{\sqrt{2}}|1\text{-}1,10\rangle$$

$$\rightarrow \quad |2\,\text{-}2\rangle = |1\text{-}1,1\text{-}1\rangle$$

$$\rightarrow \quad |1\,1\rangle = \frac{1}{\sqrt{2}}|11,10\rangle - \frac{1}{\sqrt{2}}|10,11\rangle$$

$$\hbar\sqrt{2}\,|1\,0\rangle = \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|11,1\text{-}1\rangle - \frac{\hbar\sqrt{2}}{\sqrt{2}}|1\text{-}1,11\rangle - \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle$$

$$\rightarrow \quad |1\,0\rangle = \frac{1}{\sqrt{2}}|11,1\text{-}1\rangle - \frac{1}{\sqrt{2}}|1\text{-}1,11\rangle$$

$$\rightarrow \quad |1,-1\rangle = \frac{1}{\sqrt{2}}|10,1,-1\rangle - \frac{1}{\sqrt{2}}|1,-1,10\rangle$$

We next want $|0,0\rangle$ which has the form

$$|00\rangle = \alpha|11,1,-1\rangle + \beta|10,10\rangle + \gamma|1,-1,11\rangle$$

We want $|00\rangle$ to be orthogonal to $|10\rangle$ and $|20\rangle$.

$$\langle 10|00\rangle = 0 \implies \alpha = \gamma$$

$$\langle 20|00\rangle = 0 \implies \alpha + \gamma + 2\beta = 0$$

which means: $\alpha = \gamma = -\beta = \frac{1}{\sqrt{3}}$

$$|00\rangle = \frac{1}{\sqrt{3}}|11,1,-1\rangle - \frac{1}{\sqrt{3}}|10,10\rangle + \frac{1}{\sqrt{3}}|1,-1,11\rangle$$

15.2.5

(1) First notice that $\vec{S}_1 = \vec{\sigma}_1 \frac{\hbar}{2}$ and $\vec{S}_2 = \vec{\sigma}_2 \frac{\hbar}{2}$.

$$\mathbb{P}_1 = \frac{3}{4}I + \frac{\vec{S}_1 \cdot \vec{S}_2}{\hbar^2} = \left(\frac{3}{4}I + \frac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2\right).$$

$$\mathbb{P}_1\mathbb{P}_1 = \left(\frac{9}{16}I + \frac{6}{16}\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{1}{16}(\vec{\sigma}_1 \cdot \vec{\sigma}_2)^2\right).$$

Now $(\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) = \underbrace{(\vec{\sigma}_1 \cdot \vec{\sigma}_1)}_{3} + i\underbrace{(\vec{\sigma}_1 \times \vec{\sigma}_1)}_{2i\vec{\sigma}_1} \cdot \vec{\sigma}_2$

$$= (3I - 2\vec{\sigma}_1 \cdot \vec{\sigma}_2).$$

$$\therefore \quad \mathbb{P}_1^2 = \left(\frac{3}{4}I + \frac{6}{16}\vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{2}{16}\vec{\sigma}_1 \cdot \vec{\sigma}_2\right) = \mathbb{P}_1.$$

$$\mathbb{P}_2 = \left(\frac{1}{4}I - \frac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2\right)$$

$$\mathbb{P}_2\mathbb{P}_2 = \left(\frac{1}{16}I - \frac{2}{16}\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{1}{16}(3I - 2\vec{\sigma}_1 \cdot \vec{\sigma}_2)\right) = \mathbb{P}_2$$

$$P_1 P_2 = \left(\frac{3}{4} I + \frac{1}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2\right)\left(\frac{1}{4} I - \frac{1}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2\right)$$

$$= \left(\frac{3}{16} I - \frac{2}{16}(\vec{\sigma}_1 \cdot \vec{\sigma}_2) - \frac{1}{16}(3I - 2\vec{\sigma}_1 \cdot \vec{\sigma}_2)\right)$$

$$= 0$$

(2) Now let us write $\vec{S} = \vec{S}_1 + \vec{S}_2$; $\vec{S}_1^2 = \frac{3}{4} I\hbar^2 = \vec{S}_2^2$

$$\therefore \frac{(\vec{S}_1 \cdot \vec{S}_2)}{\hbar^2} = \frac{1}{2\hbar^2}(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2) = \frac{\vec{S}^2}{2\hbar^2} - \frac{3}{4} I .$$

$$\therefore P_1 = \frac{\vec{S}^2}{2\hbar^2}$$

Now if P_1 acts on $|11\rangle$ $|10\rangle$ or $|1{-1}\rangle$ we get 1.

If it acts on $|00\rangle$ we get 0.

$$P_2 = I - \frac{\vec{S}^2}{2\hbar^2}$$

If P_2 acts on $|11\rangle$, $|10\rangle$, $|1{-1}\rangle$ we get a 0 and when it acts on $|00\rangle$ we get 1.

Thus P_1 projects into the spin-1 space and P_2 projects into spin-0.

PHY 6646 - Quantum Mechanics II - Spring 2011
Homework set # 3, due February 1

1. An electron in a solid has orbital angular momentum $l = 1$ and spin $s = 1/2$. Its total angular momentum is $\vec{J} = \vec{L} + \vec{S}$.
a. How many possible states can the electron be in?
b. What are the possible j values for the total angular momentum, and how many possible states exist for each j?
c. What is the state $|j\ m\ l\ s>$ with maximum values of j and m?
d. Find all the remaining states $|j\ m\ l\ s>$ with positive values of m, expressing them as linear combinations of the direct product states $|l\ m_l> |s\ m_s>$.

2. Consider three spin 1/2 systems, $j_1 = j_2 = j_3 = \frac{1}{2}$. There are a total of eight direct product states $|j_1\ m_1> |j_2\ m_2> |j_3\ m_3>$. Obtain eight linear combinations of the direct product states which are eigenstates of the total angular momentum $J^2 = (\vec{J}_1 + \vec{J}_2 + \vec{J}_3)^2$ and the z-component of total angular momentum $J_z = J_{1z} + J_{2z} + J_{3z}$. Hint: First add two spins together, then add the third to the result. The answers to problem 1 may be useful.

3. Problems 15.2.1 (both parts) and 15.2.2(2) (second part only).

2. a. $(2l+1)(2s+1) = 3 \times 2 = 6$

b. $j = l - s = 1/2$

and $j = l + s = 3/2$

c. $|3/2\ \ 3/2\ \ 1\ \ 1/2\rangle = |1\ \ 1\rangle |1/2\ \ 1/2\rangle$

d. $J_- = L_- + S_-$

$J_- |3/2\ \ 3/2\ \ 1\ \ 1/2\rangle$

$= \sqrt{(3/2 - 3/2 + 1)(3/2 + 3/2)}\ |3/2\ \ 1/2\ \ 1\ \ 1/2\rangle$

$= \sqrt{3}\ |3/2\ \ 1/2\ \ 1\ \ 1/2\rangle$

$= \sqrt{(1 - 1 + 1)(1 + 1)}\ |1\ \ 0\rangle |1/2\ \ 1/2\rangle$

$+ \sqrt{(1/2 - 1/2 + 1)(1/2 + 1/2)}\ |1\ \ 1\rangle |1/2, -1/2\rangle$

$= \sqrt{2}\ |1\ \ 0\rangle |1/2\ \ 1/2\rangle + |1\ \ 1\rangle |1/2, -1/2\rangle$

$\therefore |3/2\ \ 1/2\ \ 1\ \ 1/2\rangle = \sqrt{\frac{2}{3}}\ |1\ \ 0\rangle |1/2\ \ 1/2\rangle$

$+ \frac{1}{\sqrt{3}} |1\ \ 1\rangle |1/2\ \ -1/2\rangle$

$$\therefore \quad |\tfrac{1}{2}\ \tfrac{1}{2}\ 1\ \tfrac{1}{2}\rangle = \sqrt{\frac{2}{3}}\ |1\ 1\rangle\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$- \sqrt{\frac{1}{3}}\ |1\ 0\rangle\ |\tfrac{1}{2}\ +\tfrac{1}{2}\rangle$$

There are no other states with positive values of m.

3. First and third equations

$$|1\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$J_- = S_{1-} + S_{2-}$$

$$\sqrt{2}\ |1\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$+ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\therefore \quad |1\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$+ \frac{1}{\sqrt{2}}\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\therefore \quad |1\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = -\frac{1}{\sqrt{2}}\,|1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,|1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

Using

$$\langle j_1\ m_1\ j_2\ m_2 \mid j\ m\rangle$$

$$= (-1)^{j_1+j_2-j}\,\langle j_1\ -m_1\ j_2\ -m_2 \mid j\ -m\rangle$$

we have more

$$\langle \tfrac{1}{2}\ -\tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2} \mid 1\ -1\rangle$$

$$= (-1)^{0}\,\langle \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2} \mid 1\ 1\rangle$$

$$\therefore \quad |1\ -1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = +\,|1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\ -\tfrac{1}{2}\rangle$$

Next add the third spin:

$(1/2) \times (1/2) = (1) + (0)$

$(1/2) \times (1/2) \times (1/2) = ((1) + (0)) \times (1/2)$

$= (3/2) + (1/2)' .$

Using the results of problem 1

$|3/2\ 3/2\rangle = |1\ 1\rangle\ |1/2\ 1/2\rangle$

$= |1/2\ 1/2\rangle\ |1/2\ 1/2\rangle\ |1/2\ 1/2\rangle$

$|3/2\ 1/2\rangle = \sqrt{\frac{2}{3}}\ |1\ 0\rangle\ |1/2\ 1/2\rangle + \frac{1}{\sqrt{3}}\ |1\ 1\rangle\ |1/2\ -1/2\rangle$

$= \frac{1}{\sqrt{3}}\ |1/2\ -1/2\rangle\ |1/2\ 1/2\rangle\ |1/2\ 1/2\rangle$

$+ \frac{1}{\sqrt{3}}\ |1/2\ 1/2\rangle\ |1/2\ -1/2\rangle\ |1/2\ +1/2\rangle$

$+ \frac{1}{\sqrt{3}}\ |1/2\ 1/2\rangle\ |1/2\ 1/2\rangle\ |1/2\ -1/2\rangle$

Since $(-1)^{j_1 + j_2 - j} = (-1)^{1 + 1/2 - 3/2} = +1$

$|3/2\ -3/2\rangle = |1/2\ -1/2\rangle\ |1/2\ -1/2\rangle\ |1/2\ -1/2\rangle$ and

$$|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$+\frac{1}{\sqrt{3}}\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$+\frac{1}{\sqrt{3}}\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|1\,1\rangle\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$-\sqrt{\frac{1}{3}}\,|1\,0\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$= \sqrt{\frac{2}{3}}\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$-\frac{1}{\sqrt{6}}\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$-\frac{1}{\sqrt{6}}\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$\text{Ex: } (-1)^{\phi_1+\phi_2-\phi} = (-1)^{1+\frac{1}{2}-\frac{1}{2}} = -1$$

$$|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = -\sqrt{\frac{2}{3}}\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ +\tfrac{1}{2}\rangle$$

$$+\frac{1}{\sqrt{6}}\Big[\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + |1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,\Big].$$

$$|1/2\ 1/2\rangle' = |0\ 0\rangle\ |1/2\ 1/2\rangle$$

$$= -\frac{1}{\sqrt{2}}\,|1/2\ -1/2\rangle\ |1/2\ 1/2\rangle\ |1/2\ 1/2\rangle$$

$$+\frac{1}{\sqrt{2}}\,|1/2\ 1/2\rangle\ |1/2\ -1/2\rangle\ |1/2\ +1/2\rangle$$

Since $(-1)^{j_1+j_2-j} = (-1)^{0+1/2-1/2} = +1$

$$|1/2\ -1/2\rangle = -\frac{1}{\sqrt{2}}\,|1/2\ -1/2\rangle\ |1/2\ 1/2\rangle\ |1/2\ -1/2\rangle$$

$$+\frac{1}{\sqrt{2}}\,|1/2\ 1/2\rangle\ |1/2\ -1/2\rangle\ |1/2\ -1/2\rangle$$

In summary, the coefficients are

	↑↑↑	↑↑↓	↑↓↑	↓↑↑	↑↓↓	↓↑↓	↓↓↑	↓↓↓
$\|3/2\ 3/2\rangle$	1	0	0	0	0	0	0	0
$\|3/2\ 1/2\rangle$	0	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	0	0	0	0
$\|3/2\ -1/2\rangle$	0	0	0	0	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	0
$\|3/2\ -3/2\rangle$	0	0	0	0	0	0	0	1
$\|1/2\ 1/2\rangle$	0	$\sqrt{2/3}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$	0	0	0	0
$\|1/2\ -1/2\rangle$	0	0	0	0	$1/\sqrt{6}$	$1/\sqrt{6}$	$-\sqrt{2/3}$	0
$\|1/2\ +1/2\rangle'$	0	0	$+1/\sqrt{2}$	$-1/\sqrt{2}$	0	0	0	0
$\|1/2\ -1/2\rangle'$	0	0	0	0	$+1/\sqrt{2}$	$-1/\sqrt{2}$	0	0

§15.2.1

i. $J^2 |j_1, j_1; j_2, j_2\rangle$

$= \left(J_1^2 + J_2^2 + 2 J_{1z} J_{2z} + J_{1+} J_{2-} + J_{1-} J_{2+} \right) |j_1, j_1; j_2, j_2\rangle$

$= \left[j_1(j_1+1) + j_2(j_2+1) + 2 j_1 j_2 \right] |j_1, j_1; j_2, j_2\rangle$

since $J_{1+} |j_1, j_1; j_2, j_2\rangle = 0$

$J_{2+} |j_1, j_1; j_2, j_2\rangle = 0$

$j_1(j_1+1) + j_2(j_2+1) + 2 j_1 j_2$

$= (j_1 + j_2)(j_1 + j_2 + 1)$

ii. $J^2 |j_1, j_1; j_2, j_2 - 1\rangle$

$= \left[j_1(j_1+1) + j_2(j_2+1) + 2 j_1 (j_2 - 1) \right] |j_1, j_1; j_2, j_2 - 1\rangle$

$+ \sqrt{2 j_1 (2 j_2 + 1)} \; |j_1, j_1 - 1; j_2, j_2\rangle$

Since

$$J_{2+}\,|j_2, j_2^{-1}\rangle = \sqrt{2j_2}\;|j_2, j_2\rangle$$

$$J_{1-}\,|j_1, j_1\rangle = \sqrt{2j_1}\;|j_1, j_1-1\rangle$$

$$J^2\left[\sqrt{\frac{j_1}{j_1+j_2}}\,|j_1, j_1; j_2, j_2^{-1}\rangle - \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_1^{-1}; j_2, j_2\rangle\right]$$

$$= \left[\sqrt{\frac{j_1}{j_1+j_2}}\left(j_1(j_1+1) + j_2(j_2+1) + 2j_1(j_2-1)\right)\right.$$

$$\left. - \sqrt{\frac{j_2}{j_1+j_2}}\,\sqrt{2j_1\cdot 2j_2}\right]|j_1, j_1; j_2, j_2^{-1}\rangle$$

$-$ Same with $1 \leftrightarrow 2$

$$= \sqrt{\frac{j_1}{j_1+j_2}}\left[j_1(j_1+1) + j_2(j_2+1) + 2j_1(j_2-1)\right.$$

$$\left. - 2j_2\right]|j_1, j_1; j_2, j_2^{-1}\rangle$$

$-$ Same with $1 \leftrightarrow 2$

$$j_1(j_1+1) + j_2(j_2+1) + 2j_1(j_2-1) - 2j_2$$

$$= j_1(j_1+1) + j_2(j_2+1) + 2j_1 j_2 - 2(j_1+j_2)$$

$$= (j_1+j_2)(j_1+j_2-1)$$

which is symmetric under $1 \leftrightarrow 2$

Hence

$$J^2\left(\sqrt{\frac{j_1}{j_1+j_2}}\,|j_1, j_1; j_2, j_2-1\rangle - \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_1-1; j_2, j_2\rangle\right)$$

$$= (j_1+j_2)(j_1+j_2-1)\left(\sqrt{\frac{j_1}{j_1+j_2}}\,|j_1, j_1; j_2, j_2-1\rangle\right.$$

$$\left. - \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_1-1; j_2, j_2\rangle\right)$$

§ 15.2.2

a.

$$|1,1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle$$

$$J_-|\tfrac{3}{2},\tfrac{3}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$= \hbar\left(\sqrt{2}\,|1,0\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + |1,1\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1,1\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{3}}\,|1,0\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,1\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\hbar}\,J_-\,|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\,|1,-1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$-\tfrac{2}{\sqrt{3}}\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\,|1,-1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}\,|1,-1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|3/2, -3/2\rangle = |1,-1\rangle\,|1/2,-1/2\rangle$$

6. $|1,1\rangle\,|1,1\rangle = |2,2\rangle$

$$\frac{1}{\hbar} J_- |2,2\rangle = 2\,|2,1\rangle$$

$$= \sqrt{2}\,|1,0\rangle\,|1,1\rangle + \sqrt{2}\,|1,1\rangle\,|1,0\rangle$$

$$|2,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle + \frac{1}{\sqrt{2}}|1,1\rangle|1,0\rangle$$

$$|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle - \frac{1}{\sqrt{2}}|1,1\rangle|1,0\rangle$$

$$\frac{1}{\hbar} J_- |2,1\rangle = \sqrt{6}\,|2,0\rangle$$

$$= (1+1)|1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle + |1,1\rangle|1,-1\rangle$$

$$|2,0\rangle = \frac{1}{\sqrt{6}}|1,-1\rangle|1,1\rangle + \frac{2}{\sqrt{6}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{6}}|1,1\rangle|1,-1\rangle$$

$$\frac{1}{\hbar} J_- |1,1\rangle = \sqrt{2}\,|1,0\rangle$$

$$= (1-1)|1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle - |1,1\rangle|1,-1\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle|1,1\rangle - \frac{1}{\sqrt{2}}|1,1\rangle|1,-1\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{3}}|1,-1\rangle|1,1\rangle - \frac{1}{\sqrt{3}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{3}}|1,1\rangle|1,-1\rangle$$

$$\frac{1}{\hbar}J_-|1,0\rangle = \sqrt{2}\,|1,-1\rangle$$

$$= |1,-1\rangle|1,0\rangle + |1,0\rangle|1,-1\rangle$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle|1,0\rangle - \frac{1}{\sqrt{2}}|1,0\rangle|1,-1\rangle$$

$$|2,-1\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle|1,0\rangle + \frac{1}{\sqrt{2}}|1,0\rangle|1,-1\rangle$$

$$|2,-2\rangle = |1,-1\rangle|1,-1\rangle$$

Solutions Problem set 4

① We have three spin-½ particles. The total spin can be constructed by adding the first two spins and then the third spin.

