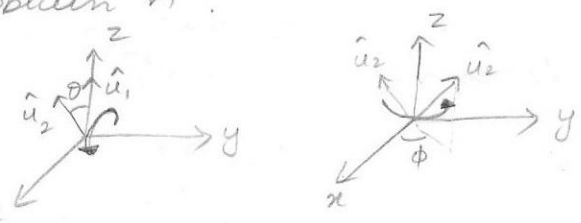
Shankar 14.3.6.

(1) $\hat{n}$ points in the direction (θ, ϕ).

Consider a unit vector $\hat{u}_1$ along z-axis. Rotate it by θ about y-axis to obtain $\hat{u}_2$. Rotate $\hat{u}_2$ by ϕ about z-axis to obtain $\hat{n}$.



Similar reasoning can be applied to $|S_z = \hbar/2\rangle$ and $|\hat{n}, +\rangle$ so: $|\hat{n}, +\rangle = U[R(\phi\hat{k})]\, U[R(\theta\hat{j})]\, |S_z = \hbar/2\rangle$

(2) Using:

$$U[R(\theta)] = \cos(\theta/2)\, I - i \sin(\theta/2)\, \hat{\theta} \cdot \vec{\sigma}$$

So:

$$U[R(\theta\hat{j})] = \cos(\theta/2)\, I - i \sin(\theta/2)\, \sigma_y$$

$$= \cos\left(\frac{\theta}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i \sin\left(\frac{\theta}{2}\right) \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$U[R(\phi\hat{k})] = \cos(\phi/2)\, I - i \sin(\phi/2)\, \sigma_z$$

$$= \cos\left(\frac{\phi}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i \sin\left(\frac{\phi}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

Now:

$$U[R(\theta\hat{j})]\, |S_z = \hbar/2\rangle = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

$$U[R(\phi\hat{k})]\, U[R(\theta\hat{j})]\, |S_z = \hbar/2\rangle$$

$$= \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{bmatrix} = |\hat{n}, +\rangle$$