Physics 827: Homework Set No. 1

Due date: Thursday, September 29, 2011, 3:00pm in the grader's office (see course web site)

Total point value of set: 55 points

Problem 1 (5 pts.): Exercise 1.1.2 (Shankar p.3)

Problem 2 (5 pts.): Exercise 1.1.3 (Shankar p.4)

Problem 3 (10 pts.): Orthonormalize the vectors $|I\rangle \leftrightarrow (0,3,4)$, $|II\rangle \leftrightarrow (2,0,0)$, $|III\rangle \leftrightarrow (0,-1,3)$, using the Gram-Schmidt procedure and starting with $|I\rangle$.

Problem 4 (5 pts.): Exercise 1.3.4 (Shankar p.17, proof of triangle inequality)

Problem 5 (15 pts.): Exercise 1.4.1 (Shankar p.18)

Problem 6 (10 pts.): Exercise 1.4.2 (Shankar p.18)

Problem 7 (5 pts.): Exercise 1.6.1 (Shankar p.22)

HW 1

Physics 827 (QM1)

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Exercise 1.1.2 (p. 3)

The rule given for addition and scalar multiplication is that for the cartesian components of 3-vectors in $\mathbb{V}^3(\mathbb{R})$.

- The null vector is $|0\rangle = (0,0,0)$; it obviously satisfies $|V\rangle + |0\rangle = (v_1, v_2, v_3) + (0,0,0) = (v_1+0, v_2+0, v_3+0) = (v_1, v_2, v_3) = |V\rangle$
- The inverse of $|V\rangle = (a,b,c)$ is $|-V\rangle = (-a,-b,-c) = -(a,b,c) = -|V\rangle$
- The vectors of the form $(a,b,1)$ do not include the null vector (i.e. $\alpha(a,b,1) \notin \mathbb{V}$ for $\alpha = 0$), so $\mathbb{V} = \{(a,b,1) \mid a,b \in \mathbb{R}\}$ is not a vector space.

Exercise 1.1.3 (p. 4)

- $\mathbb{V}_1 = \{f(x) \mid 0 \le x \le L, f(0) = f(L) = 0\}$

  If I add two such functions or multiply such a function with a constant, the result will satisfy the boundary condition ✓

  The null vector $f(x) = 0 \ \forall x$ satisfies the boundary condition ✓

  The inverse $-f(x)$ satisfies the boundary condition ✓

  Distributive, commutative and associative properties are satisfied independent of boundary conditions ✓

  $\Rightarrow \mathbb{V}_1$ is a vector space.

- $\mathbb{V}_2 = \{f(x) \mid 0 \le x \le L, f(0) = f(L)\}$ (periodic functions)

  Adding two periodic functions or multiplying them by a constant results again in periodic functions ✓

  The null function is periodic ✓

- The inverse $-f(x)$ of a periodic function is periodic ✓
- Distribution, commutation and associative properties hold independent of boundary conditions

$\Rightarrow \mathbb{V}_2$ is a vector space

- $\mathbb{V}_3 = \{ f(x) \mid 0 \le x \le L,\ f(0) = 4 \}$

This is not a vector space since $g(x) = \alpha f(x)$, $\alpha \neq 1$, does not satisfy $g(0) = 4$.

- Exercise 1.3.2 (modified) (p. 16)

Orthonormalize the 3 vectors $|\mathrm{I}\rangle = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $|\mathrm{II}\rangle = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $|\mathrm{III}\rangle = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

$$|1\rangle = \frac{|\mathrm{I}\rangle}{\sqrt{\langle \mathrm{I}|\mathrm{I}\rangle}} = \frac{1}{\sqrt{4+0+0}} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|2'\rangle = |\mathrm{II}\rangle - |1\rangle \underbrace{\langle 1|\mathrm{II}\rangle}_{0} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$ (since $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ are already orthogonal)

$$|2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{1}{\sqrt{0+9+16}} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$$

$$|3'\rangle = |\mathrm{III}\rangle - |1\rangle \underbrace{\langle 1|\mathrm{III}\rangle}_{0} - |2\rangle\langle 2|\mathrm{III}\rangle = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix} \left(0 - \frac{3}{5} + \frac{12}{5}\right)$$

$$= \begin{bmatrix} 0 \\ -1 - \frac{27}{25} \\ 3 - \frac{36}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{52}{25} \\ +\frac{39}{25} \end{bmatrix}$$

$$|3\rangle = \frac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}} = \begin{bmatrix} 0 \\ -\frac{52}{25} \\ +\frac{39}{25} \end{bmatrix} \frac{1}{\frac{1}{25}\sqrt{0 + 4^2 \cdot 13^2 + 3^2 \cdot 13^2}} = \begin{bmatrix} 0 \\ -52 \\ +39 \end{bmatrix} \frac{1}{13 \cdot 5} = \begin{bmatrix} 0 \\ -4/5 \\ 3/5 \end{bmatrix}$$

• Exercise 1.3.4 (p. 17)

Prove $|V+W| \leq |V| + |W|$.