We get

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \oplus 0) \otimes \frac{1}{2}$$

$$= \left(1 \otimes \frac{1}{2}\right) \oplus \left(0 \otimes \frac{1}{2}\right)$$

$$\frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$$

∴ The eigenvalues of $\vec{S}^2$ are $\hbar^2 s(s+1)$

where $s = \frac{3}{2}, \frac{1}{2}$

The degeneracy for $s = \frac{3}{2} \Rightarrow 4$

" $s = \frac{1}{2} \Rightarrow 4$

② We have the dipole interaction

$$H = \frac{\gamma_1 \gamma_2}{r^3} \left[\vec{S}_1 \cdot \vec{S}_2 - 3 (\vec{S}_1 \cdot \hat{n})(\vec{S}_2 \cdot \hat{n}) \right]$$

Now $r = R$, $\hat{n} = \hat{z}$ and hence

$$H = \frac{\gamma_1 \gamma_2}{R^3} \left[\vec{S}_1 \cdot \vec{S}_2 - 3 S_{1z} S_{2z} \right]$$

$$H = \frac{\gamma_1 \gamma_2}{R^3} \left[\frac{1}{2} (\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2) - 3 S_{1z} S_{2z} \right]$$

The total angular mom states $|s,m\rangle$ are eigenstates of $\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2$ with eigenvalue $\hbar^2 (s(s+1) - \frac{3}{2})$

Now the total angular momentum states are also eigenstates of $S_{1z} S_{2z}$ for this problem. To see this note

$|s\,m\rangle \qquad |s_1 m_1, s_2 m_2\rangle$

$$|1\,1\rangle = \left|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{-1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\tfrac{-1}{2}, \tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

$$|1\,{-1}\rangle = \left|\tfrac{1}{2}\tfrac{-1}{2}, \tfrac{1}{2}\tfrac{-1}{2}\right\rangle$$

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{-1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\tfrac{-1}{2}, \tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

$$S_{1z} S_{2z} |1\,1\rangle = \frac{\hbar^2}{4} |1\,0\,1\rangle$$

$$S_{1z} S_{2z} |1\,0\rangle = -\frac{\hbar^2}{4} |1\,0\rangle$$

$$S_{1z} S_{2z} |1\,{-1}\rangle = \frac{\hbar^2}{4} |1\,{-1}\rangle$$

$$S_{1z} S_{2z} |0\,0\rangle = -\frac{\hbar^2}{4} |0\,0\rangle .$$

$\therefore$ $|s,m\rangle$ are eigenstates of the Hamiltonian.

eigenstate	eigenval
$\lvert 1\,1\rangle$	$\frac{\hbar^2 \gamma_1 \gamma_2}{R^3}\left[\frac{1}{4} - \frac{3}{4}\right] = -\frac{\hbar^2 \gamma_1 \gamma_2}{2R^3}$
$\lvert 1\,0\rangle$	$\frac{\hbar^2 \gamma_1 \gamma_2}{R^3}\left[\frac{1}{4} + \frac{3}{4}\right] = \frac{\hbar^2 \gamma_1 \gamma_2}{R^3}$
$\lvert 1\,{-1}\rangle$	$-\frac{\hbar^2 \gamma_1 \gamma_2}{2R^3}$
$\lvert 0\,0\rangle$	$\frac{\hbar^2 \gamma_1 \gamma_2}{R^3}\left[0 + \frac{3}{4}\right] = \frac{3}{4}\frac{\hbar^2 \gamma_1 \gamma_2}{R^3}$

(3) When we add $\vec{L}$ and $\vec{S}$, the eigen states of $\vec{J}^2$ and J_z can be written as (3)

$$|j = l+\tfrac{1}{2}, m\rangle = \alpha\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle + \beta\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$|j = l-\tfrac{1}{2}, m\rangle = \alpha'\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle + \beta'\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle$$

Since both these vectors are orthonormal we expect

$$\alpha^2+\beta^2 = 1 \qquad \text{(Note: } \alpha, \beta, \alpha', \beta' \text{ can be chosen real)}$$

$$\alpha'^2+\beta'^2 = 1 \qquad \alpha\alpha'+\beta\beta' = 0$$

Now $\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + \vec{S}^2 + 2L_zS_z + L_-S_+ + L_+S_-$

where we used the fact that

$\vec{L}\cdot\vec{S} = L_zS_z + L_xS_x + L_yS_y$ and $L_-S_+ + L_+S_- = 2S_xL_x + 2S_yL_y$.

We apply $\vec{J}^2$ on both sides of first eq. we get

$$\vec{J}^2\,|j = l+\tfrac{1}{2}, m\rangle = \hbar^2\left(l+\tfrac{1}{2}\right)\left(l+\tfrac{3}{2}\right)|j = l+\tfrac{1}{2}, m\rangle$$

In order to apply $\vec{J}^2$ on the R.H.S we note that

$$\vec{L}^2\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle = \hbar^2 l(l+1)\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$\vec{L}^2\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle = \hbar^2 l(l+1)\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$\vec{S}^2\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{3}{4}\hbar^2\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$\vec{S}^2\,|l(m+\tfrac{1}{2})\ \tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{3}{4}\hbar^2\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$L_zS_z\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{\hbar^2}{2}(m-\tfrac{1}{2})\,|l(m-\tfrac{1}{2}), \tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$L_zS_z\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle = -\frac{\hbar^2}{2}(m+\tfrac{1}{2})\,|l(m+\tfrac{1}{2}), \tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$L_- S_+ \left| l\left(m-\tfrac{1}{2}\right), \tfrac{1}{2}\tfrac{1}{2}\right\rangle = 0$$

$$L_+ S_- \left| l\left(m-\tfrac{1}{2}\right), \tfrac{1}{2}\tfrac{1}{2}\right\rangle = \hbar^2 \sqrt{\left(l-m+\tfrac{1}{2}\right)\left(l+m+\tfrac{1}{2}\right)} \left| l\left(m+\tfrac{1}{2}\right), \tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$L_- S_+ \left| l\left(m+\tfrac{1}{2}\right), \tfrac{1}{2}-\tfrac{1}{2}\right\rangle = \hbar^2 \sqrt{\left(l+m+\tfrac{1}{2}\right)\left(l-m+\tfrac{1}{2}\right)} \left| l\left(m-\tfrac{1}{2}\right), \tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

$$L_+ S_- \left| l\left(m+\tfrac{1}{2}\right), \tfrac{1}{2}-\tfrac{1}{2}\right\rangle = 0$$

Now for convenience let us define

$$\left| l\left(m-\tfrac{1}{2}\right), \tfrac{1}{2}\tfrac{1}{2}\right\rangle = |1\rangle$$

$$\left| l\left(m+\tfrac{1}{2}\right), \tfrac{1}{2}-\tfrac{1}{2}\right\rangle = |2\rangle$$

Then the first eq. becomes

$$\left| j = l+\tfrac{1}{2}, m\right\rangle = \alpha|1\rangle + \beta|2\rangle$$

applying $\vec{J}^2$ on both sides we see that—

$$\vec{J}^2 \left| j = l+\tfrac{1}{2}, m\right\rangle = \left(\vec{L}^2 + \vec{S}^2 + 2L_z S_z + L_- S_+ + L_+ S_-\right)\left(\alpha|1\rangle + \beta|2\rangle\right)$$

$$\Rightarrow \hbar^2\left(l+\tfrac{1}{2}\right)\left(l+\tfrac{3}{2}\right)\left(\alpha|1\rangle+\beta|2\rangle\right) = \left(\hbar^2 l(l+1) + \tfrac{3}{4}\hbar^2\right)\left(\alpha|1\rangle+\beta|2\rangle\right)$$

$$+ \frac{2\hbar}{2}\left(m-\tfrac{1}{2}\right)\alpha|1\rangle - \frac{2\hbar}{2}\left(m+\tfrac{1}{2}\right)\beta|2\rangle$$

$$+ \hbar^2\sqrt{\left(l+m+\tfrac{1}{2}\right)\left(l-m+\tfrac{1}{2}\right)}\,\beta|1\rangle$$

$$+ \hbar^2\sqrt{\left(l-m+\tfrac{1}{2}\right)\left(l+m+\tfrac{1}{2}\right)}\,\alpha|2\rangle$$

$$\Rightarrow \left(l+\tfrac{1}{2}-m\right)\alpha|1\rangle + \left(l+\tfrac{1}{2}+m\right)\beta|2\rangle = \sqrt{\left(l+m+\tfrac{1}{2}\right)\left(l-m+\tfrac{1}{2}\right)}\left(\beta|1\rangle+\alpha|2\rangle\right)$$

$$\Rightarrow \frac{\beta}{\alpha} = \left(\frac{l+\tfrac{1}{2}-m}{l+\tfrac{1}{2}+m}\right)^{1/2}$$

Let us call $\alpha = A$

$$\beta = \left(\frac{l+\frac{1}{2}-m}{l+\frac{1}{2}+m}\right)^{1/2} A \,.$$

Now $A^2 + \left(\frac{l+\frac{1}{2}-m}{l+\frac{1}{2}+m}\right) A^2 = 1 \Rightarrow A = \frac{(l+\frac{1}{2}+m)^{1/2}}{(2l+1)^{1/2}}$

$$\therefore \quad \alpha = \frac{(l+\frac{1}{2}+m)^{1/2}}{(2l+1)^{1/2}} \qquad \beta = \frac{(l+\frac{1}{2}-m)^{1/2}}{(2l+1)^{1/2}}$$

$$\beta' = \alpha \,, \qquad \alpha' = -\beta \,.$$

↑ keeps the sign convention for CG coeff. (atleast for $l>0$)!

(4) A tensor operator T^q_κ transforms under rotation as given by

$$U[R]\, T^q_\kappa\, U^\dagger[R] = \sum_{q'} D^{(\kappa)}_{q'q}\, T^{q'}_\kappa$$

where $U[R] = \exp\left[-\frac{i}{\hbar}\theta\, \hat{n}\cdot\vec{J}\right]$

$$D^{(\kappa)}_{q'q} = \langle \kappa q' |\, U[R]\, | \kappa q \rangle \,.$$

For an infinitesimal transformation we have

$$U\, T^q_\kappa\, U^\dagger \longrightarrow \left(1 - \frac{i}{\hbar}\theta\, \hat{n}\cdot\vec{J}\right) T^q_\kappa \left(1 + \frac{i}{\hbar}\theta\, \hat{n}\cdot\vec{J}\right)$$

$$\simeq T^q_\kappa - \frac{i}{\hbar}\theta\, [\hat{n}\cdot\vec{J},\, T^q_\kappa]$$

$$D^{(\kappa)}_{q'q} \longrightarrow \delta_{q'q} - \frac{i}{\hbar}\theta\, \langle \kappa q' |(\hat{n}\cdot\vec{J})| \kappa q' \rangle \,.$$

This means that T_k^q must satisfy.

$$[\hat{n}\cdot\vec{J}, T_k^q] = \sum_{q'} \langle kq'|\, \hat{n}\cdot\vec{J}\,|kq\rangle\, T_k^{q'}$$

$$\Rightarrow \quad [J_i, T_k^q] = \sum_{q'} \langle kq'|\, J_i\,|kq\rangle\, T_k^{q'}$$

which means

$$[J_z, T_k^q] = \hbar q\, T_k^q$$

$$[J_+, T_k^q] = \sum_{q'} \langle kq'|\, J_+\,|kq\rangle\, T_k^{q'}$$

$$= \hbar\sqrt{(k-q)(k+q+1)}\; T_k^{q+1}$$

$$[J_-, T_k^q] = \sum_{q'} \langle kq'|\, J_-\,|kq\rangle\, T_k^{q'}$$

$$= \hbar\sqrt{(k+q)(k-q+1)}\; T_k^{q-1}$$

where we used

$$\langle kq'|\, J_\pm\,|kq\rangle = \hbar\sqrt{(k\mp q)(k\pm q+1)}\;\delta_{q',q\pm1}$$

(5) We need to show that $\sum_q (-1)^q S_k^q T_k^{-q}$ is a scalar operator.

(1) For $k=1$ $\qquad S^0_{k=1} = S_z \qquad S^1_{k=1} = -\dfrac{(S_1+iS_2)}{\sqrt{2}}$

$$S^{-1}_{k=1} = +\frac{(S_1-iS_2)}{\sqrt{2}}$$

where S_1, S_2, S_3 are Cartesian components and $S^1_{k=1}\ S^{-1}_{k=1}\ S^0_{k=1}$ are Spherical tensor components.

For $k=1$ we have $\sum_q (-1)^q S_k^q T_k^{-q}$ can be written as

$$S_1^0 T_1^0 - S_1^1 T_1^{-1} - S_1^{-1} T_1^1$$

$$= S_z T_z + \frac{(S_1 + iS_2)}{\sqrt{2}} \frac{(T_1 - iT_2)}{\sqrt{2}} + \frac{(S_1 - iS_2)}{\sqrt{2}} \frac{(T_1 + iT_2)}{\sqrt{2}}$$

$$= S_z T_z + S_1 T_1 + S_2 T_2 = \vec{S} \cdot \vec{T} \quad \text{(as expected!)}$$

(2) General Proof: We need to show

$$U \left(\sum_q (-1)^q S_k^q T_k^{-q} \right) U^\dagger = \sum_q (-1)^q S_k^q T_k^{-q}$$

Left Hand Side:

$$\Rightarrow \sum_{q'} (-1)^q \underbrace{U S_k^q U^\dagger}_{\sum_{q'} D^{(k)}_{q'q} S_k^{q'}} \underbrace{U T_k^{-q} U^\dagger}_{\sum_{q''} D^{(k)}_{-q''-q} T_k^{-q''}}$$

Note:

$$U T_k^q U^\dagger = \sum_{q''} D^{(k)}_{q''q} T_k^{q''}$$

$$U T_k^{-q} U^\dagger = \sum_{q''} D^{(k)}_{q''-q} T_k^{q''} = \sum_{q''} D_{-q''-q} T_k^{-q''}$$

see proof

Now

$$\Rightarrow \sum_q (-1)^q \sum_{q'} D^{(k)}_{q'q} S_k^{q'} \sum_{q''} D^{(k)}_{-q''-q} T_k^{-q''}$$

$$= \sum_{q'q''} \sum_q \left\{ D^{(k)}_{q'q} \underbrace{D^{(k)}_{-q''-q}}_{\downarrow \text{Hint in Shankar}} (-1)^q \right\} S_k^{q'} T_k^{-q''}$$

$$D^{*(k)}_{q''q} (-1)^{q''-q}$$

L.H.S

$$\sum_{q'q''} \left(\underbrace{\sum_{q} D^{(k)}_{q'q} D^{*(k)}_{q''q}}_{\delta_{q'q''} \text{ (see proof below)}} \right) (-1)^{q''} S_k^{q'} T_k^{-q''}$$

$$= \sum_{q'} (-1)^{q'} S_k^{q'} T_k^{-q'} = \text{R.H.S}$$

Let us now prove $\sum_q D^{(k)}_{q'q} D^{*(k)}_{q''q} = \delta_{q'q''}$

Now

$$D^{(k)}_{q'q} = \langle kq' | U[R] | kq \rangle$$

$$D^{*(k)}_{q'q} = \langle kq | U^{\dagger}[R] | kq' \rangle$$

$$D^{*(k)}_{q''q} = \langle kq | U^{\dagger}[R] | kq'' \rangle$$

$$\therefore \sum_q D^{(k)}_{q'q} D^{*(k)}_{q''q} = \sum_q \langle kq' | U[R] | kq \rangle \langle kq | U^{\dagger}[R] | kq'' \rangle$$

($\sum_q |kq\rangle\langle kq|$ → Identity)

$$= \langle kq' | U[R] U^{\dagger}[R] | kq'' \rangle$$

$$= \delta_{q'q}.$$

PHY 615-2012
PROBLEM SET #6

Problem 1:

$$V(x) = \frac{\lambda}{4}\left(x^4 + ax^3 - \frac{1}{2}a^2x^2\right)$$

$$V'(x) = \frac{\lambda}{4}\left(4x^3 + 3ax^2 - a^2x\right) = 0$$

$$V''(x) = \frac{\lambda}{4}\left(12x^2 + 6ax - a^2\right) = 0$$

Extrema:

$x = 0$ $\qquad 4x^2 + 3ax - a^2 = 0$

$$(4x - a)(x + a) = 0$$

ie $\quad x = 0,\ a/4,\ -a.$

$$V''(0) = -\frac{\lambda a^2}{4} < 0 \quad \Rightarrow \text{maxima}$$

$$V''(-a) = \frac{\lambda}{4}\left(12a^2 - 6a^2 - a^2\right) = \frac{5\lambda a^2}{4} > 0$$

$$V''(a/4) = \frac{\lambda}{4}\,\frac{5a^2}{4} > 0$$

$$V(-a) = -\frac{\lambda a^4}{8} \qquad V\left(\frac{a}{4}\right) = -\frac{3\lambda a^4}{1024}$$

Thus $V(-a) < V(a/4)$.

$\Rightarrow$ Global minimum is $x = -a$

The trial wavefunction is

$$\psi(x) = \left(\frac{\beta}{\pi}\right)^{1/4} \exp\left[-\beta(x+a)^2/2\right]$$

$$\left\langle \frac{P^2}{2m} \right\rangle = -\frac{1}{2m}\left(\frac{\beta}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} dx\, e^{-\beta(x+a)^2/2}\,\frac{d^2}{dx^2}e^{-\beta(x+a)^2/2}$$

In integral shift $x \to x-a$

$$\left\langle \frac{P^2}{2m} \right\rangle = -\frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-\beta x^2/2}\,\frac{d^2}{dx^2}e^{-\beta x^2/2}$$

$$= \frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\left(\frac{d}{dx}e^{-\beta x^2/2}\right)^2$$

$$= \frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\,\beta^2\int_{-\infty}^{\infty} dx\, x^2 e^{-\beta x^2}$$

$$= \frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\,\beta^2\,\frac{1}{\beta^{3/2}}\int_{-\infty}^{\infty} dy\cdot y^2 e^{-y^2}$$

$$= \frac{\beta}{2m}\,\frac{1}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2} = \frac{\beta}{4m}$$

For the potential

$$\langle V \rangle = \frac{\lambda}{4}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-\beta(x+a)^2}\left[x^4 + ax^3 - \frac{1}{2}a^2x^2\right]$$

$$= \frac{\lambda}{4}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-\beta x^2}\left[(x-a)^4 + a(x-a)^3 - \frac{1}{2}a^2(x-a)^2\right]$$

$$= \frac{\lambda}{4}\sqrt{\frac{\beta}{\pi}}\sqrt{\frac{\pi}{\beta}}\left[\frac{3}{4\beta^2} + \frac{5a^2}{4\beta} - \frac{1}{2}a^4\right]$$

Thus

$$\langle E \rangle = \frac{\beta}{4m} + \frac{\lambda}{4}\left(\frac{3}{4\beta^2} + \frac{5a^2}{4\beta} - \frac{1}{2}a^4\right)$$

$$= \frac{\beta}{4m} + \frac{\lambda}{16\beta^2}\left(3 + 5a^2\beta - 2\beta^2 a^4\right)$$

Thus a natural variable is $\beta a^2 = y$

$$\langle E \rangle = \frac{1}{4ma^2}\left[y + \frac{\lambda a^6 m}{4y^2}\left(3 + 5y - 2y^2\right)\right]$$

Thus eqn determining y is

Define $\alpha = \lambda m a^6$

$$1 - \frac{6\alpha}{4y^3} - \frac{5\alpha}{4y^2} = 0$$

$$4y^3 - 6\alpha - 5\alpha y = 0$$

For $\alpha = 1$ $\quad (2y-3)(2y^2+3y+2) = 0$

Solutions of $2x^2+3x+2=0$ are complex. $(3^2 - 4.4 < 0)$

$\Rightarrow$ only real solution is $y = 3/2$

For this y

$$E_0 = \frac{1}{2ma^2}\left(\frac{13}{12}\right)$$

Problem 2: (16.2.1)

$$W(E) = \int_{x'}^{x} dx'' \sqrt{2M(E - V(x''))}$$

$$\frac{\partial W(E)}{\partial E} = \int_{x'}^{x} dx'' \sqrt{\frac{M}{2}} \frac{1}{\sqrt{E - V(x'')}}$$

$$= \int_{x'}^{x} dx'' \frac{dt}{dx''} = \int_{t'}^{t} dt$$

$$= (t - t')$$

Problem 3:

(i) Consider a trial wave function

$$\psi_\alpha(x) = \left(\frac{\alpha^2}{2\pi}\right)^{1/4} \exp\left[-\frac{\alpha^2 x^2}{4}\right]$$

Then

$$\langle \psi_\alpha | \frac{\hat{p}^2}{2m} | \psi_\alpha \rangle = \frac{\hbar^2 \alpha^2}{8m} \geq 0$$

Now consider

$$\langle \psi_\alpha | V(\hat{x}) | \psi_\alpha \rangle = -\frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |V(x)| e^{-\alpha^2 x^2/2} \, dx$$

Since $V(\hat{x})$ is negative everywhere.