Proof: Since both sides are real and positive, it's sufficient to prove $|V+W|^2 \leq (|V| + |W|)^2$

or $|V|^2 + 2|V||W| + |W|^2 - \langle V+W | V+W \rangle \geq 0$

Work on l.h.s.:

$$|V|^2 + 2|V||W| + |W|^2 - \underbrace{\langle V|V\rangle}_{|V|^2} - \langle V|W\rangle - \underbrace{\langle W|V\rangle}_{=\langle V|W\rangle^*} - \underbrace{\langle W|W\rangle}_{|W|^2}$$

$$= 2\left(|V||W| - \underbrace{\mathrm{Re}\langle V|W\rangle}_{\leq |\langle V|W\rangle|}\right)$$

Since $|\langle V|W\rangle|^2 = (\mathrm{Re}\langle V|W\rangle)^2 + (\mathrm{Im}\langle V|W\rangle)^2$

$$\geq (\mathrm{Re}\langle V|W\rangle)^2$$

$$\geq 2\left(|V||W| - |\langle V|W\rangle|\right) \geq 0 \quad \diamond$$

↑ Schwarz inequality

• Exercise 1.4.1 (p. 18)

In $\mathbb{V}^n$, prove that $\mathbb{V}_\perp^{n-1} = \{|W\rangle \mid \langle V|W\rangle = 0, |V\rangle \neq 0\}$ is a subspace of dimension $n-1$.

Proof: Since $\mathbb{V}^n$ has dimension n, we can find n linearly independent vectors in $\mathbb{V}^n$, $\{|W_i\rangle \mid i=1,\ldots,n\}$. Using the Gram-Schmidt algorithm, we can orthonormalize these vectors to obtain an orthonormal basis

$\{|i\rangle \,|\, i=1,\dots,n\}$. Now choose an arbitrary vector $|V\rangle \neq |0\rangle$ in $\mathbb{V}^n$. We can decompose it in this basis as

$$|V\rangle = \sum_{i=1}^{n} v_i |i\rangle, \qquad (*)$$

where at least one of the components v_i is non-zero. Let us assume without loss of generality that $v_1 \neq 0$. (We can always achieve that by renumbering the basis states accordingly.). Eq. (*) can then be rewritten as

$$|V\rangle = v_1|1\rangle + \sum_{i=2}^{n} v_i|i\rangle \;\Rightarrow\; |1\rangle = \sum_{i=2}^{n} \frac{v_i}{v_1}|i\rangle - \frac{1}{v_1}|V\rangle \qquad (**)$$

Since the basis was orthonormal, $|1\rangle$ is orthogonal to all states $|i\rangle, i=2,\dots,n$. For (**) to hold, $|V\rangle$ must therefore have a part that is orthogonal to all states $|i\rangle, i=2,\dots,n$. Hence $|V\rangle$ is linearly independent of $\{|i\rangle \,|\, i=2,\dots,n\}$, since the part of $|V\rangle$ that is orthogonal to these vectors cannot be obtained from any linear superposition of these vectors.

Let us now look at this new set of n linearly independent vectors given by $\{|V\rangle, |2\rangle, \dots, |n\rangle\}$. Using Gram-Schmidt again, starting from $|V\rangle$, we obtain a new orthonormal basis of $\mathbb{V}^n$,

$\{|1'\rangle, |2'\rangle, \dots, |n'\rangle\}$ whose first unit vector $|1'\rangle = \dfrac{|V\rangle}{\sqrt{\langle V|V\rangle}}$ is in the direction of $|V\rangle$.