If $\int_{-\infty}^{\infty} dx\, V(x)$ converges, $\langle \psi_\alpha | V | \psi_\alpha \rangle$ has a finite limit as $\alpha \to 0$. In this case,

$$\langle E \rangle = \frac{\hbar^2 \alpha^2}{8m} - \alpha c$$

where c is some $+ve$ number.

$$\frac{\partial \langle E \rangle}{\partial \alpha} = 0 \Rightarrow \frac{\hbar^2 \alpha_0}{4m} = c \qquad \alpha_0 = \frac{4mc}{\hbar^2}$$

$$\Rightarrow \langle E(\alpha_0) = -\frac{2mc^2}{\hbar^2} < 0.$$

Since $E_0 \leq \langle E \rangle(\alpha)$ (E_0 is ground state energy)
This shows $E_0 < 0$. Thus there is at least one bound state

If $\int_{-\infty}^{\infty} dx\, V(x)$ is divergent, clearly $\langle \psi_\alpha | H | \psi_\alpha \rangle < 0$, so that $E_0 < 0$ as well

(ii) In 3d, consider once again the trial wave fn.

$$\psi_\alpha = \left(\frac{\alpha^2}{2\pi}\right)^{3/4} \exp\left[-\frac{\alpha^2 r^2}{4}\right]$$

Then

$$\langle \psi_\alpha | V(x) | \psi_\alpha \rangle \sim -\alpha^3$$

If the integral of V is now finite

$$\langle E \rangle = \frac{\hbar^2 \alpha^2}{8M} - \alpha^3 C$$

and now the value of $\langle E \rangle$ at extremum is zero or +ve — so we cannot use this argument to show there is a bound state

In fact generically there may or may not be bound states

Problem 4

(i) $$\left\langle \frac{p^2}{2m} \right\rangle = -\frac{\alpha}{2m} \int_{-\infty}^{\infty} dx\, e^{-\alpha|x|} \left(e^{-\alpha|x|}\right)''$$

$$\left(e^{-\alpha|x|}\right)'' = \left(-\alpha\left(\theta(x) - \theta(-x)\right) e^{-\alpha|x|}\right)'$$

$$= \left(-2\alpha\,\delta(x) + \alpha^2\right) e^{-\alpha|x|}$$

$$\left(\theta'(x) = \delta(x)\right)$$

Thus

$$\left\langle \frac{p^2}{2m} \right\rangle = -\frac{\alpha}{2m}\left[-2\alpha + \alpha^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha|x|}\right]$$

$$= -\frac{\alpha}{2m}\left[-2\alpha + 2\alpha^2 \int_{0}^{\infty} dx\, e^{-2\alpha x}\right]$$

$$= -\frac{\alpha}{2m}\left(-2\alpha + \alpha\right)^0$$

$$= \alpha^2/2m$$

The potential energy is

$$V = qE\left[\theta(x) - \theta(-x)\right] x$$

$$= qE|x|$$

Thus

$$\langle V \rangle = qE \int_{-\infty}^{\infty} dx\, |x| e^{-2\alpha|x|} = \frac{qE}{2\alpha}$$

$$\Rightarrow$$

$$\langle E \rangle = \frac{\alpha^2}{2m} + \frac{qE}{2\alpha}$$

$$\frac{\partial \langle E \rangle}{\partial \alpha} = \frac{\alpha}{m} - \frac{qE}{2\alpha^2} \Rightarrow \alpha = \left(\frac{qEm}{2}\right)^{1/3}$$

$$E_0 = \frac{3}{2^{5/3}} \left[\frac{(qE)^2}{m}\right]^{1/3}$$

(ii) Bohr Sommerfeld

Turning points are $x_0 = \pm \frac{e}{qE}$

where e is the energy

Thus

$$\int_{-e/qE}^{e/qE} dx \sqrt{2m(e - qE|x|)} = \left(n+\tfrac{1}{2}\right)\pi$$

$$\int_{-e/qE}^{0} dx \sqrt{2m(e+qEx)} + \int_{0}^{e/qE} dx \sqrt{2m(e-qEx)} = \left(n+\tfrac{1}{2}\right)\pi$$

$$\frac{2}{3}\frac{e^{3/2}\sqrt{2m}}{qE} + \frac{2}{3}\frac{e^{3/2}\sqrt{2m}}{qE} = \left(n+\tfrac{1}{2}\right)\pi$$

Thus gives for $n = 0$ (ground state)

$$e^{3/2} = \frac{3qE\pi}{8\sqrt{2m}} = \frac{3\pi}{8\sqrt{2}}\left(\frac{qE}{\sqrt{m}}\right)$$

$$e = \left(\frac{3\pi}{8\sqrt{2}}\right)^{2/3}\left(\frac{(qE)^2}{m}\right)^{1/3}$$

Problem #5

For energy E the turning points are $\pm x_0$ where

$$E = V(x_0) = \frac{1}{2} m\omega^2 x_0^2$$

$$\Rightarrow \quad x_0 = \sqrt{\frac{2E}{m\omega^2}}$$

(1) In the allowed region the WKB wave function is (for energy E)

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{-x_0}^{x} p(x')\,dx' - \frac{\pi}{4}\right)$$

where $\quad p(x) = \sqrt{2m\left(E - \frac{1}{2}m\omega^2 x^2\right)}$

$$= \sqrt{2m\,\frac{1}{2}m\omega^2\left(x_0^2 - x^2\right)}$$

$$= m\omega\sqrt{x_0^2 - x^2}$$

Quantization condition is

$$\int_{-x_0}^{x_0} dx'\, p(x') = \left(n + \frac{1}{2}\right)\hbar$$

$$\Rightarrow \quad m\omega x_0^2 \int_{-1}^{1} dx\,\sqrt{1-x^2} = \left(n + \frac{1}{2}\right)\hbar\pi$$

$$\frac{2E}{\omega}\cdot\frac{\pi}{2} = \left(n + \frac{1}{2}\right)\hbar$$

$$\Rightarrow \quad \boxed{E = \left(n + \frac{1}{2}\right)\hbar\omega}$$

Problem Set 21

Problem 17.2.1: With $H^0 = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2$ and $H^1 = \lambda X^4$, and using that $X = \sqrt{\hbar/(2m\omega)}(a + a^\dagger)$, we have:

(1):

$$\begin{aligned}
E_n^1 &= \langle n|H^1|n\rangle = \frac{\lambda\hbar^2}{4m^2\omega^2}\langle n|(a+a^\dagger)^4|n\rangle \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\langle n|(a^2a^{\dagger 2} + aa^\dagger aa^\dagger + a^\dagger a^2 a^\dagger + aa^{\dagger 2}a + a^\dagger aa^\dagger a + a^{\dagger 2}a^2)|n\rangle \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\langle n|(6a^2a^{\dagger 2} - 12aa^\dagger + 3)|n\rangle \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\left\{6\langle n+2|(n+1)(n+2)|n+2\rangle - 12\langle n+1|(n+1)|n+1\rangle + 3\langle n|n\rangle\right\} \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\left\{6(n+1)(n+2) - 12(n+1) + 3\right\} = \frac{3\hbar^2\lambda}{4m^2\omega^2}(2n^2+2n+1),
\end{aligned}$$

where in the first line we dropped the zero superscripts from the unperturbed eigenstates; in the second line we kept only terms with equal numbers of a's and $a^\dagger$'s; in the third line we used $[a, a^\dagger] = 1$; and in the fourth line we used $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(2): For any finite value of λ, as n gets large

$$\frac{E_n^1}{\Delta E} \sim \frac{\hbar^2\lambda n^2/(m^2\omega^2)}{\hbar\omega} \sim n^2\frac{\hbar\lambda}{m^2\omega^3} \gg 1.$$

Physically, at large x, no matter how small λ is,

$$V(x) = \frac{m\omega^2}{2}x^2 + \lambda x^4 \sim \lambda x^4$$

for $\lambda \neq 0$.

Problem 17.2.2: $H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = H^0 + H^1$ with $H^0 = -\gamma B_0 S_z$ and $H^1 = -\gamma B S_x$. The H^0 eigenvalues are $E_\pm^0 = \mp\gamma B_0\hbar/2$ with eigenstates $|\pm\rangle^0$, the S_z eigenstates. Then

$$E_\pm^1 = \langle\pm|^0 H^1|\pm\rangle^0 = -\gamma B\langle\pm|^0 S_x|\pm\rangle^0 = -\frac{\gamma B}{2}\langle\pm|^0(S_+ + S_-)|\pm\rangle^0 = 0,$$

and

$$\begin{aligned}
E_\pm^2 &= \sum_m{}' \frac{|\langle\pm|^0 H^1|m\rangle^0|^2}{E_\pm^0 - E_m^0} = \frac{|\langle\pm|^0 H^1|\mp\rangle^0|^2}{E_\pm^0 - E_\mp^0} = \frac{\gamma^2B^2}{4}\frac{|\langle\pm|^0(S_+ + S_-)|\mp\rangle^0|^2}{\mp\gamma B_0\hbar} \\
&= \mp\frac{\gamma B^2}{4B_0\hbar}|\langle\pm|^0 S_\pm|\mp\rangle^0|^2 = \mp\frac{\gamma B^2}{4B_0\hbar}|\langle\pm|^0\hbar\sqrt{(\tfrac{1}{2}\mp(\mp\tfrac{1}{2}))(\tfrac{3}{2}\pm(\mp\tfrac{1}{2}))}|\mp\rangle^0|^2 = \mp\frac{\gamma\hbar B^2}{4B_0},
\end{aligned}$$

and

$$|\pm\rangle' = \sum_m{}' \frac{|m\rangle^0\langle m|^0 H^1|\pm\rangle^0}{E^0_\pm - E^0_m} = \frac{|\mp\rangle^0\langle\mp|^0 H^1|\pm\rangle^0}{E^0_\pm - E^0_\mp} = -\frac{\gamma B}{2}\frac{|\mp\rangle^0\langle\mp|^0 S_\mp|\pm\rangle^0}{\mp\gamma B_0\hbar} = \pm\frac{B}{2B_0}|\mp\rangle^0,$$

where we used that $\langle\mp|^0 S_\mp|\pm\rangle^0 = \hbar$, as computed in the previous equation. Therefore

$$E_\pm = E^0_\pm + E^1_\pm + E^2_\pm + \mathcal{O}(B^3) = \mp\frac{\gamma B_0\hbar}{2} \mp \frac{\gamma B^2\hbar}{4B_0} = \mp\frac{\gamma B_0\hbar}{2}\left(1 + \frac{B^2}{2B_0^2}\right),$$

and

$$|\pm\rangle = |\pm\rangle^0 + |\pm\rangle^1 + \mathcal{O}(B^2) = |\pm\rangle^0 \pm \frac{B}{2B_0}|\mp\rangle^0.$$

Now compare to the exact answer. $H = -\gamma(B_0 S_z + B S_x) = -\gamma\sqrt{B_0^2 + B^2}\,\widehat{n}\cdot\vec{S}$, where $\widehat{n} = (B_0\widehat{k} + B\widehat{i})/\sqrt{B_0^2 + B^2}$. The eigenvalues of $\widehat{n}\cdot\vec{S}$ are $\pm\hbar/2$ with eigenvectors $|\pm\rangle$ with

$$\begin{aligned} |+\rangle &= \cos\tfrac{\theta}{2}\, e^{-i\phi/2}|+\rangle^0 + \sin\tfrac{\theta}{2}\, e^{i\phi/2}|-\rangle^0, \\ |-\rangle &= -\sin\tfrac{\theta}{2}\, e^{-i\phi/2}|+\rangle^0 + \cos\tfrac{\theta}{2}\, e^{i\phi/2}|-\rangle^0, \end{aligned} \tag{1}$$

where $|\pm\rangle^0$ are the S_z eigenvectors. Eqn. (1) can be rewritten as

$$|\pm\rangle = \cos\tfrac{\theta}{2}\, e^{\mp i\phi/2}|\pm\rangle^0 \pm \sin\tfrac{\theta}{2}\, e^{\pm i\phi/2}|\mp\rangle^0,$$

and

$$\begin{aligned} \cos\tfrac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+n_z}{2}} = \sqrt{\frac{\sqrt{B_0^2+B^2}+B_0}{2\sqrt{B_0^2+B^2}}} = 1 + \mathcal{O}(B^2), \\ \sin\tfrac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-n_z}{2}} = \sqrt{\frac{\sqrt{B_0^2+B^2}-B_0}{2\sqrt{B_0^2+B^2}}} = \frac{B}{2B_0} + \mathcal{O}(B^3), \end{aligned}$$

and $\phi = 0$, since $\widehat{n}$ is in the x-z plane. This implies

$$|\pm\rangle = |\pm\rangle^0 \pm \frac{B}{2B_0}|\mp\rangle^0 + \mathcal{O}(B^2),$$

in agreement with the perturbation theory result. Similarly, the exact eigenvalues are

$$E_\pm = \mp\frac{\gamma\hbar}{2}\sqrt{B_0^2 + B^2} = \mp\frac{\gamma\hbar B_0}{2}\left(1 + \frac{B^2}{B_0^2}\right)^{1/2} = \mp\frac{\gamma\hbar B_0}{2}\left(1 + \frac{B^2}{2B_0^2} + \mathcal{O}(B^4)\right),$$

again in agreement with the perturbation theory result.

Problem 17.3.2: $H = AS_z^2 + B(S_x^2 - S_y^2)$ on the spin-1 Hilbert space. In the S_z basis, $|m\rangle$ $(m = \pm 1, 0)$, we have

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(see p. 328 of the text). This implies

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}. \tag{2}$$

Clearly

$$|m\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is the eigenvector with $E = 0$. So we only need to look at the $|m\rangle = |\pm 1\rangle$ subspace where $H = H^0 + H^1$ with

$$H^0 = \hbar^2 A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H^1 = \hbar^2 B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{on the } |m\rangle = \{|\pm 1\rangle\} \text{ subspace.}$$

H^0 is degenerate: $H^0|m\rangle = \hbar^2 A|m\rangle$ for $m = \pm 1$. The basis stable under H^1 is the one which diagonalizes H^1. Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

the eigenvectors of H^1 are

$$\left|\overline{1}\right\rangle \equiv \frac{1}{\sqrt{2}} \left(|1\rangle + |-1\rangle\right) \qquad \text{and} \qquad \left|\overline{-1}\right\rangle \equiv \frac{1}{\sqrt{2}} \left(|1\rangle - |-1\rangle\right).$$

To order $\mathcal{O}(B)$, the energy shifts of $\left|\overline{1}\right\rangle$ and $\left|\overline{-1}\right\rangle$ are

$$\begin{aligned} E^1_{\overline{1}} &= \left\langle \overline{1}\right| H^1 \left|\overline{1}\right\rangle = \hbar^2 B \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \hbar^2 B, \\ E^1_{\overline{-1}} &= \left\langle \overline{-1}\right| H^1 \left|\overline{-1}\right\rangle = \hbar^2 B \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} = -\hbar^2 B. \end{aligned}$$

So, the eigenvalues, to $\mathcal{O}(B)$, are

$$E_{\overline{1}} = \hbar^2(A + B), \qquad E_{\overline{-1}} = \hbar^2(A - B), \qquad E_0 = 0.$$

Compare this to the exact eigenvalues of (2) given by $0 = \det(H - \lambda) = \lambda(\hbar^2(B - A) + \lambda)(\hbar^2(B + A) - \lambda)$, which implies $\lambda = \{0, \hbar^2(A - B), \hbar^2(A + B)\}$. So, the $\mathcal{O}(B)$ results are exact.

Solution Set 5

Problem 1: When H^0 includes the Coulomb potential plus the spin orbit coupling we see that the energy eigen states will be given by

$$|nj\, m_j;\, l\, s\rangle \qquad \text{where} \quad l = \text{orbital angular mom}$$

$s = \frac{1}{2}$ spin angular momentum.

Note that the original degeneracy of the levels, without the spin orbit interaction was $2n^2$. This degeneracy has been lifted by the spin orbit coupling. And we know that levels with a fixed l are split into $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ levels.

We also found (see eq. 15.2.20) of Shankar.

$$|nj = l+1/2, m_j;\, ls\rangle = \sqrt{\frac{(l+\frac{1}{2}+m_j)}{2l+1}}\; |n\, l\, m_j - \tfrac{1}{2}\rangle \otimes |1/2\rangle + \sqrt{\frac{(l+\frac{1}{2}-m_j)}{2l+1}}\; |n\, l\, m_j + \tfrac{1}{2}\rangle \otimes |1 - 1/2\rangle .$$

$$|nj = l-\tfrac{1}{2}, m_j;\, ls\rangle = -\sqrt{\frac{(l+\frac{1}{2}-m_j)}{2l+1}}\; |n\, l\, m_j - \tfrac{1}{2}\rangle \otimes |1/2\rangle + \sqrt{\frac{(l+\frac{1}{2}+m_j)}{2l+1}}\; |n\, l\, m_j + \tfrac{1}{2}\rangle \otimes |1 - 1/2\rangle$$

Now adding a weak magnetic field. $\vec{B} = Bk$ we have

$$H' = \left(\frac{eB}{2mc}\right)(L_z + 2S_z)$$

Since H' operator does not change the "l"/quantum #'s (and m_j) we just have '(we don't have to worry about accidental degeneracies!)

$$E' = \langle n j m_j; l s \mid H' \mid n j m_j; l s \rangle .$$

$$(L_z + 2S_z) \mid n j = l \pm \tfrac{1}{2}\, m_j ; l s \rangle$$

$$= \pm \sqrt{\frac{(l + \frac{1}{2} \pm m_j)}{(2l+1)}} \cdot (m_j + \tfrac{1}{2}) \mid n\, l\, m_j - \tfrac{1}{2} \rangle \otimes \mid \tfrac{1}{2} \rangle$$

$$+ \sqrt{\frac{(l + \frac{1}{2} \mp m_j)}{2l+1}} \quad (m_j - \tfrac{1}{2}) \mid n l\, m_j + \tfrac{1}{2} \rangle \otimes \mid -\tfrac{1}{2} \rangle .$$

$\therefore$

$$E' = \left(\frac{eB}{2mc}\right) \cdot \left\{ (m_j + \tfrac{1}{2}) \frac{(l + \frac{1}{2} \pm m_j)}{2l+1} + (m_j - \tfrac{1}{2}) \frac{(l + \frac{1}{2} \mp m_j)}{2l+1} \right\} .$$

$$E' = \left(\frac{eB}{2mc}\right) \left\{ 1 \pm \frac{1}{2l+1} \right\} m_j$$

Problem 2 : There are four degenerate levels $|2\ell m\rangle$. These are $\ell=0\ m=0$, $\ell=1, m=1,0,-1$. Let us first construct the 4×4 matrix.

$$\langle 2\ell m|\hat{x}|2\ell' m'\rangle .$$

We see that the non-zero elements are.

$$\langle 200|\hat{x}|211\rangle \text{ and } \langle 200|\hat{x}|21-1\rangle .$$

In fact using Wigner-Eckart theorem alone may indicate

$$\langle 210|\hat{x}|211\rangle \text{ and } \langle 210|\hat{x}|21-1\rangle$$

may be non-zero. But an explicit calculation shows these are zeros.