We can now decompose any vector $|W\rangle \in \mathbb{V}^n$ as

$$|W\rangle = \sum_{i=1}^{n} \omega_n |i'\rangle .$$

Any element of the set $\mathbb{V}_\perp = \{|V_\perp\rangle \in \mathbb{V}^n, \langle V_\perp|V\rangle = 0\}$ of vectors $|V_\perp\rangle$ that are orthogonal to $|V\rangle$ are orthogonal to $|1'\rangle$ and thus can be written as

$$|V_\perp\rangle = \sum_{i=2}^{n} v_i |i'\rangle$$

with no component along $|1'\rangle$. The set $\{|i'\rangle | i=2,\dots,n\}$ forms an $(n-1)$-dimensional orthonormal basis, so the set $\mathbb{V}_\perp \equiv \mathbb{V}_\perp^{n-1}$ has dimension $n-1$. (We need to show that this subset of $\mathbb{V}^n$ is a vector space:) It contains the null vector $|0\rangle$ since $\langle 0|V\rangle = 0$ for any $|V\rangle \in \mathbb{V}^n$. For any vector $|V_\perp\rangle \in \mathbb{V}_\perp^{n-1}$, its inverse $|-V_\perp\rangle$ is also in $\mathbb{V}_\perp^{n-1}$ since $\langle -V_\perp|V\rangle = -\langle V_\perp|V\rangle = 0$, so $|-V_\perp\rangle$ is orthogonal to $|V\rangle$. Adding two vectors that are orthogonal to $|V\rangle$ or multiplying them by a constant again produces vectors that are orthogonal to $|V\rangle$, i.e. elements of $\mathbb{V}_\perp^{n-1}$.

Hence $\mathbb{V}_\perp^{n-1}$ satisfies all axioms that define a vector space; it is an $(n-1)$-dimensional subspace of $\mathbb{V}^n$.

Exercise 1.4.2 (p. 18)

- The elements of $V_1^{n_1}$ can be written as

$$|V\rangle = \sum_{i=1}^{n_1} v_i |i\rangle$$

where $\{|i\rangle, i=1,\dots,n_1\}$ is an orthonormal basis of $V_1^{n_1}$.

- The elements of $V_2^{n_2}$ can be written as

$$|W\rangle = \sum_{i=1}^{n_2} w_i |i'\rangle$$

where $\{|i'\rangle, i=1,\dots,n_2\}$ is an orthonormal basis of $V_2^{n_2}$.

Since all elements of $V_1^{n_1}$ are orthogonal to all elements of $V_2^{n_2}$, all states $|i\rangle$ $(i=1,\dots,n_1)$ are orthogonal to all states $|i'\rangle$ $(i=1,\dots,n_2)$.

Hence the set $\{|i\rangle, |j'\rangle, i=1,\dots,n_1; j=1,\dots,n_2\}$ is an orthonormal basis for an (n_1+n_2)-dimensional vector space.

By definition, the elements of $V_1^{n_1} \oplus V_2^{n_2}$ are all linear combinations

$$|Z\rangle = a|V\rangle + b|W\rangle \qquad |V\rangle \in V_1^{n_1}, \; |W\rangle \in V_2^{n_2},$$

a, b constants.

These can be written as

$$|Z\rangle = a\sum_{i=1}^{n} v_i |i\rangle + b\sum_{j=1}^{n_2} w_2 |j'\rangle = \sum_{i=1}^{n_1} (a v_i)|i\rangle + \sum_{j=1}^{n_2} (b w_j)|j'\rangle$$

i.e. they can be decomposed in the orthonormal

basis set $\{|i\rangle, |j'\rangle \mid i=1,\dots n_1; j=1,\dots,n_2\}$ which is n_1+n_2 dimensional. Hence the dimensionality of $V_1^{n_1} \oplus V_2^{n_2}$ is not larger than n_1+n_2.

Since the orthogonal basis elements themselves are also elements of $V_1^{n_1} \oplus V_2^{n_2}$, its dimensionality is at least n_1+n_2

Hence $V_1^{n_1} \oplus V_2^{n_2}$ has dimensionality equal to n_1+n_2.

Exercise 1.6.1 (p. 22)

From the columns of the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [\Omega_{ij}]$ we read off the action of $\hat{\Omega}$ on the basis states:

$\hat{\Omega}|1\rangle = |2\rangle$, $\hat{\Omega}|2\rangle = |3\rangle$, $\hat{\Omega}|3\rangle = |1\rangle$.

So $\hat{\Omega}$ cyclically permutes the states $|1\rangle, |2\rangle, |3\rangle$.

This corresponds to a rotation by 120° (⅓ of 360°) around the axis $\frac{|1\rangle + |2\rangle + |3\rangle}{\sqrt{3}}$. ($\frac{1}{\sqrt{3}}$ normalizes the unit vector pointing along this axis.).