Given $\langle 200|\hat{z}|210\rangle = -3a_0$ (from class) we see

$$\langle 200|\hat{x}|211\rangle = \frac{3}{\sqrt{2}}a_0 = -\langle 200|\hat{x}|21-1\rangle .$$

$$\hat{x} \doteq \begin{array}{c} \\ 200 \\ 210 \\ 211 \\ 21-1 \end{array} \begin{array}{c} \begin{array}{cccc} 200 & 210 & 211 & 21-1 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{3}{\sqrt{2}}a_0 & -\frac{3}{\sqrt{2}}a_0 \\ 0 & 0 & 0 & 0 \\ \frac{3}{\sqrt{2}}a_0 & 0 & 0 & 0 \\ -\frac{3}{\sqrt{2}}a_0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

($|210\rangle$ state does not change!)

Ignoring $|210\rangle$ we see

$$\doteq \frac{3}{\sqrt{2}}a_0 \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The matrix has three vectors:

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{pmatrix} 1/\sqrt{2} \\ 1/2 \\ -1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/\sqrt{2} \\ -1/2 \\ +1/2 \end{pmatrix}$$

eigenvalue 0 eigenvalue $3a_0$ eigenvalue $-3a_0$

∴ Zeroth order states are

	1st order energy shift
$\|210\rangle$	0
$\frac{1}{\sqrt{2}}(\|211\rangle + \|21{-1}\rangle)$	0
$\left(\frac{1}{\sqrt{2}}\|200\rangle + \frac{1}{2}\|211\rangle - \frac{1}{2}\|21{-1}\rangle\right)$	$+3a_0\lambda$
$\left(\frac{1}{\sqrt{2}}\|200\rangle - \frac{1}{2}\|211\rangle + \frac{1}{2}\|21{-1}\rangle\right)$	$-3a_0\lambda$

Problem 3 : Problem 17.3.4 in Shankar.

(2) $\left\langle \frac{\lambda}{r^2} \right\rangle$ is the first order correction to the Hamiltonian if we add $\frac{\lambda}{r^2}$ to the Hamiltonian.

As discussed by Shankar we can find the exact answer! by replacing.

$$\frac{\hbar^2 \ell(\ell+1)}{2mr^2} + \frac{\lambda}{r^2} = \frac{\hbar^2 \ell'(\ell'+1)}{2mr^2}.$$

Now $$E(\ell') = \frac{-me^4}{2\hbar^2(k+\ell'+1)^2} = E^0 + E' + \cdots$$

$$\left\langle \frac{\lambda}{r^2} \right\rangle = E' = \lambda \left(\frac{\partial E(\ell')}{\partial \lambda} \right)_{\lambda=0} = \lambda \cdot \left(\frac{\partial E}{\partial \ell'} \frac{\partial \ell'}{\partial \lambda} \right)_{\lambda=0}$$

$$\left\langle \frac{\lambda}{r^2} \right\rangle = \lambda \left(\frac{me^4}{\hbar^2(k+\ell'+1)^3} \frac{d\ell'}{d\lambda} \right)_{\lambda=0}$$

$$1 = \frac{\hbar^2}{2m} (2\ell'+1) \frac{d\ell'}{d\lambda}.$$

$$\therefore \left\langle \frac{\lambda}{r^2} \right\rangle = \lambda \cdot \left(\frac{me^4}{\hbar^2(k+l+1)^3} \cdot \frac{2m}{\hbar^2(2l+1)} \right) = \frac{\lambda}{\eta^3 a_0^2 (l+1/2)}$$

(3) Let us first verify that

$$\frac{P_r^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)$$

$$\left(\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right).$$

where $P_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$

$$P_r^2 = -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \cancel{\frac{1}{r^2}} + \frac{2}{r}\frac{\partial}{\partial r} - \cancel{\frac{1}{r^2}}\right)$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)$$

$$\therefore \quad \frac{P_r^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right).$$

Now the Coulomb Hamiltonian can be written as.

$$H = \left(\frac{P_r^2}{2m} + \frac{\vec{L}^2}{2mr^2} - \frac{e^2}{r}\right). \quad \left\{\begin{array}{l}\text{see eq.}\\ (12.6.1)\\ \text{and}\\ (12.5.36)\end{array}\right\}.$$

$[\vec{L}, P_r] = 0$ since $\vec{L}$ involves angular variables and P_r involves radial variables. Also

$$\left[\frac{1}{r^n}, P_r\right] = -\frac{i\hbar n}{r^{n+1}}$$

$$\therefore \quad [H, P_r] = -i\hbar\left(\frac{\vec{L}^2}{mr^3} - \frac{e^2}{r^2}\right)$$

$$\langle E_n^0 | [H, P_r] | E_n^0 \rangle = E_n^0 \langle P_r \rangle - \langle P_r \rangle E_n^0 = 0 .$$

$$\left\langle \frac{\vec{L}^2}{m r^3} \right\rangle = \left\langle \frac{e^2}{r^2} \right\rangle .$$

$$\frac{\hbar^2 \ell(\ell+1)}{m} \left\langle \frac{1}{r^3} \right\rangle = \left\langle \frac{e^2}{r^2} \right\rangle$$

$$\therefore \left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a_0 \ell(\ell+1)} \left\langle \frac{1}{r^2} \right\rangle .$$

Liboff 13.29) Perturbation field: $\hat{H}' = 2\hat{H}'(\vec{r})\cos\omega t$

Let it act for a long time, and let $\omega \simeq \omega_{kl}$.

Since we are dealing with absorbtion,

$$P_{lk} = \frac{4|H'_{kl}|^2}{\hbar^2(\omega_{kl}-\omega)^2}\sin^2\left[\tfrac{1}{2}(\omega_{kl}-\omega)t\right]$$

Since $\hat{H}'$ acts for a long time, taking the limit $t\to\infty$ we have

$$P_{lk} \to \frac{2\pi t}{\hbar^2}|H'_{kl}|^2\delta(\omega_{kl}-\omega)$$

Thus the δ-function indicates that the only state to be excited is the kth state.

The Fourier decomposition of H' in time:

$$\int_{-\infty}^{\infty}\hat{H}'e^{i\omega t}\,dt = 2\hat{H}'(\vec{r})\int\cos\omega_0 t\, e^{i\omega t}\,dt$$

$$= \hat{H}'(\vec{r})\int(e^{i\omega_0 t}+e^{-i\omega_0 t})e^{i\omega t}\,dt$$

$$= \hat{H}'(\vec{r})\int(e^{i(\omega_0+\omega)t}+e^{i(\omega-\omega_0)t})\,dt$$

Recall $\int_{-\infty}^{\infty}e^{i(a+b)t} = 2\pi\delta(a+b)$

$$\Rightarrow = 2\pi\hat{H}'(\vec{r})\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right),$$

Picking out $\omega_0\to-\omega$ or $\omega_0\to\omega$.

Liboff

13.33 1-D Harmonic Oscillator. Change to mass e/m, spring constant K in its ground state.

We have $\mathcal{E}(t) = 2\mathcal{E}\cos\omega_0 t$. Thus $V(t) = -2\mathcal{E}\hat{x}\cos\omega_0 t$.

We write $\hat{H}' = -2e\mathcal{E}\hat{x}\cos(\omega_0 t)$

$= |\hat{H}'(x)\, 2\cos(\omega_0 t) \Rightarrow \hat{H}'(x) = -e\mathcal{E}\hat{x}$

Since $(\omega_{n0} - \omega_0)t \ll 1$ use

$$P_{on} = \frac{t^2 |H'_{n0}|^2}{\hbar^2}. \quad \text{Now, } H_{n0} = -e\mathcal{E}\langle n|x|0\rangle$$

$$= -e\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\langle n|a + a'|0\rangle$$

$$= -e\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\langle n|1\rangle \quad \leftarrow \delta_{n,1}$$

Thus

$$P_{on} = \delta_{n,1}\left|\frac{e\mathcal{E}}{\sqrt{2}\hbar\beta}\right|^2 t^2 \quad \text{with} \quad \beta^2 \equiv \frac{m\omega_0}{\hbar}$$

(3)

Shankar 18.2.1

$$C_0(t) = \frac{-i}{\hbar}\int_{-\infty}^{\infty} dt\, e^{i\omega_{10}t}(-e\mathcal{E})\langle 1|\hat{x}|0\rangle \frac{1}{1+\left(\frac{t}{\tau}\right)^2}$$

$\langle 1|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$

where $\omega_{10} = \frac{E_1^{(0)} - E_0^{(0)}}{\hbar} = \frac{\hbar\omega\left(\frac{3}{2}-\frac{1}{2}\right)}{\hbar} = \omega$

$$C_0(t) = \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}(-e\mathcal{E})\int_{-\infty}^{\infty} dt\, e^{i\omega t}\frac{1}{1+\left(\frac{t}{\tau}\right)^2}$$

$$= \frac{+ie\mathcal{E}\tau^2}{\sqrt{2m\omega\hbar}}\int dt\, \frac{e^{i\omega t}}{t^2+\tau^2}$$

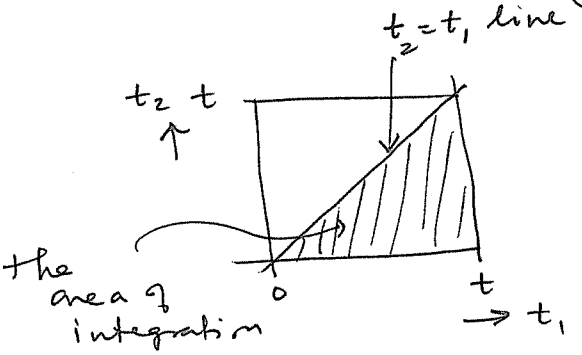
Using integral table: $= \frac{ie\mathcal{E}\tau^2}{\sqrt{2m\omega\hbar}}\frac{\pi}{\tau}e^{-\omega\tau}$

$$P = |C|^2 = \frac{e^2\mathcal{E}^2\tau^2}{2m\omega\hbar}e^{-2\omega\tau} \qquad \text{QED.}$$

Solution Set 7

(1) Consider the integral

$$I = \int_0^t dt_1 \int_0^{t_1} dt_2 \; f(t_1, t_2)$$



where $f(t_1, t_2)$ is some quantity. The same integral can be written as

$$I = \int_0^t dt_2 \int_{t_2}^t dt_1 \; f(t_1, t_2)$$

(If you don't understand this change of integration stop and think about it. It is important to understand that!).

Since t_1 and t_2 are dummy variables we can relabel

$$I = \int_0^t dt_1 \int_{t_1}^t dt_2 \; f(t_2, t_1) .$$

Now

$$I = \frac{1}{2} \left\{ \int_0^t dt_1 \int_0^{t_1} dt_2 \, f(t_1, t_2) + \int_0^t dt_1 \int_{t_1}^t dt_2 \, f(t_2, t_1) \right\}$$

Now if we choose

$$f(t_1, t_2) = H(t_1) H(t_2)$$

we get

$$I = \frac{1}{2} \left\{ \int_0^t dt_1 \int_0^{t_1} dt_2 \, H(t_1) H(t_2) + \int_0^t dt_1 \int_{t_1}^{t_2} dt_2 \, H(t_2) H(t_1) \right\} .$$

Now since t_1 and t_2 always appear in a time ordered way we can replace

$$H(t_1)\,H(t_2) \longrightarrow \hat{T}\{H(t_1)\,H(t_2)\} \quad \text{in } 1^{st} \text{ integral}$$

$$H(t_2)\,H(t_1) \longrightarrow \hat{T}\{H(t_1)\,H(t_2)\} \quad \text{in } 2^{nd} \text{ integral}$$

and thus.

$$I = \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2) = \frac{1}{2}\left\{\int_0^t dt_1 \int_0^{t_1} dt_2\, \hat{T}\{H(t_1)H(t_2)\} + \int_0^t dt_1 \int_{t_1}^t dt_2\, \hat{T}\{H(t_1)H(t_2)\}\right\}$$

$$= \frac{1}{2}\hat{T}\left\{\int_0^t dt_1\, \hat{H}(t_1) \int_0^t dt_2\, \hat{H}(t_2)\right\}$$

$$= \hat{T}\left\{\frac{1}{2}\left(\int_0^t dt_1\, \hat{H}(t_1)\right)^2\right\}.$$

18.2.2 We want to calculate

$$A_{i\to f} = \left(\frac{-i}{\hbar}\right)\langle 2\ell m|-e\mathcal{E}\hat{Z}|100\rangle \underbrace{\int_{-\infty}^{\infty} dt\, e^{i(\omega_{21}t)}\, e^{-t^2/\tau^2}}_{\sqrt{\pi\tau^2}\, e^{-\tau^2\omega_{21}^2/4}}.$$

only $\ell=1$ $m=0$ contributes.

$$\therefore P(n=2) = \frac{e^2\mathcal{E}^2}{\hbar^2}\underbrace{|\langle 210|\hat{Z}|100\rangle|^2}_{\frac{2^{15}}{3^{10}}a_0^2 \text{ (see prob 17.2.6 in Set 6)}}\pi\tau^2\, e^{-\tau^2\omega_{21}^2/2}$$

where

$$\omega_{21} = \frac{E_{210}-E_{100}}{\hbar}.$$

18.2.3 The velocity of the particle is

$$v = \frac{\pi\hbar}{mL} \Rightarrow T = \frac{L}{v} = \frac{mL^2}{\pi\hbar}$$

If the box expands symmetrically to twice its size the sudden approximation can be used if $\tau \ll T$.

Thus.

$$\psi_{init} = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} \qquad -\frac{L}{2} \le x \le \frac{L}{2}$$

$$= 0 \qquad \text{outside.}$$

$$\psi_{final} = \psi_{init}.$$

The wave fn of the new ground state

$$\psi_{init} = \sqrt{\frac{1}{L}} \cos \frac{\pi x}{2L} \qquad -L \le x \le L$$

$$= 0 \qquad \text{outside.}$$

$$\therefore \quad Prob = \left\{ \int_{-L/2}^{L/2} dx \, \frac{\sqrt{2}}{L} \underbrace{\left(\cos \frac{\pi x}{2L} \cos \frac{\pi x}{L} \right)}_{\frac{1}{2}\left(\cos \frac{3\pi x}{2L} + \cos \frac{\pi x}{2L}\right)} \right\}^2$$

$$= \left\{ \frac{1}{\sqrt{2}L} \int_{-L/2}^{L/2} dx \left(\cos \frac{3\pi x}{2L} + \cos \frac{\pi x}{2L} \right) \right\}^2$$

$$= \left\{ \frac{1}{\sqrt{2}L} \left\{ \frac{2L}{3\pi} \left(\sin \frac{3\pi x}{2L} \right)_{-L/2}^{L/2} + \frac{2L}{\pi} \left(\sin \frac{\pi x}{2L} \right)_{-L/2}^{L/2} \right\} \right\}^2$$

$$= \left[\frac{\sqrt{2}}{\pi} \left\{ \frac{2}{3} + 2 \right\} \sin \frac{\pi}{4} \right]^2 = \left(\frac{8}{3\pi} \right)^2.$$

18.2.4 In the β-decay of H^3

$$H^3 \longrightarrow He^3 + e^-$$

The K.E of e^- is 16 keV.

Mass of the electron 511 keV

The size of H^3 $a_0 = \frac{\hbar^2}{me^2}$ $\qquad \frac{v}{c} \simeq \sqrt{\frac{16}{511}}$

∴ Time for the β-particle to leave H^3 is

$$\tau = \frac{\hbar^2}{me^2} \cdot \frac{1}{v} = \frac{\hbar^2}{me^2} \sqrt{\frac{511}{16}} \frac{1}{c}$$

Time period of the electron in H^3

$$T = \frac{\hbar}{me^2} \frac{1}{\alpha c}$$

$$\therefore \quad \frac{\tau}{T} = \sqrt{\frac{511}{16}} \cdot \frac{1}{137} \simeq 0.04 \ll 1.$$

So we can use the sudden approx.

The Hamiltonian of the Helium atom is

$$H_{He} = \frac{p^2}{2\mu} - \frac{Ze^2}{r} \qquad \text{where } Z = 2.$$

ground state of He: $\psi_{1,00} = \frac{Z^{3/2}}{(\pi a_0^3)^{1/2}} e^{-rZ/a_0}$

ground state of H^3: $\psi_{100} = \frac{1}{(\pi a_0^3)^{1/2}} e^{-r/a_0}$.

$$A_{i\to f} \text{ (amplitude)} = \int d^3r \, \frac{Z^{3/2}}{(\pi a_0^3)} \, e^{-r(Z+1)/a_0}$$

$$= \frac{4\pi Z^{3/2}}{\pi a_0^3} \cdot \int_0^\infty dr \, r^2 \, e^{-r(Z+1)/a_0}$$

$$= \frac{8 Z^{3/2}}{(Z+1)^3} = \frac{8 \cdot 2\sqrt{2}}{3^3} .$$

$\therefore$ Amplitude to be in the ground state $= 16\sqrt{2}/27$.

Probability to be in some other state. $|n\ell m\rangle$ is

$$\psi_{n\ell m} = R_{n\ell}(r, Z) \, Y_{\ell m}(\theta, \phi) .$$

So probability to be in $|n=16, \ell=3, m=0\rangle$ state of He^{3+} is

$$\langle n=16, \ell=3, m=0 |_{He} \; | n=1, \ell=0, m=0 \rangle_{H^3} = 0$$

The $Y_{\ell m}(\theta, \phi)$ integration will give zero!

18.2.6 If $H_1(t) = H_1 \delta(t)$, then we see that-

$$|\psi(t=\tfrac{\varepsilon}{2})\rangle = |\psi(t=-\tfrac{\varepsilon}{2})\rangle + \left(\frac{-i}{\hbar}\right) \int_{-\varepsilon/2}^{\varepsilon/2} dt \; H_1 \delta(t) \; |\psi(t=-\tfrac{\varepsilon}{2})\rangle$$

$$|\psi(t=\tfrac{\varepsilon}{2})\rangle = |i^0\rangle + \left(\frac{-i}{\hbar}\right) H_1 |i^0\rangle$$

$\therefore$ amplitude to go to $|f^0\rangle$ $f \neq i$ is

$$A_{i\to f} = \left(\frac{-i}{\hbar}\right) \langle f^0 | H_1 | i^0 \rangle .$$

Problem Set 22

Problem 18.2.2: To first order, the amplitude $d_{2\ell m}(\infty)$ for the atom to be in the $|n=2,\ell,m\rangle$ state is

$$d_{2\ell m}(\infty) = -\frac{i}{\hbar}\int_{-\infty}^{\infty} \langle 2\ell m|(e\mathcal{E}Z)|100\rangle e^{-t^2/\tau^2} e^{i\omega_{21}t} dt$$

since the potential $H^1(t) = -\vec{\mu}\cdot\vec{E} = eZ\mathcal{E}e^{-t^2/\tau^2}$. Here $\omega_{21} = (E_2 - E_1)/\hbar$, which from now on we'll just call ω. So we need to evaluate $\langle 2\ell m|Z|100\rangle$. Z is a component of a vector irreducible tensor operator, $Z = T_1^0$, so by angular momentum selection rules, only the $\langle 210|Z|100\rangle$ matrix element is non-vanishing. This matrix element is easily evaluated using the wave functions for the the $|210\rangle$ and $|100\rangle$ states, and that $Z = r\cos\theta$:

$$\begin{aligned}
\langle 210|Z|100\rangle &= \int_0^\infty r^2 dr \int d\Omega \left(\frac{1}{2^5\pi a_0^3}\right)^{1/2} \left(\frac{r}{a_0}e^{-r/2a_0}\cos\theta\right)(r\cos\theta)\left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0} \\
&= 2\pi\left(\int_{-1}^{1} d(\cos\theta)\cos^2\theta\right)\cdot\left(\int_0^\infty dr\, r^4 e^{-3r/2a_0}\right)\cdot\frac{1}{2^{5/2}\pi a_0^4} \\
&= 2\pi\cdot\frac{2}{3}\cdot\left(\frac{2a_0}{3}\right)^5\left(\int_0^\infty d\rho\,\rho^4 e^{-\rho}\right)\cdot\frac{1}{2^{5/2}\pi a_0^4} \;=\; 2^{15/2}\cdot 3^{-5}\cdot a_0.
\end{aligned}$$

Therefore

$$\begin{aligned}
d_{2\ell m}(\infty) &= \delta_{\ell 1}\delta_{m0}\left(\frac{-i}{\hbar}\right)e\mathcal{E}\cdot 2^{15/2}\cdot 3^{-5}\cdot a_0\int_{-\infty}^{\infty} e^{-t^2/\tau^2}e^{i\omega t}dt \\
&= \delta_{\ell 1}\delta_{m0}\left(\frac{-ie\mathcal{E}}{\hbar}\right)2^{15/2}\cdot 3^{-5}\cdot a_0\sqrt{\pi\tau^2}e^{-\omega^2\tau^2/2},
\end{aligned}$$

and so the probability for the transition is

$$P(n{=}2) = \sum_{\ell,m}|d_{2\ell m}|^2 = \left(\frac{e\mathcal{E}}{\hbar}\right)^2\left(\frac{2^{15}a_0^2}{3^{10}}\right)\pi\tau^2 e^{-\omega^2\tau^2/2}.$$

This answer does not depend on the electron spin: since S_z is conserved by H^0 and H^1, there is still only a single final state it can go into.

Problem 18.2.4: The kinetic energy of the emitted electron is 16 keV $= \frac{1}{2}mv_e^2 = \frac{1}{2}mc^2(v_e/c)^2 = \frac{1}{2}(511\text{keV})(v_e/c)^2$, which implies that $v_e/c \sim 1/4$. Therefore the time, τ, for emission is $\tau \sim a_0/v_e = (a_0/c)/(v_e/c) \sim 4a_0/c$, since the typical size of the 1s electron orbit is $r \sim a_0$, the Bohr radius. In comparison, the characteristic time scale of the 1s electron, T, is $T \sim a_0/v_s = (a_0/c)/(v_s/c) = (a_0/c)/\alpha \sim 140a_0/c$, since

the typical velocity of an electron bound in the hydrogen atom is $v_s/c = \alpha$, the fine structure constant. Therefore, $T \gg \tau$, and the sudden approximation is appropriate.

In the sudden approximation, right after emission the 1s electron will be in the same state, the $|100\rangle_{(Z=1)}$ state of hydrogen. Therefore, the amplitude for the electron to be in the $|100\rangle_{(Z=2)}$ state of $(\mathrm{He}^3)^+$ is given by the overlap

$$\begin{aligned} {}_{(Z=2)}\langle 100|100\rangle_{(Z=1)} &= \int_0^\infty r^2 dr \int d\Omega \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-rZ/a_0} \cdot \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0} \\ &= 4\pi \cdot \frac{2^{3/2}}{\pi a_0^3} \int_0^\infty r^2 dr e^{-3r/a_0} = \frac{2^{7/2}}{a_0^3}\left(\frac{a_0}{3}\right)^3 \int_0^\infty \rho^2 d\rho e^{-\rho} = 2^{9/2}\cdot 3^{-3}, \end{aligned}$$

where in the first line $Z = 2$ and I used the fact that under changing the hydrogen nucleus charge from 1 to Z, all that changes is $a_0 \to a_0/Z$.

Finally, ${}_{(Z=2)}\langle 16, 3, 0|100\rangle_{(Z=1)} = 0$ since $\ell = 3$ states are orthogonal to $\ell = 0$ ones (and the radial part, and therefore Z, does not affect this).

PHYS 235 HW 8
SOLUTIONS

T. PRATT
11/19/08

8.4.2

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi$$

$$B = \nabla \times A$$

now, take $\begin{cases} \phi \to \phi' = \phi + \frac{1}{c}\partial_t \Lambda \\ A \to A' = A - \nabla\Lambda \end{cases}$

$$\Rightarrow E' = -\frac{1}{c}\partial_t A' - \nabla\phi'$$

$$= -\frac{1}{c}\partial_t A + \cancel{\frac{1}{c}\partial_t \nabla\Lambda} - \nabla\phi - \cancel{\frac{1}{c}\nabla\partial_t\Lambda}$$

$$= -\frac{1}{c}\partial_t A - \nabla\phi$$

and $B' = \nabla \times A'$

$$= \nabla \times (A - \nabla\Lambda)$$

$$= \nabla \times A - \underbrace{(\nabla \times \nabla)\Lambda}_{0 \text{ by vector identity}}$$

$$= \nabla \times A$$

So the fields computed using $(\phi, \vec{A})$ and $(\phi', \vec{A}')$ are identical.

(1) $\phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}$

$= \phi + \frac{1}{c}\partial_t\left(-c\int_{-\infty}^{t}\phi(r,t')dt'\right)$ $\quad\left(\text{recall: } \partial_x \int_a^x f(x')dx' = f(x)\right)$

$= \phi - (\phi(r,t))$

$= 0$

$A' = A - \nabla\Lambda = A + c\int_{-\infty}^{t}\nabla\phi(r,t')dt'$

$\nabla\cdot A' = \nabla\cdot A + c\int_{-\infty}^{t}\nabla^2\phi(r,t')dt'$

$\neq 0$ in general.

5.1 If we had left in the $e^{i\vec{k}\cdot\vec{r}}$, Eqn 18.5.19 would read

$$H'_{fi} = N\int e^{-i\vec{p}_f\cdot\vec{r}/\hbar}\, A_0\cdot(-i\hbar\nabla)\, e^{-r/a_0}\, d^3r\, e^{i\vec{k}\cdot\vec{r}}$$

$$= N\int e^{-i(\vec{p}_f-\hbar\vec{k})\cdot\vec{r}/\hbar}\, A_0\cdot(-i\hbar\nabla) e^{-r/a_0}\, d^3r$$

Integrating by parts, 18.5.21 becomes

$$H'_{fi} = N A_0\cdot(p_f-\hbar k)\int e^{-i(\vec{p}_f-\hbar\vec{k})\cdot\vec{r}/\hbar}\, e^{-r/a_0}\, d^3r \qquad \text{(choose } \hat{z} \text{ along } \vec{p}_f$$

$$= \frac{8\pi/a_0}{\left(\left(\frac{1}{a_0}\right)^2 + \left|\frac{\vec{p}_f-\hbar\vec{k}}{\hbar}\right|^2\right)^2} \qquad (18.5.22)$$

$$\Rightarrow R_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{1}{8\pi^3\hbar^3}\,\frac{1}{\pi a_0^3}\,\frac{|\vec{A}_0\cdot(\vec{p}_f-\hbar\vec{k})|^2\, 64\pi^2 a_0^6}{\left[1+\left(\frac{a_0|\vec{p}_f-\hbar\vec{k}|}{\hbar}\right)^2\right]^4}\,\delta(E_f^0-E_i^0-\hbar$$

(18.5.23)

This is the average transition rate after a long exposure to coherent light into a precise free particle state $|p_f\rangle$. Realistically, we are more interested in the rate of production of plane wave states pointed in directions lying in a cone of solid angle $d\Omega$, with any allowed energy $E_f = \frac{p_f^2}{2m}$.

$\Rightarrow$ compute $d\Omega\int dp_f\, p_f^2\, R_{i\to f}$. Now $\int\delta(f(x))\,g(x)\,dx = \int\delta(y)\, g(f^{-1}(y))\,\frac{df^{-1}(y)}{dy}\,d$

$$= g(f^{-1}(0))\,\frac{df^{-1}(y)}{dy}\bigg|_{y=0}$$

$\Rightarrow$ for $f(x) = \frac{p_f^2}{2m} - E_i^0 - \hbar\omega$, $\frac{df^{-1}(y)}{dy} = \frac{m}{p_f}$ $\Rightarrow$ $g(p_f)\,\frac{m}{p_f}\Big|_{p_f=\sqrt{2m(E_i^0+\hbar\omega}}$

$$\Rightarrow \boxed{R_{i\to d\Omega} = \frac{4e^2a_0^3 p_f\,|\vec{A}_0\cdot(\vec{p}_f-\hbar\vec{k})|^2\,d\Omega}{m\pi\hbar^4c^2\left[1+\left(\frac{a_0|\vec{p}_f-\hbar\vec{k}|}{\hbar}\right)^2\right]^4}\Bigg|_{|\vec{p}_f|=\sqrt{2m(E_i^0+\hbar\omega)}}}$$

but we worked in the Coulomb gauge, where $\vec{A}_0\cdot\vec{k} = 0$

$$\therefore \boxed{R_{i\to d\Omega} = \frac{4e^2a_0^3 p_f\,|\vec{A}_0\cdot\vec{p}_f|^2\,d\Omega}{m\pi\hbar^4c^2\left[1+\frac{a_0^2}{\hbar^2}|\vec{p}_f-\hbar\vec{k}|^2\right]^4}\Bigg|_{|\vec{p}_f|=\sqrt{2m(E_i^0+\hbar\omega)}}}$$

This is maximized when $\vec{p}_f$ and $\vec{k}$ are parallel, reflecting the fact that the electron has an increased component in the high energy plane wave state after the perturbation if the initial e^- momentum and the photon momentum could add constructi

(1) $\sigma = \frac{128 a_0^3 \pi e^2 p_f^3}{3m\hbar^3 \omega c\left[1+\frac{p_f^2 a_0^2}{\hbar^2}\right]^4}$ $\qquad E_f = \frac{p_f^2}{2m} = 10R_y = \frac{10me^4}{2\hbar^2}$

Now $a_0 = \frac{\hbar^2}{me^2}$, so $\frac{p_f^2 a_0^2}{\hbar^2} = \frac{2m \cdot 10me^4\hbar^4}{\hbar^2 \cdot 2\hbar^2 m^2 e^4} = 10$

$$\Rightarrow \sigma = \frac{128\pi e^2}{3m\omega c} \cdot 10^{3/2} \cdot \frac{1}{(1+10)^4}$$

And the e^- started in the Hydrogen ground state, $E_i = -R_y$

$$\Rightarrow 10R_y = -R_y + \hbar\omega$$
$$\Rightarrow \hbar\omega = 11R_y = \frac{11me^4}{2\hbar^2}$$

So $\sigma = \frac{128\pi \cdot 10^{3/2}}{3 \cdot 11^4} \cdot \underbrace{\frac{e^2\hbar}{mc} \cdot \frac{2\hbar^2}{11me^4}}$

$$\frac{2\hbar^3}{11m^2ce^2} = \frac{2\hbar^4 e^2}{11m^2c e^4 \hbar} = \frac{2}{11} a_0^2 \frac{e^2}{\hbar c} = \frac{2}{11}\alpha a_0^2$$

$$\Rightarrow \sigma = \frac{128 \cdot 10^{3/2}}{3 \cdot 11^4} \cdot \frac{2}{11}\alpha \cdot \pi a_0^2 \quad , \quad \alpha = \frac{1}{137} \quad , \quad \sqrt{10} \simeq 3$$

$$\simeq (128\alpha)\left(\frac{10^{1/2}}{3}\right)\left(\frac{10}{11}\right)\cdot\left(\frac{2}{11^4}\right) = \boxed{(\pi a_0^2)\cdot 10^{-4}}$$
$$\simeq 1 \quad \simeq 1 \quad \simeq 1 \quad \simeq 10^{-4}$$

(2) In the limit $\frac{p_f a_0}{\hbar Z} \gg 1$, $\sigma \to \frac{128\pi e^2 \hbar^5}{p_f^5 a_0^5 \cdot 3m\omega c}$. The Z could appear either with e^2 or a_0. But the e^2 came from $(H'_{fi})^2$, the interaction between light & the electron - this doesn't care about the nucleus charge. On the other hand, we know the Bohr radius is tightened to a_0/Z. So $\sigma \sim Z^5$, because $\sigma \sim a_0^{-5}$.

Problem S18.4.3 (Shankar, page 494):

(1) Show that a gauge transformation on potentials ϕ_0 and A_0 using

$$\Lambda_0(r,t) = -c\int_{-\infty}^{t}\phi(r,t')dt' \tag{1.1}$$

gives $\phi_1(r,t) = 0$ and $A_1(r,t) = A_0 - \nabla\Lambda_0$

Solution: This is problem 1 of HW8.

(2) Show that if we transform once more to ϕ_2 and A_2 using

$$\Lambda_1 = -\frac{1}{4\pi}\int\frac{\nabla'\bullet A_1(r',t)}{|r-r'|}d^3r' \tag{1.2}$$

then $\nabla\bullet A_2(r,t) = 0$.

Solution: Using equation (18.4.12) from Shankar ($A' = A - \nabla\Lambda$),we have

$$A_2(r,t) = A_1(r,t) + \frac{1}{4\pi}\int\vec{\nabla}\left[\frac{\nabla'\bullet A_1(r',t)}{|r-r'|}\right]d^3r' \tag{1.3}$$

Note that the integrand is the gradient of a scalar function, which produces a vector. If we now take the divergence of this, we get

$$\nabla\bullet A_2(r,t) = \nabla\bullet A_1(r,t) + \frac{1}{4\pi}\int\nabla^2\left[\frac{\nabla'\bullet A_1(r',t)}{|r-r'|}\right]d^3r' \tag{1.4}$$

Since the only r-dependence in the integrand comes from the denominator, we can use the identity

$$\nabla^2\left(1/|r-r'|\right) = -4\pi\delta^3(r-r') \tag{1.5}$$

Substituting this into (1.4) and doing the integrals over d^3r gives

$$\nabla\bullet A_2(r,t) = \nabla\bullet A_1(r,t) + \frac{1}{4\pi}\left[-4\pi\nabla'\bullet A_1(r',t)\right]_{r'=r} = \nabla\bullet A_1(r,t) - \nabla\bullet A_1(r,t) = 0 \tag{1.6}$$

(3) Verify that ϕ_2 is also zero using $\nabla\bullet E_0 = 0$.

Solution:

$$\phi_2(r,t) = \phi_1 + \frac{1}{c}\frac{\partial\Lambda}{\partial t} = \phi_1 - \frac{1}{4\pi c}\frac{\partial}{\partial t}\left\{\int\frac{\nabla'\bullet A_1(r',t)}{|r-r'|}d^3r'\right\} \tag{1.7}$$

Incorporating the results of part (1), this becomes

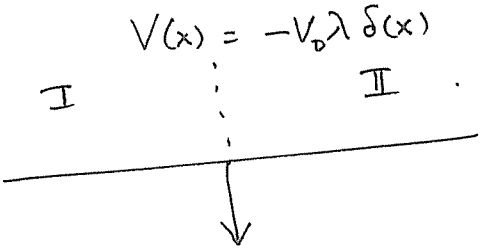
$$\phi_2(r,t) = \phi_1 + \frac{1}{c}\frac{\partial \Lambda}{\partial t} = 0 - \frac{1}{4\pi c}\frac{\partial}{\partial t}\left\{\int \frac{\nabla' \bullet \left[\frac{\partial A_0(r',t)}{\partial t} + c\nabla'\phi_0\left(r',t\right)\right]}{\left|r-r'\right|} d^3r'\right\} \qquad (1.8)$$

$$= \frac{1}{4\pi c}\frac{\partial}{\partial t}\left\{\int \frac{\nabla' \bullet \left[-cE_0\left(r',t\right)\right]}{\left|r-r'\right|} d^3r'\right\}$$

If we are in free space, so $\rho = 0$ (a condition that was implied but not explicitly stated), then Maxwell's first equation tells us that the numerator of the integrand is zero; so $\phi_2(r,t) = 0$.

Solution Set 8

(1)



Particle energy $E = \frac{\hbar^2 k^2}{2m}$. The wave function must satisfy the equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\lambda\,\delta(x)\,\psi = \frac{\hbar^2 k^2}{2m}\psi .$$

Integrating on both sides from $-\varepsilon$ to ε (ε = small) we get

$$\lim_{\varepsilon\to 0}\left.\frac{d\psi}{dx}\right|_{+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{-\varepsilon} = -\frac{2mV_0}{\hbar^2}\lambda\,\psi(0)$$

$\therefore$ If $\psi_I(x)$ is the wave function in region I and $\psi_{II}(x)$ is the wavefunction in region II.

$$\left.\frac{d\psi_{II}}{dx}\right|_{x=0} - \left.\frac{d\psi_I}{dx}\right|_{x=0} = -\frac{2mV_0}{\hbar^2}\lambda\,\psi_I(0)$$

and $\psi_I(0) = \psi_{II}(0)$ (continuity)

} B.C of the wavefunction.

Now $\psi_I(x) = A e^{ikx} + B e^{-ikx}$

$\psi_{II}(x) = C e^{ikx}$

$\therefore$ $A + B = C$

$$Cik - (Aik - Bik) = -\frac{2mV_0}{\hbar^2}\lambda\, C$$

Let us define $\frac{2mV_0}{\hbar^2}\lambda = k_0$, then we have

$$C(ik+k_0) = ik(A-B)$$

$$C = A+B$$

$$\therefore \quad A = C\left(1+\frac{k_0}{2ik}\right)$$

$$B = -\frac{Ck_0}{2ik}.$$

$$R = \frac{|B|^2}{|A|^2} = \frac{k_0^2/4k^2}{(1+k_0^2/4k^2)}.$$

$$T = \frac{|C|^2}{|A|^2} = \frac{1}{(1+k_0^2/4k^2)}.$$

(2) The probability current $\vec{J}_{sc}$ is defined as

$$\vec{J}_{sc} = \frac{\hbar}{2\mu i}\left(\psi_{sc}^* \vec{\nabla}\psi_{sc} - \nabla\psi_{sc}^* \psi_{sc}\right).$$

where $\psi_{sc} = \frac{e^{\pm ikr}}{r} f(\theta,\phi)$.

In spherical coordinates

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right).$$

In the $r\to\infty$ limit the $\hat{e}_\theta$ and $\hat{e}_\phi$ are subleading contributions and so can be neglected.

The leading contribution comes in the $\hat{e}_r$ direction when the "$\frac{\partial}{\partial r}$" acts on the $e^{\pm ikr}$ part of the wave function. Thus.

$$\vec{J}_{sc} = \pm \frac{\hbar k}{\mu} \frac{|f|^2}{r^2} \hat{e}_r + O(\text{higher orders})$$

The + sign leads to the outgoing current.

" incoming ".

(3) The free particle energy eigenfunctions can be chose to be simultaneous eigenfunctions of $\vec{L}^2$ and L_z. since the problem is spherically symmetric.

$$\psi_{klm}(\vec{r}) = R_l(kr)\, Y_l^m(\theta,\phi)$$

where the function $R_l(kr)$ satisfies the equation

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] R_l(kr) = \frac{\hbar^2 k^2}{2\mu} R_l(kr)$$

(Shankar eq. 12.6.3)

where we assume $E = \frac{\hbar^2 k^2}{2\mu}$.

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2\right] R_l(kr) = 0$$

(b) We see that $R_\ell(\rho)$ satisfies the eq.

$$\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho} - \frac{\ell(\ell+1)}{\rho^2} + 1\right] R_\ell(\rho) = 0$$

The solutions are the Spherical Bessel function $j_\ell(\rho)$ and Spherical Neumann functions $n_\ell(\rho)$.

Note if we write $R_\ell(\rho) = \frac{U_\ell(\rho)}{\rho}$ then we get the eq.

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + 1\right] U_\ell(\rho) = 0.$$

whose solutions are $\rho j_\ell(\rho)$ and $\rho n_\ell(\rho)$.

(c) The large distance behavior of $j_\ell(\rho)$ and $n_\ell(\rho)$ is given by:

$$\lim_{\rho\to\infty} j_\ell(\rho) = \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right)$$

$$\lim_{\rho\to\infty} n_\ell(\rho) = -\frac{1}{\rho}\cos\left(\rho - \frac{\ell\pi}{2}\right).$$

Thus these are <u>super positions</u> of spherically incoming and outgoing waves. To get an out going wave need to combine $j_\ell(\rho)$ and $n_\ell(\rho)$. But for the free particle $n_\ell(\rho)$ is not an allowed function since it blows up at the origin. So we cannot construct a purely outgoing spherical wave for the free particle! (Indeed that would imply there is a source at the center!)

19.3.1 In the class we showed that if

$$V(r) = \frac{g\, e^{-\mu_0 r}}{r}$$

then

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2 g^2}{\hbar^4 [\mu_0^2 + 4k^2 \sin^2\theta/2]^2}$$

To obtain the total scattering cross section we need

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_0^{\pi} d\theta \sin\theta \frac{4\mu^2 g^2}{\hbar^4 [\mu_0^2 + 4k^2 - 2k^2 \cos\theta]^2}$$

$$= 16\pi r_0^2 \left(\frac{g\mu r_0}{\hbar^2}\right)^2 \frac{1}{(1 + 4k^2 r_0^2)} .$$

where $r_0 = \frac{1}{\mu_0}$. Compare this with the Geometric cross section $\simeq \pi r_0^2$

19.3.2 Given a spherically symmetric potential $V(r)$ we found in class (see (eq. 19.3.8) is Shankar)

$$f(\theta) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty \sin qr'\, V(r')\, r'\, dr'$$

Given $V(r) = -V_0\,\theta(r_0 - r)$ we see that-

$$f(\theta) = +\frac{2\mu V_0}{\hbar^2 q}\int_0^{r_0} r' \sin qr'\, dr'$$

$$= \frac{2\mu V_0}{\hbar^2 q}\left\{\frac{(\sin qr_0 - (qr_0)\cos qr_0)}{q^2}\right\}$$

$$\therefore \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = 4r_0^2\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{(\sin qr_0 - (qr_0)\cos qr_0)^2}{(qr_0)^6}$$

where $q^2 = 2k^2(1-\cos\theta)$.

As $kr_0 \to 0$ $\quad qr_0 \to 0$ we see that-

$$\frac{(\sin qr_0 - (qr_0)\cos qr_0)^2}{(qr_0)^6} \simeq \frac{1}{9}\left[1 - \mathcal{O}(qr_0)^2\right]$$

$$\therefore \quad \sigma \simeq \frac{16\pi r_0^2}{9}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2$$

19.3.3 For the Gaussian Potential $V(r) = V_0 e^{-r^2/r_0^2}$ (9)

$$f(\theta) = \frac{-2\mu V_0}{\hbar^2 q}\int_0^\infty r' \sin qr'\, e^{-r'^2/r_0^2}\, dr'$$

$$= \left(\frac{-2\mu V_0}{\hbar^2 q}\right)\left(-\frac{\partial}{\partial q}\right)\int_0^\infty \underbrace{\cos qr'}_{\frac{e^{iqr'}+e^{-iqr'}}{2}}\, e^{-r'^2/r_0^2}\, dr'$$

$$\left[\int_0^\infty e^{-iqr'} e^{-r'^2/r_0^2}\, dr' = \int_{-\infty}^{0} e^{iqr'} e^{-r'^2/r_0^2}\, dr'\right]$$

$$= \left(\frac{-2\mu V_0}{\hbar^2 q}\right)\left(-\frac{\partial}{\partial q}\right)\left\{\frac{\sqrt{\pi r_0^2}}{2}\left(e^{-\frac{q^2 r_0^2}{4}}\right)\right\}$$

$$= \left(\frac{-2\mu V_0}{\hbar^2 q}\right)\sqrt{\pi r_0^2}\,\frac{1}{4}\, q r_0^2 e^{-q^2 r_0^2/4}$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{\pi r_0^2}{4}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 e^{-q^2 r_0^2/2}$$

The total cross section.

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = (2\pi)\frac{\pi r_0^2}{4}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \int_0^\pi d\theta \sin\theta\, e^{-q^2 r_0^2/2}$$

Let $q^2 = 2k^2(1-\cos\theta)$

$$\frac{d(q^2)}{2k^2} = \sin\theta\, d\theta$$

Hence

$$\sigma = \frac{\pi^2 r_0^2}{2}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \underbrace{\int_0^{4k^2} \frac{d(q^2)}{2k^2} e^{-q^2 r_0^2/2}}_{\frac{2\left(1 - e^{-2k^2 r_0^2}\right)}{2k^2 r_0^2}}$$

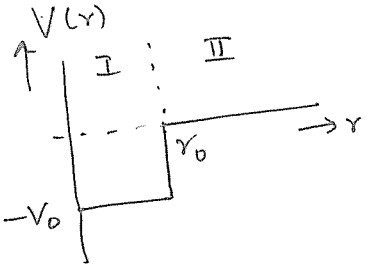
$$\therefore \quad \sigma = \frac{\pi^2}{2k^2}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \left(1 - e^{-2k^2 r_0^2}\right)$$

Solution set 11

Problem 19.5.4 (Shankar):

For S-wave scattering we need to solve.

$$\left\{ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left[E - V(r)\right] \right\} U(r) = 0 .$$



We have $V(r) = -V_0 \quad r < r_0$

$= 0 \quad r > r_0$

Thus if we define $\frac{2m(E+V_0)}{\hbar^2} = k'^2$, $\frac{2mE}{\hbar^2} = k^2$

we see that for

$r < r_0$ $U_I(r)$ obeys

$$\left(\frac{d^2}{dr^2} + k'^2 \right) U_I(r) = 0 .$$

$r > r_0$ $U(r)$ obeys

$$\left(\frac{d^2}{dr^2} + k^2 \right) U(r) = 0 .$$

$\therefore$ Region I

$$U(r) = A' \sin k'r + B' \cos k'r$$

Region II

$$U(r) = A \sin kr + B \cos kr$$

$$= R \sin(kr + \delta_0) .$$

→ Wave function must vanish at $r=0$. $\Rightarrow$ $B' = 0$.

→ Wave function must be continuous at $r = r_0$

$$A \sin k' r_0 = R \sin(k r_0 + \delta_0)$$

→ First derivative of the wave function must be continuous at r_0.

$$A k' \cos k' r_0 = R k \cos(k r_0 + \delta_0)$$

∴ We see that-

$$k \sin k' r_0 = k' \tan(k r_0 + \delta_0)$$

$$\therefore \quad \delta_0 = -k r_0 + \tan^{-1}\left(\frac{k}{k'} \tan k' r_0\right).$$

Neglecting $k r_0$ we see that-

$$\delta_0 = \tan^{-1}\left(\frac{k}{k'} \tan k' r_0\right).$$

When $k' \simeq k'_n = (2n+1)\pi/2r_0$

then $\tan k'_n r_0 = \tan\left((2n+1)\pi/2\right) = \infty$.

Thus $\tan(k' r_0) \simeq \dfrac{1}{(k'_n - k')r_0} = \dfrac{k'_n + k'}{(k'^2_n - k'^2) r_0}$.

$$= \frac{\frac{\hbar^2}{2m r_0}(k'_n + k')}{(E_n - E)} \simeq \frac{\frac{\hbar^2}{m r_0} k'_n}{(E_n - E)} = \frac{\Gamma/2}{(E_R - E)}$$

$$\therefore \quad \frac{\Gamma}{2} = \frac{\hbar^2}{m r_0} k'_n .$$

Let us now consider bound states with energy E. Then we can find a condition.

if the bound states are of energy $E = 0$. Then we must have

$$A \sin k' r_0 = R e^{-k r_0}$$

$$A k' \cos k' r_0 = -R k e^{-k r_0}$$

$$\therefore \quad \tan k' r_0 = \frac{-k'}{k}$$

Now,

$$\begin{cases} \frac{\hbar^2 k^2}{2m} = |E| \\ \frac{\hbar^2 k'^2}{2m} = (V_0 - |E|) \end{cases}$$

If ($E \to 0$ or) $k \to 0$ then $k' r_0 = (2n+1)\pi/2 = k'_n r_0$

$\therefore \quad k' = k'_n$ is the point where we get bound states of energy $\neq 0$!

Problem 19.5.6 (Shankar).

(1) We know that the full wavefunction of scattering is

$$\psi_k(\vec{r}) = e^{ikz} + f(\theta, \phi) \frac{e^{ikr}}{r}$$

$$= e^{ikr\cos\theta} + f(\theta, \phi) \frac{e^{ikr}}{r}.$$

Now the probability current is

$$\vec{J} = \frac{\hbar}{2mi} \left[\psi^* \vec{\nabla} \psi - (\vec{\nabla} \psi) \psi^* \right]$$

$$= \frac{\hbar}{m} \mathrm{Im} \left[\psi^* \vec{\nabla} \psi \right].$$

$$\lim_{r \to \infty} \vec{J} = \frac{\hbar}{m} \Big[\underbrace{k \hat{z}}_{J_{inc}} + |f|^2 \frac{k \hat{r}}{r^2} + \underbrace{\frac{1}{r} \mathrm{Im} \{ ik e^{ikr(\cos\theta - 1)} f^* \hat{z} + ik e^{-ikr(\cos\theta - 1)} f \hat{r} \}}_{\vec{J}_{interference}} \Big]$$

$\vec{J}_{sc}$

$$\lim_{r\to\infty} \vec{j}_{int} = \frac{\hbar k}{m} \mathrm{Im}\left\{ i \frac{e^{ikr(\cos\theta - 1)}}{r} f^* \hat{z} + e^{-ikr(\cos\theta - 1)} \frac{f}{r} \hat{r} \right\}.$$

↑ interference term

The radial component is given by

$$\lim_{r\to\infty} j_{int,r} = \frac{\hbar k}{m} \frac{1}{r} \mathrm{Im}\left[i e^{ikr(\cos\theta - 1)} f^*(\theta) \cos\theta + i e^{-ikr(\cos\theta - 1)} f(\theta) \right].$$

(2) When we integrate over a small solid angle at $\theta \neq 0$ we see $(\cos\theta - 1) \neq 0$. Hence $e^{ikr(\cos\theta - 1)}$ oscillates rapidly (even) over the small integration region and gives zero!

(3) When $\theta \approx 0$ then $(\cos\theta - 1) \approx -\frac{\theta^2}{2}$. Then the integral survives! We now see that $\cos\theta \approx 1$ $\sin\theta\, d\theta \approx \theta\, d\theta$

$$\int_0^\theta e^{-ikr\theta^2/2} f^*(\theta)\, \theta\, d\theta = \frac{f^*(0)}{kr}$$

$$\int_0^\theta e^{ikr\theta^2/2} f(\theta)\, \theta\, d\theta = -\frac{f(0)}{kr}.$$

$$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = \frac{\hbar k}{\mu} \frac{1}{r} \mathrm{Im}\left[i \frac{f^*(0)}{kr} - \frac{i f(0)}{kr} \right] 2\pi r^2$$

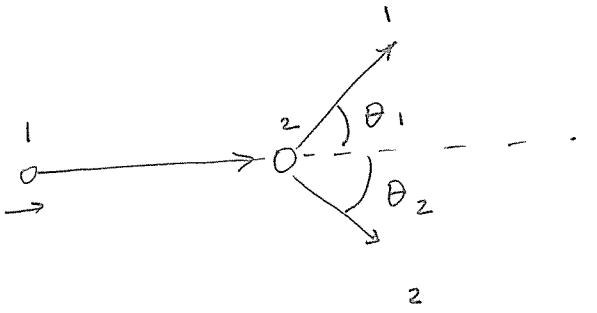
$$= -\frac{\hbar k}{\mu} \left(\frac{4\pi}{k}\right) \mathrm{Im}\, f(0).$$

(2) · We can see that- $\theta_1 \leq \pi/2$ from energy - momentum conservation.

To see this note that-

$$mv = mv_1 \cos\theta_1 + mv_2 \cos\theta_2$$

$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2$$



clearly we can assume without loss of generality that- $\theta_1 > 0$ and $\theta_2 > 0$. (because the reverse situation $\theta_1 < 0$ and $\theta_2 < 0$ is a symmetric situation).

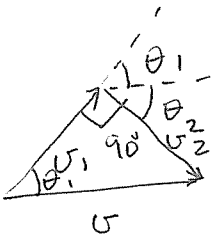
Thus,

$$v = v_1 \cos\theta_1 + v_2 \cos\theta_2$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 .$$

and Energy conservation $\Rightarrow$ $v^2 = v_1^2 + v_2^2$.

Geometrically



$\therefore$ v_1 v_2 and v form sides of a right triangle.

$\Rightarrow$ $\theta_1 + \theta_2 = \pi/2$. $\Longrightarrow$ Along with $\theta_1 > 0$ $\theta_2 > 0$ means $\underline{\theta_1 \leq \pi/2}$

$$\therefore \quad \frac{\int_{\text{forward cone}} j_r^{\text{int}} r^2 d\Omega}{j_{\text{inc}}} = -\left(\frac{4\pi}{k}\right) \text{Im } f(0).$$

$-$ ↑ deficit (depletion).

Now σ_{tot} = total scattering cross section.

$$\therefore \quad \sigma_{tot} = \left(\frac{4\pi}{k}\right) \text{Im } f(0) \quad \Longleftarrow \text{ Optical theorem.}$$

Problem 19.6.1 (Shankar).

(1) We know that

$$\left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right) \cdot \left(\frac{d\Omega_{CM}}{d\Omega_L}\right)$$

Now $d\Omega_{CM} = \sin\theta_{CM}\, d\theta_{CM}\, d\phi_{CM}$.

$d\Omega_L = \sin\theta_L\, d\theta_L\, d\phi_L$.

From eq. (19.6.16) / 6.17 we see that $2\theta_L = \theta_{CM}$ $\quad \phi_L = \phi_{CM}$

$$\therefore \quad \left(\frac{d\Omega_{CM}}{d\Omega_L}\right) = \frac{\sin 2\theta_L}{\sin\theta_L} \underbrace{\frac{d\theta_{CM}}{d\theta_L}}_{2} \underbrace{\frac{d\phi_{CM}}{d\phi_L}}_{1}$$

$$= 4\cos\theta_L$$

$$\therefore \quad \left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right)_{\theta_{CM}=2\theta_L} 4\cos\theta_L$$

(3) For unequal mass eqs. (19.6.14), (19.6.15) change as follows

$$\tan\theta = \frac{P_\perp'}{P_z'} \qquad \tan\phi = \frac{P_y'}{P_x'} \qquad \text{(same as } m_1 = m_2\text{)}$$

$$\tan\theta_L = \frac{P_\perp'}{\left(P_z' + \frac{m_1}{m_2}P\right)} \qquad \tan\phi_L = \frac{P_y'}{P_x'} \; . \longrightarrow$$

$$\therefore \quad \phi_L = \phi \quad \text{but} \quad \tan\theta_L = \frac{P_\perp'/P}{\left(\frac{P_z'}{P} + \frac{m_1}{m_2}\right)} \; .$$

Now $P_\perp'/P = \sin\theta \qquad P_z'/P = \cos\theta$.

$$\therefore \quad \tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}} \; .$$

Problem 19.6.2 (Shankar).

We start with eq. (19.3.16) for $f_c(\theta)$.

$$f_c(\theta) = \frac{-\gamma}{2k(\sin\theta/2)^2} e^{-i\gamma \ln \sin^2\theta/2 + iA} \qquad (A = \text{some const.}) \; .$$

purely real.

Now $\frac{d\sigma}{d\Omega} = |f(\theta,\phi) + f(\pi-\theta, \phi+\pi)|^2$

$$f_c(\pi-\theta) = \frac{-\gamma}{2k(\cos\theta/2)^2} e^{-i\gamma \ln \cos^2\theta/2 + A} \; .$$

$$|f_c(\theta)|^2 = \frac{\gamma^2}{4k^2 \sin^4\theta/2} \; . \qquad |f_c(\pi-\theta)|^2 = \frac{\gamma^2}{4k^2\cos^4\theta/2}$$

$$f_c^*(\theta) f_c(\pi-\theta) = \frac{\gamma^2}{4k^2 \sin^2\theta/2 \cos^2\theta/2} \cdot e^{i\gamma \ln \tan^2\theta/2}$$

$$f_c(\theta) f_c^*(\pi-\theta) = \frac{\gamma^2}{4k^2 \sin^2\theta/2 \cos^2\theta/2} e^{-i\gamma \ln \tan^2\theta/2}$$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{\gamma^2}{4k^2} \left[\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} + \frac{2\cos(\gamma \ln \tan^2\theta/2)}{\sin^2\theta/2 \cos^2\theta/2} \right]$$

$$\text{If } \gamma = \frac{e^2\mu}{\hbar^2 k} \qquad \frac{\gamma}{k} = \frac{e^2}{2E} \cdot \qquad \frac{\gamma^2}{4k^2} = \left(\frac{e^2}{4E}\right)^2$$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4E}\right)^2 \left[\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} + \frac{2\cos(\gamma \ln \tan^2\theta/2)}{\sin^2\theta/2 \cos^2\theta/2} \right]$$

Solution Set 9

(1.) Let us solve the differential equation (1.)

$$[\nabla^2 + k^2]\, G(\vec{r}) = \delta^3(\vec{r})$$

We see that except at the origin the above equation is the same as the equation one would solve for the wave function of free particles!

$$[\nabla^2 + k^2]\, G(\vec{r}) = 0 \qquad \text{for } |\vec{r}| \neq 0 .$$

We know that the general solution to this equation (in spherical polar coordinates) is given by

$$G(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A_l j_l(kr) + B_l n_l(kr)]\, Y_l^m(\theta, \phi)$$

where $j_l(kr)$ and $n_l(kr)$ are spherical Bessel and spherical Neumann functions.

Now for $r \to \infty$ we want the "radial" part of the Green's function to be of the form $\frac{e^{ikr}}{r}$.

We know for $r \to \infty$

$$j_l(kr) \longrightarrow \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right)$$

$$n_l(kr) \longrightarrow \frac{-1}{kr} \cos\left(kr - \frac{l\pi}{2}\right).$$

Thus we need $A_l = -B_l$ to cancel the $\frac{e^{-ikr}}{r}$ part. Thus

$$G(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l [j_l(kr) - n_l(kr)]\, Y_l^m(\theta, \phi).$$

Now $\delta^3(\vec{r})$ has an integrable singularity as $r \to 0$! which means $\delta^3(\vec{r}) \to \frac{1}{r^2}$ as $r \to 0$! This means the coefficient of $n_\ell(kr)$ must be zero for all $\ell > 0$!

Remember

$$n_\ell(kr) \sim \frac{1}{(kr)^{\ell+1}}$$

and $$\nabla^2 n_\ell(kr) \sim \frac{1}{(kr)^{\ell+2}}$$

Hence

$$G(\vec{r}) = A_0 \frac{e^{ikr}}{kr}$$

To determine A_0 we integrate the orginal eq. over V (a small sphere of radius ε) i.e...

$$\int_V d^3r \, [\nabla^2 + k^2] G(\vec{r}) = \int_V d^3\vec{r} \, \delta^3(\vec{r})$$

Substituting we get

$$\int_V d^3r \, \vec{\nabla} \cdot \vec{\nabla} G(\vec{r}) + k^2 \int_V d^3r \, G(\vec{r}) = 1 .$$

$\int_V d^3r \, G(\vec{r}) \to 0$ (taking $\varepsilon \to 0$)

$$= \int_S d\Omega \, \hat{e}_r \cdot \vec{\nabla} G(\vec{r}) = 1$$

S ↑ Surface of the sphere

$$\Rightarrow \quad A_0 = -\frac{k}{4\pi} \quad \Rightarrow \quad \boxed{G(\vec{r}) = -\frac{1}{4\pi} \frac{e^{ikr}}{r}}$$

(2) The Born approximation is valid as long as (3)

$$\frac{2\mu}{\hbar^2 k}\left|\int_0^\infty e^{ikr'}\sin kr'\,V(r)\,dr'\right| << 1 .$$

for a spherically symmetric potential. For the Yukawa potential this condition becomes

$$\frac{2\mu|V_0|}{\hbar^2 k\alpha}\left|\int_0^\infty e^{ikr'}\sin kr'\,\frac{e^{-\alpha r'}}{r'}\,dr'\right| << 1$$

When $\frac{k}{\alpha} << 1$ we can write the above condition as

$$\frac{2\mu V_0}{\hbar^2 k\alpha}\left|\int_0^\infty e^{i\frac{k}{\alpha}t}\sin\frac{k}{\alpha}t\;e^{-t}\,\frac{dt}{t}\right| << 1$$

where we substituted $\alpha r' = t$

which then leads to

$$\boxed{\frac{2\mu|V_0|}{\hbar^2\alpha^2} << 1} \quad \text{(Low energies)} .$$

When $\frac{k}{\alpha} >> 1$ we can the write the condition as

$$\frac{2\mu|V_0|}{\hbar^2 k\alpha}\left|\int_0^\infty e^{it}\,\frac{\sin t}{t}\;e^{-\frac{\alpha}{k}t}\,dt\right|$$

Now

$$\int_0^\infty e^{it}\,\frac{\sin t}{t}\,e^{-\frac{\alpha}{k}t}\,dt = \int_0^\infty \left\{\frac{1}{2}\,\frac{\sin 2t}{t}\,e^{-\frac{\alpha}{k}t} + i\,\frac{\sin^2 t}{t}\,e^{-\frac{\alpha}{k}t}\right\} dt$$

From the "table of integrals" (Gradshteyn & Ryzhik) we see that-

$$\int_0^\infty \frac{\sin 2t}{t}\,e^{-\frac{\alpha}{k}t} = \tan^{-1}\left(\frac{2k}{\alpha}\right)$$

$$\int_0^\infty \frac{\sin^2 t}{t}\,e^{-\frac{\alpha}{k}t} = \frac{1}{4}\ln\left(\frac{(\alpha/k)^2+4}{(\frac{\alpha}{k})^2}\right)$$

In the $\frac{\alpha}{k} \to 0$ limit we see that-

$$\int_0^\infty e^{it}\,\frac{\sin t}{t}\,e^{-\frac{\alpha}{k}t}\,dt \simeq -\frac{i}{2}\ln\left(\frac{\alpha}{k}\right) \simeq \frac{i}{2}\ln\left(\frac{k}{\alpha}\right)$$

* Also see the end of Problem

∴ The condition is given by.

$$\frac{\mu|V_0|}{\hbar^2 k\alpha}\ln\left(\frac{k}{\alpha}\right) \ll 1$$

{ You can roughly estimate the above integral as follows:

$$\int_0^\infty \frac{e^{it}\sin t}{t}\,e^{-\frac{\alpha}{k}t}\,dt \simeq \int_0^{k/\alpha} \frac{e^{it}\sin t}{t}\,dt \xrightarrow[k/\alpha \to \infty]{\text{diverges for}} \text{as}\left(\ln\frac{k}{\alpha}\right) \}.$$

(5)

When there are bound states

$$\frac{2m|V_0|}{\hbar^2\alpha^2} \geq 2.7 .$$

Thus the Born approximation is violated at

necessarily
low energies but not at high energies! This is

reasonable since the bound states affect only low energies.

Thus the sign of V_0 should not matter at high energies!

---

* We can also estimate the integral $\int_0^\infty e^{it}\frac{\sin t}{t}e^{-\frac{\alpha}{k}t}dt$ more easily! since the exponential is simply cutting off the t integral we can approx it by

$$\simeq \int_0^{k/\alpha} e^{it}\frac{\sin t}{t}dt \approx \int_0^{k/\alpha}\frac{e^{2it}}{2it} - \int_0^{k/\alpha}\frac{1}{2it}dt$$

We only want the singular part as $k/\alpha \to \infty$. this is obtained from the 2nd integral (first one is convergent at ave k/α due to oscillating part)
The second integral is $\frac{i}{2}\ln k/\alpha$ as expected!

(3) The Lippmann-Schwinger formalism in 1-d can be formulated as follows:

We want to solve

$$\hat{H}|\psi_{E_k}\rangle = E_k|\psi_{E_k}\rangle$$

where $\hat{H} = \hat{H}_0 + \hat{V}$

we know $\hat{H}_0|\phi_k\rangle = E_k|\phi_k\rangle$. (Free problem) .

We want to solve

$$|\psi_{E_k}\rangle = |\phi_{E_k}\rangle + |\psi_{sc}\rangle .$$

$$\therefore \quad \cancel{H_0|\phi_{E_k}\rangle} + V|\phi_{E_k}\rangle + H_0|\psi_{sc}\rangle + V|\psi_{sc}\rangle$$

$$= \cancel{E_k|\phi_{E_k}\rangle} + E_k|\psi_{sc}\rangle .$$

$$(E_k - H_0)\,|\psi_{sc}\rangle = V|\psi_{E_k}\rangle .$$

$$\therefore \quad |\psi_{sc}\rangle = \frac{1}{E_k - H_0} V|\psi_{E_k}\rangle .$$

$$\therefore \quad \boxed{|\psi_{E_k}\rangle = |\phi_k\rangle + \frac{1}{E_k - H_0} V|\psi_{E_k}\rangle} .$$

Thus we need to find the prescription for handling the singularity in $\frac{1}{E_k - H_0}$.

In 1-d we have

$$\psi_{E_k}(x) = \phi_{E_k}(x) + \langle x| \frac{1}{E_k - H_0} \hat{V}|\psi_{E_k}\rangle .$$

Let us assume that the prescription $E_k \to E_k + i\varepsilon$ works and find out if the Green's function

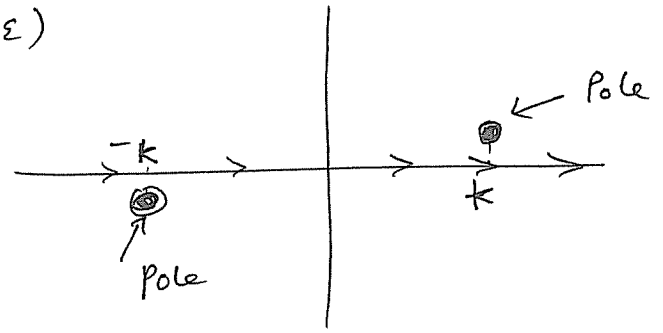
$$G^{(+)}(x,x') = \frac{\hbar^2}{2m} \langle x| \frac{1}{E_k - H_0 + i\varepsilon} |x'\rangle .$$

has the right boundary condition.

Now $G^{(+)}(x,x') = \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{\frac{i}{\hbar}p(x-x')} \frac{1}{\left(E_k - \frac{p^2}{2m} + i\varepsilon\right)}$

Let us substitute $\frac{p}{\hbar} = q$ and $E_k = \frac{\hbar^2 k^2}{2m}$. Then we get

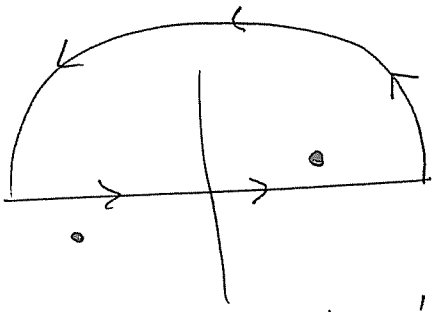
$$G^{(+)}(x,x') = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iq(x-x')}}{(q^2 - k^2 - i\varepsilon)}$$

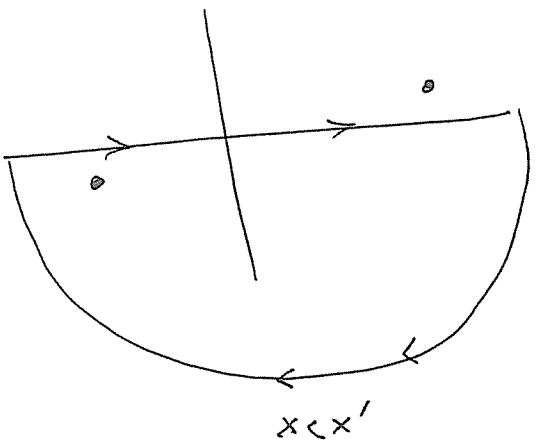


The poles are at

$$q = \pm (k + i\varepsilon')$$

Now if $x > x'$ we can close the contour in the upper half plane and when $x < x'$ we can close the contour in the lower half plane.





Thus

$$G^{(+)}(x,x') = -\oint \frac{dq}{2\pi} \frac{e^{iq(x-x')}}{(q-k-i\varepsilon')(q+k+i\varepsilon')}$$

$$= \begin{cases} -\frac{i}{2k} e^{ik(x-x')} & x > x' \\ -\frac{i}{2k} e^{-ik(x-x')} & x < x' . \end{cases}$$

Thus $G^{(+)}(x,x') = \frac{-1}{2k} e^{ik|x-x'|}$

Thus the integral equation for $\psi_k^+(x)$ is

$$\psi_k^+(x) = \psi_{inc}(x) + \int dx' G(x,x') V(x') \psi_k(x') .$$

(b) Now we are given $V(x)$, so

$$\psi_k^+(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{2m}{\hbar^2}\int dx' \left(-\frac{i}{2k} e^{ik|x-x'|}\right)\left(-\frac{\gamma\hbar^2}{2m}\right)\delta(x')\,\psi_k(x')$$

$$\therefore \psi_k^+(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k} e^{ik|x|}\psi_k^+(0) .$$

Now substituting $x = 0$ we see

$$\psi_k^+(0)\left[1 - \frac{i\gamma}{2k}\right] = \frac{1}{\sqrt{2\pi}}$$

$$\therefore \psi_k^+(0) = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{i\gamma}{2k}\right)^{-1} .$$

Hence we see

$$\psi_k^+(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k}\frac{1}{\sqrt{2\pi}}\cdot\left(1 - \frac{i\gamma}{2k}\right)^{-1} e^{ik|x|}$$

for $x \to +\infty$ we see

$$\psi_k^+(x) = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{\frac{i\gamma}{2k}}{1 - \frac{i\gamma}{2k}}\right) e^{ikx}$$

for $x \to -\infty$ we see

$$\psi_k^+(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{1}{\sqrt{2\pi}} \frac{\frac{i\gamma}{2k}}{\left(1 - \frac{i\gamma}{2k}\right)} e^{-ikx}$$

$$\therefore \quad R = \left| \frac{\frac{i\gamma}{2k}}{\left(1 - \frac{i\gamma}{2k}\right)} \right|^2 = \left(\frac{\gamma^2}{4k^2 + \gamma^2} \right)$$

and $$T = \left| \frac{1}{1 - \frac{i\gamma}{2k}} \right|^2 = \left(\frac{4k^2}{4k^2 + \gamma^2} \right) .$$

Exercise 19.5.6 (Optical Theorem)

(1) The wavefunction is given by

$$\psi(\vec{r}) = \psi_{inc} + \psi_{sc}$$

where
$$\left.\begin{aligned} \psi_{inc} &= e^{ikr\cos\theta} \\ \psi_{sc} &= \frac{e^{ikr}}{r} f(\theta) . \end{aligned}\right\} \text{ for } r \to \infty .$$

The current density due to the interference term is given by

$$(\vec{J})^{int} = \frac{\hbar}{2\mu i}\left[\psi_{inc}^* \vec{\nabla}\psi_{sc} + \psi_{sc}^* \vec{\nabla}\psi_{inc} - \vec{\nabla}\psi_{inc}^* \psi_{sc} - \vec{\nabla}\psi_{sc}^* \psi_{inc} \right] .$$

$$= \frac{\hbar}{\mu} \mathrm{Im}\left[\psi_{inc}^* \vec{\nabla}\psi_{sc} + \psi_{sc}^* \vec{\nabla}\psi_{inc} \right] .$$

If we look at the radial part at $r \to \infty$ we just consider

$$\vec{\nabla}\psi_{sc} \to \frac{\partial}{\partial r}\psi_{sc} \quad \text{and} \quad \vec{\nabla}\psi_{inc} \to \frac{\partial}{\partial r}\psi_{inc}$$

Substituting we get

$$j_r^{int} \underset{r\to\infty}{\sim} \frac{\hbar k}{\mu r} \mathrm{Im}\left[i e^{ikr(1-\cos\theta)} f(\theta) + e^{ikr(\cos\theta - 1)} f^*(\theta)\cos\theta \right]$$

(2) As long as $\theta \neq 0$ we see that $(1-\cos\theta)$ is not zero.

This means j_r^{int} oscillates wildly over any small change in the angle (when $r \to \infty$) and hence integrates to zero!

(3) When $\theta \simeq 0$ $\quad (1-\cos\theta) \simeq \frac{\theta^2}{2}$. This means

$$\int_{\substack{\text{forward} \\ \text{cone}}} j_r^{int}\, r^2 d\Omega \simeq \left(\frac{\hbar k}{\mu}\right) \cdot r \cdot \mathrm{Im}\left[i \int_0^{2\pi}\int_0^{\theta} e^{-ikr\theta^2/2}\, \theta\, d\theta\, d\phi\, f^*(0) + i \int_0^{2\pi}\int_0^{\theta} e^{ikr\theta^2/2}\, \theta\, d\theta\, d\phi\, f(0) \right]$$

$\theta = 0$ to θ (small)
$\phi = 0$ to 2π

$$\simeq \left(\frac{\hbar k}{\mu}\right) r\, 2\pi\, \mathrm{Im}\left[i f^*(0) \left(\frac{e^{-ikr\theta^2/2}}{ikr} \right)_0^{\theta} + i f(0) \left(\frac{e^{ikr\theta^2/2}}{ikr} \right)_0^{\theta} \right]$$

Now in the $r \to \infty$ limit we can set $e^{\pm ikr\theta^2/2} \to 0$ (using argument in Shankar page 66 on evaluating rapidly oscillating functions).

$$\simeq -\left(\frac{\hbar k}{\mu}\right) \cdot \frac{4\pi}{k}\, \mathrm{Im}\, f(0)$$

Exercise 19.6.1 ① We know from (19.6.16) and (19.6.17) that

$$\theta_L = \theta_{CM}/2 \qquad \phi_L = \phi_{CM}.$$

Further

$$\left(\frac{d\sigma}{d\Omega_L}\right) \sin\theta_L \, d\theta_L \, d\phi_L = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \sin\theta_{CM} \, d\theta_{CM} \, d\phi_{CM}.$$

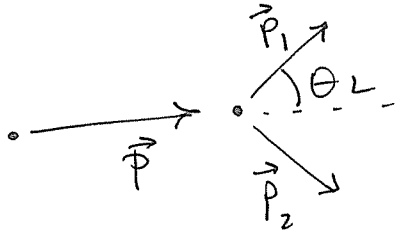
From the transformation above we see

$$\left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right) \cdot \frac{\sin\theta_{CM}}{\sin\theta_L} \cdot \left[\frac{(d\theta_{CM}\, d\phi_{CM})}{(d\theta_L\, d\phi_L)}\right]$$

↑ Jacobian of the transformation

$$\left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right)\Bigg|_{\substack{\theta_{CM}=2\theta_L \\ \phi_{CM}=\phi_L}} \cdot \, 4\cos\theta_L$$

② Let $\vec{P}$ be the momentum of the projectile before collision.



Let $\vec{P}_1$ be the momentum of the projectile after collision and $\vec{P}_2$ be the momentum of the target after collision.

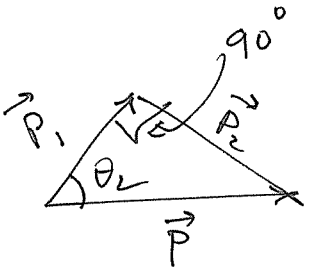
From momentum conservation

$$\vec{P} = \vec{P}_1 + \vec{P}_2.$$

From Energy (kinetic Energy) conservation (Non-relativistic)

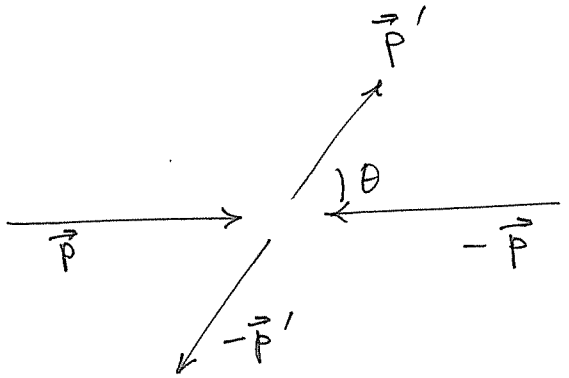
$$P^2 = P_1^2 + P_2^2.$$

Thus $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}$ form a right triangle as shown below



Clearly $0 \le \theta_L \le \pi/2$!

(3) For scatteng of unequal masses.



$$\tan\theta = \frac{\sqrt{P_x'^2 + P_y'^2}}{P_z'} \qquad \tan\phi = \frac{P_y'}{P_x'} \qquad P = \sqrt{P_x^2 + P_y^2 + P_z'^2}$$

The velocity of the target in CM frame $= -\frac{\vec{P}}{m_2}$.

In the Lab frame

$$\vec{P}_{Proj} = \vec{P} + \frac{m_1}{m_2}\vec{P}$$

$$\vec{P}_{target} = -\vec{P} + \frac{m_2}{m_2}\vec{P} = 0$$

$$\vec{P}'_{Proj} = \vec{P}' + \frac{m_1}{m_2}\vec{P}$$

$$\vec{P}'_{target} = -\vec{P}' + \vec{P}$$

$$\left.\begin{aligned} (\vec{P}'_{proj})_x &= P_x' \\ (\vec{P}'_{proj})_y &= P_y' \\ (\vec{P}'_{proj})_z &= P_z' + \frac{m_1}{m_2}P \end{aligned}\right\}$$

which means

$$\tan\theta_L = \frac{\sqrt{(\vec{P}'_{Proj})_x^2 + (\vec{P}'_{Proj})_y^2}}{(\vec{P}'_{Proj})_z}$$

$$\tan\theta_L = \frac{\sqrt{P_x'^2 + P_y'^2}}{P_z' + \frac{m_1}{m_2}P} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

Exercise 19.6.3 Consider two fermions (spin-$\frac{1}{2}$) in the triplet state scattering off of each other. Then the wave function will be antisymmetric in space. This means.

$$\frac{d\sigma}{d\Omega} = \left| f(\theta,\phi) - f(\pi-\theta, \phi+\pi) \right|^2$$

If $f(\theta,\phi) \simeq f(\theta)$ then.

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) - f(\pi-\theta) \right|^2$$

Clearly at $\theta = \pi/2$ $\left(\frac{d\sigma}{d\Omega}\right) = 0$!

Physics 215C Homework #1 Solutions

Richard Eager
Department of Physics
University of California; Santa Barbara, CA 93106

Shankar 20.1.1

Derive the continuity equation

$$\frac{\partial P}{\partial t} + \nabla \cdot j = 0$$

where $P = \psi^\dagger \psi$ and $j = c\psi^\dagger \alpha \psi$

The Dirac equation takes the equivalent forms

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \left(c\alpha \cdot P + \beta mc^2\right)|\psi\rangle$$

$$\frac{\partial \psi}{\partial t} = \left(-c\alpha \cdot \nabla - \frac{i}{\hbar}\beta mc^2\right)\psi$$

The conjugate equation is

$$\frac{\partial \psi^\dagger}{\partial t} = \left(-c\alpha \cdot \nabla + \frac{i}{\hbar}\beta mc^2\right)\psi^\dagger$$

$$\begin{aligned}\frac{\partial P}{\partial t} &= \psi^\dagger \frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t}\psi \\ &= -c\left(\psi^\dagger \alpha \cdot \nabla \psi + (\alpha \cdot \nabla \psi^\dagger)\psi\right)\end{aligned}$$

$$\nabla \cdot j = c\left(\nabla \psi^\dagger \alpha \psi + \psi^\dagger \alpha \cdot \nabla \psi\right)$$

Adding both equations together yields the desired result

$$\frac{\partial P}{\partial t} + \nabla \cdot j = 0$$

Both terms in the Hamiltonian, $c\alpha \cdot P$ and βmc^2 are Hermitian, so where did the relative minus sign come from? To show that P is a Hermitian operator you need to integrate by parts. If you are working with the L^2 inner product you can drop boundary terms, but when working locally to derive the continuity equation we don't integrate by parts and get a relative minus sign.

2

Show that the probability current j of the previous exercise reduces in the non-relativistic limit to Eq.(5.3.8) [which is the same as Sakurai Eq.(2.4.16)].

The probability current $j = c\psi^\dagger \alpha \psi$ and $\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$ We can write the wave function $\psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ in terms of its relativistic and non-relativistic components, Φ and χ respectively.

$$j = c\begin{pmatrix} \chi^\dagger & \Phi^\dagger \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \Phi \end{pmatrix} \tag{0.1}$$

$$= c\left(\chi^\dagger \sigma \Phi + \Phi^\dagger \sigma \chi\right) \tag{0.2}$$

In the non-relativistic limit (20.2.13)

$$\Phi \approx \frac{\sigma \cdot \pi}{2mc}$$

$$j = \chi^\dagger \frac{\hat{p}}{2m}\chi + (\frac{\hat{p}}{2m}\chi^\dagger)\chi$$

which is the non-relativistic current (5.3.8)

$$j = \frac{\hbar}{2mi}\left(\psi^* \nabla\psi - \psi \nabla\psi^*\right)$$

from the identification $\hat{p} = -i\hbar\nabla$.

Shankar 20.2.1

Show that

$$\pi \times \pi = \frac{iq\hbar}{c}B$$

where $\pi = P - \frac{qA}{c}$.

$$\begin{aligned} \pi \times \pi &= P \times P - \frac{qA}{c} \times P - P \times \frac{qA}{c} + \frac{qA}{c} \times \frac{qA}{c} \\ &= -\frac{q}{c}\left(A \times P + P \times A\right) \\ &= \frac{iq\hbar}{c}\left(A \times \nabla + \nabla \times A\right) \end{aligned}$$

The simplest way to manipulate operators is to act on a test function ψ

$$\begin{aligned} \frac{iq\hbar}{c}\left(A \times \nabla + \nabla \times A\right)\psi &= \frac{iq\hbar}{c}\left(A \times \nabla\psi + \nabla \times (\psi A)\right) \\ &= \frac{iq\hbar}{c}\left(A \times \nabla\psi + (\nabla\psi)A + (\nabla \times A)\psi\right) \\ &= \frac{iq\hbar}{c}\left(\nabla \times A\right)\psi \\ &= \frac{iq\hbar}{c}B\psi \end{aligned}$$

Therefore

$$\pi \times \pi = \frac{iq\hbar}{c}B$$

Shankar 20.1.1

Solve for the 4 spinors w that satisfy Shankar Eq. 20.3.3. You may assume that the 3- momentum $\vec{p}$ is along the z-axis. Normalize them to unity, and show that they are mutually orthogonal.

Equation (20.3.3) is

$$Ew = (\alpha \cdot p + \beta m)w$$

In terms of the relativistic and non-relativistic components,

$$\begin{pmatrix} E-m & -\sigma \cdot p \\ -\sigma \cdot p & E+m \end{pmatrix} \begin{pmatrix} \chi \\ \Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solutions are given in equations (20.3.7) and (20.3.8). Choosing the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for Φ and letting p be in the z direction,

$$w_{1,3} = \begin{pmatrix} p/(\pm E - m) \\ 0 \\ 1 \\ 0 \end{pmatrix} w_{2,4} = \begin{pmatrix} 0 \\ p/(\pm E - m) \\ 0 \\ 1 \end{pmatrix}$$

Orthogonality of the spinors is easy to see using $E^2 = p^2 + m^2$.

5

The five terms in 20.2.28 are

$\frac{P^2}{2m}$	Hermitian
V	Hermitian
$-\frac{P^4}{8m^3c^2}$	Hermitian
$\frac{i\sigma \cdot P \times [P,V]}{4m^2c^2}$	Hermitian
$\frac{P \cdot [P,V]}{4m^2c^2}$	anti-Hermitian

Recall that $\hat{P}$ is a Hermitian operator, the potential V is assumed to be Hermitian (for conservation of probability) and the Pauli matrices σ are Hermitian. The commutator $[X, Y]$ of two Hermitian operators is anti-Hermitian since $[X,Y]^\dagger = (XY)^\dagger - (YX)^\dagger = Y^\dagger X^\dagger - X^\dagger Y^\dagger = YX - XY = -[X,Y]$. The cross product of two Hermitian (vector) operators is again Hermitian since $(X \times Y)^\dagger = X^\dagger \times Y^\dagger = X \times Y$.

Honors Physics 632 WI 11 HW6 Solutions

1) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|100\rangle - |111\rangle\right)$; $r \le a$, $E_{n\ell} = \frac{\hbar^2 k_{\ell n}^2}{2m}$

(a) $P(E_{n\ell}) = \sum_{m=-\ell}^{\ell} \left|\langle n\ell m|\psi(t)\rangle\right|^2$

$= \frac{1}{2}\delta_{n1}\left(\delta_{\ell 0} + \delta_{\ell 1}\right)$

$P(E_{10}) = \frac{1}{2}$, $P(E_{11}) = \frac{1}{2}$, all others zero

(b) $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_{100}t/\hbar}|100\rangle - e^{-iE_{111}t/\hbar}|111\rangle\right)$

$\langle \vec{r}|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_{100}t/\hbar}\phi_{100}(\vec{r}) - e^{-iE_{111}t/\hbar}\phi_{111}(\vec{r})\right)$

$\phi_{100}(\vec{r}) = A_{10}\, j_0(k_{01}r)\, Y_0^0(\theta,\phi)$

$\phi_{111}(\vec{r}) = A_{11}\, j_1(k_{11}r)\, Y_1^1(\theta,\phi)$

(c) $\langle \hat{R}\rangle = \int_{r\le a} d^3r\, r\, |\psi(\vec{r},t)|^2$

In the cross terms we encounter $Y_0^{0*} Y_1^1$ etc. These lead to:

$\int_0^{2\pi} d\phi\, e^{i\phi} = 0 \Rightarrow$ No cross terms survive.

Also $\int d\Omega\, Y_\ell^{m*} Y_{\ell'}^{m'} = \delta_{\ell\ell'}\delta_{mm'}$.

(c) cont. $\langle \hat{R} \rangle = \frac{1}{2} \int_0^a r^3 dr \left[|A_{10}|^2 \left(j_0(k_{01} r) \right)^2 + |A_{11}|^2 \left(j_1(k_{11} r) \right)^2 \right]$

- need to evaluate further only if given tables.

(d) Prob that $z > 0$: $P_{z>0} = \int_0^a r^2 dr \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, |\psi(\vec{r}, t)|^2$

- cross terms do not survive because $\int d\Omega \, Y_0^* Y_1 = 0$...

$$P_{z>0} = \frac{2\pi}{2} \int_0^a r^2 dr \int_0^{\pi/2} \sin\theta \, d\theta \left\{ |A_{10}|^2 \left(j_0(k_{01} r) \right)^2 \left[\frac{1}{4\pi} \right] + |A_{11}|^2 \left(j_1(k_{11} r) \right)^2 \left[\frac{1}{4} \left(\frac{3}{2\pi} \right) \sin^2\theta \right] \right\}$$

Use $\int_0^a r^2 dr \, |A_{nl}|^2 \left(j_l(k_{ln} r) \right)^2 = 1$ and

$$\rightarrow \int_0^{\pi/2} \sin\theta \, d\theta = \int_0^1 dx = 1 \; ; \int_0^{\pi/2} \sin\theta \, d\theta \, \sin^2\theta = \int_0^1 dx \, (1 - x^2) = \frac{2}{3}$$

$$P_{z>0} = \frac{2\pi}{2} \left\{ 1 \times \frac{1}{4\pi} + \frac{2}{3} \times \left[\frac{1}{4} \left(\frac{3}{2\pi} \right) \right] \right\}$$

$$P_{z>0} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

2) hydrogen; $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{\sqrt{3}}|210\rangle + \frac{1}{\sqrt{6}}|21\text{-}1\rangle$

(a) $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \quad ; \quad \hbar = 1$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_1 t}|100\rangle + \frac{1}{\sqrt{3}} e^{-iE_2 t}|210\rangle + \frac{1}{\sqrt{6}} e^{-iE_2 t}|21\text{-}1\rangle$

(b) $P_{z>0} = \int d^3r \, |\psi(\vec{r},t)|^2 \, \Theta(z)$

Again we still have $\int_0^{2\pi} d\phi \rightarrow m = m'$ in any surviving cross-terms. $\Rightarrow$ only $|100\rangle + |210\rangle$

With $m = m'$, $\int_0^{2\pi} d\phi \, e^{i(m-m')\phi} = \int_0^{2\pi} d\phi = 2\pi$

$$P(z>0) = 2\pi \int_0^\infty r^2 dr \int_0^{\pi/2} \sin\theta \, d\theta \left\{ \frac{1}{2} R_{10}^2(r) \left(\frac{1}{4\pi}\right) + \right.$$

$$\frac{1}{3} R_{21}^2(r) \left(\frac{1}{4} \cdot \frac{3}{\pi} \cos^2\theta\right) + \frac{1}{6} R_{21}^2(r) \left(\frac{1}{4} \cdot \frac{3}{2\pi} \sin^2\theta\right)$$

$$\left. + \frac{1}{\sqrt{6}} R_{10}(r) R_{21}(r) \left(\frac{1}{\sqrt{4\pi}} \cdot \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta\right)\left(e^{i(E_2-E_1)t} + e^{-i(E_2-E_1)t}\right) \right\}$$

Use: $\int_0^\infty r^2 dr \, R_{n\ell}^2(r) = 1$, $\int_0^1 dx \, x^2 = \frac{1}{3}$, $\int_0^1 dx \, (1-x^2) = \frac{2}{3}$

$\int_0^1 dx \, x = \frac{1}{2}$

$$P(z>0) = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{4\pi} \cdot 1 + 2\pi \cdot \frac{1}{3} \cdot \frac{3}{4\pi} \cdot \frac{1}{3} + 2\pi \cdot \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{3}{2\pi} \cdot \frac{2}{3}$$

$$+ 2\pi \cdot \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{3}{4}} \frac{1}{2\pi} \cdot \frac{1}{2} \cdot 2\cos\left((E_2 - E_1)t\right) \int_0^\infty r^2 dr \, R_{10}(r) R_{21}(r)$$

We need one final integral:

$$\int_0^\infty r^2 dr \, \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \cdot \frac{1}{\sqrt{24 a_0^3}} \frac{r}{a_0} e^{-r/2a_0}$$

$$= \frac{1}{\sqrt{6} a_0^4} \int_0^\infty dr \, r^3 e^{-3r/2a_0}$$

$$\int_0^\infty dr \, r^n e^{-kr} = \left(-\frac{d}{dk}\right)^n \int_0^\infty dr \, e^{-kr} = \left(-\frac{d}{dk}\right)^n \frac{1}{k} = \frac{n!}{k^{n+1}}$$

$$\frac{1}{\sqrt{6} a_0^4} \cdot \frac{3!}{\left(\frac{3}{2a_0}\right)^4} = \frac{\sqrt{6} \cdot 2^4}{3^4}$$

$$P(z>0) = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} + \cos\left[(E_2 - E_1)t\right]$$

$$\frac{1}{2}\left[1 + \frac{8}{27}\cos\left((E_2 - E_1)t\right)\right]$$

(c) $P(r > l_{nm}) = \int_{l_{nm}}^{\infty} r^2 dr \int d\Omega \left|\psi(\vec{r},t)\right|^2$

$\int d\Omega$ kills all cross terms with differing l, m.

↳ no phases for $|\psi(0)\rangle$ above

$$P(r > l_{nm}) = \int_{l_{nm}}^{\infty} r^2 dr \left[\frac{1}{2} R_{10}^2(r) + \frac{1}{3} R_{21}^2(r) + \frac{1}{6} R_{21}^2(r)\right]$$

$$= \frac{1}{2} \int_{l_{nm}}^{\infty} r^2 dr \left[\frac{4}{a_0^3} e^{-2r/a_0} + \frac{1}{24 a_0^4} r e^{-r/a_0}\right]$$

$$\int_{l_{nm}}^{\infty} dr\, r^n e^{-kr} = \left(-\frac{d}{dk}\right)^n \frac{e^{-kr}}{-k}\Big|_{l_{nm}}^{\infty} = \left(-\frac{d}{dk}\right)^n \frac{e^{-k(l_{nm})}}{k}$$

$$-\frac{d}{dk}\frac{e^{-kR}}{k} = \frac{e^{-kR}}{k^2} + \frac{R e^{-kR}}{k} = \left(\frac{1}{k^2} + \frac{R}{k}\right) e^{-kR}$$

$$\left(-\frac{d}{dk}\right)^2 \frac{e^{-kR}}{k} = \left(+\frac{1}{k^3} + \frac{R}{k^2}\right) e^{-kR} + \left(\frac{R}{k^2} + \frac{R^2}{k}\right) e^{-kR}$$

$$= \left(\frac{1}{k^3} + \frac{2R}{k^2} + \frac{R^2}{k}\right) e^{-kR}$$

$$\left(-\frac{d}{dk}\right)^3 \frac{e^{-kR}}{k} = \left(\frac{1}{k^4} + \frac{4R}{k^3} + \frac{R^2}{k^2}\right) e^{-kR} + \left(\frac{R}{k^3} + \frac{2R^2}{k^2} + \frac{R^3}{k}\right) e^{-kR}$$

$$= \left(\frac{1}{k^4} + \frac{5R}{k^3} + \frac{3R^2}{k^2} + \frac{R^3}{k}\right) e^{-kR}$$

$R = 1\,nm$

$$P(r > 1\,nm) = \frac{2}{a_0^3}\left(\frac{1}{\left(\frac{2}{a_0}\right)^3} + \frac{2R}{\left(\frac{2}{a_0}\right)^2} + \frac{R^2}{\left(\frac{2}{a_0}\right)}\right)e^{-2R/a_0} + \frac{1}{48a_0^4}\left(\frac{1}{\left(\frac{1}{a_0}\right)^4} + \frac{5R}{\left(\frac{1}{a_0}\right)^3} + \frac{3R^2}{\left(\frac{1}{a_0}\right)^2} + \frac{R^3}{\left(\frac{1}{a_0}\right)}\right)e^{-R/a_0}$$

We can simplify this a bit more:

$$P(r > 1\,nm) = \left(\frac{1}{4} + \frac{R}{a_0} + \frac{R^2}{a_0^2}\right)e^{-2R/a_0} + \left(\frac{1}{48} + \frac{5R}{48a_0} + \frac{R^2}{16a_0^2} + \frac{R^3}{48a_0^3}\right)e^{-R/a_0}$$

$$\frac{R}{a_0} = \frac{10^{-9}\,m}{0.5 \times 10^{-10}\,m} \simeq \frac{1}{5}$$

$$P(r > 1\,nm) \simeq \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{25}\right)e^{-2/5} + \left(\frac{1}{48} + \frac{1}{48} + \frac{1}{16 \cdot 25} + \frac{1}{(48)(125)}\right) \times e^{-1/5}$$

$$\simeq \frac{1}{2}e^{-2/5} + \frac{1}{50}e^{-1/5} \simeq 0.35$$

(d) $P(m_x) = \sum_{n\ell} \left| \langle n\ell m_x | \psi(t) \rangle \right|^2$

We encounter $\langle n\ell m_x | n'\ell' m' \rangle \propto \delta_{nn'} \delta_{\ell\ell'}$,

so most terms drop out.

For $\ell = 0$, $\langle 00_x | 00 \rangle = 1$; only one state.

From table we can get $\langle 10_x | 10 \rangle = 0$, $\langle 10_x | 1,-1 \rangle = -\frac{1}{\sqrt{2}}$,

$\langle 11_x | 10 \rangle = \frac{1}{\sqrt{2}}$, $\langle 11_x | 1,-1 \rangle = \frac{1}{2}$, $\langle 1,-1_x | 10 \rangle = -\frac{1}{\sqrt{2}}$

and $\langle 1,-1_x | 1,-1 \rangle = \frac{1}{2}$.

$$P(m_x = 1) = \left| \langle 211_x | \psi(t) \rangle \right|^2 = \left| \frac{1}{\sqrt{3}} e^{-iE_2 t} \langle 211_x | 210 \rangle + \frac{1}{\sqrt{6}} e^{-iE_2 t} \langle 211_x | 21,-1 \rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \cdot \frac{1}{2} \right|^2 = \left| \frac{3}{2\sqrt{6}} \right|^2 = \frac{9}{24}$$

$$\boxed{P(m_x = 1) = \frac{3}{8}}$$

$$P(m_x = -1) = \left| \langle 21,-1_x | \psi(t) \rangle \right|^2 = \left| \frac{1}{\sqrt{3}} e^{-iE_2 t} \langle 21,-1_x | 210 \rangle + \frac{1}{\sqrt{6}} e^{-iE_2 t} \langle 21,-1_x | 21,-1 \rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{3}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{6}} \left(\frac{1}{2}\right) \right|^2 = \left| \frac{1}{2\sqrt{6}} \right|^2 = \frac{1}{24}$$

$$P(m_x=0) = \left|\langle 100_x | \psi(t)\rangle\right|^2 + \left|\langle 210_x | \psi(t)\rangle\right|^2$$

$$= \frac{1}{2} + \left|\frac{1}{\sqrt{3}}\langle 10_x | 10\rangle + \frac{1}{\sqrt{6}}\langle 10_x | 1,-1\rangle\right|^2$$

only $l+m$ matter

$$= \frac{1}{2} + \left|\frac{1}{\sqrt{3}}\cdot 0 + \frac{1}{\sqrt{6}}\cdot\left(-\frac{1}{\sqrt{2}}\right)\right|^2 = \frac{1}{2} + \frac{1}{12}$$

$$\boxed{P(m_x=0) = \frac{7}{12}}$$

Probs $\rightarrow$ $\frac{3}{8} + \frac{1}{24} + \frac{7}{12} = 1$ ✓