Physics 827: Homework Set No. 2

Due date: Thursday, October 6, 2011, 3:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 60 points

Problem 1 (5 pts.): Exercise 1.8.3 (Shankar p.41)

Problem 2 (5 pts.): Exercise 1.8.6 (Shankar p.42)

Problem 3 (5 pts.): Exercise 1.8.8 (Shankar p.43) [Note: in part (2) it should read $M^iM^j = -M^jM^i$]

Problem 4 (5 pts.): Exercise 1.9.2 (Shankar p.55)

Problem 5 (5 pts.): Exercise 1.9.3 (Shankar p.55)

Problem 6 (5 pts.): Exercise 1.10.2 (Shankar p.63)

Problem 7 (5 pts.): Exercise 1.10.3 (Shankar p.63)

Problem 8 (15 pts.): For $\hat{H}$ Hermitean, $\hat{U}, \hat{V}$ unitary, show:

(i) $\hat{U}\hat{H}\hat{U}^{-1}$ is Hermitean

(ii) $(\hat{U})^n$ is unitary

(iii) $\hat{U}\hat{V}$ is unitary

(iv) If $f(x)$ is real and has no poles, $f(\hat{H})$ is Hermitean

(v) $\mathrm{i}\dfrac{\hat{U}-\hat{I}}{\hat{U}+\hat{I}}$ is Hermitean

(vi) $\dfrac{\hat{I}-\mathrm{i}\hat{H}}{\hat{I}+\mathrm{i}\hat{H}}$ is unitary

Problem 9 (10 pts.): Show that $e^{\hat{L}}\hat{M}e^{-\hat{L}} = \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{L},\hat{M}]_{(n)}$ where $[\hat{L},\hat{M}]_{(0)} = \hat{M}$, $[\hat{L},\hat{M}]_{(n)} = [\hat{L},[\hat{L},\hat{M}]_{(n-1)}]$.
(We will need this later for $\hat{L} = i\hat{H}t$ where $\hat{H}$ is the Hamiltonian.)
Hint: Consider $\hat{F}(\alpha) = e^{\alpha\hat{L}}\hat{M}e^{-\alpha\hat{L}}$ and expand it into a Taylor series around the origin.

Exercise 1.8.3 (p. 41)

(1) Characteristic polynomial: $\det(\hat{\Omega} - \omega \hat{I}) = 0$

$$\begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & 3/2-\omega & -1/2 \\ 0 & -1/2 & 3/2-\omega \end{vmatrix} = (1-\omega)\left[\left(\tfrac{3}{2}-\omega\right)^2 - \tfrac{1}{4}\right] = (1-\omega)\left(2 - 3\omega + \omega^2\right)$$

$$= (1-\omega)\big((2-\omega)(1-\omega)\big) = 0 \quad \Rightarrow \quad \boxed{\omega_1 = \omega_2 = 1,\ \omega_3 = 2.}\ \checkmark$$

(2) Eigenvectors: represent $|\omega = 2\rangle$ as column vector $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 2\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \begin{matrix} a_1 = 2a_1 \Rightarrow a_1 = 0 \\ 3a_2 - a_3 = 4a_2 \Rightarrow -a_3 = a_2 \\ -a_2 + 3a_3 = 4a_3 \Rightarrow -a_2 = a_3 \end{matrix} \Big\} \text{redundant.}$$

$\Rightarrow |\omega = 2\rangle \leftrightarrow \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$ $(a \in \mathbb{R})$; The normalized eigenvector is $|\omega = 2\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ ✓ but any vector of the form $\begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$ is an eigenvector, too, with the same eigenvalue.

(This differs from the general form given in the problem statement which is, in fact, not the general solution, but the normalized one.)

(3) $|\omega = 1\rangle$ eigenvectors:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \begin{matrix} a_1 = a_1 \\ 3a_2 - a_3 = 2a_2 \Leftrightarrow a_2 = a_3 \\ -a_2 + 3a_3 = 2a_3 \Leftrightarrow a_3 = a_2 \end{matrix} \Big\} \text{redundant}$$

$\Rightarrow |\omega = 1\rangle \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix}$ general eigenvector;

normalized eigenvector $|\omega = 1\rangle \leftrightarrow \begin{pmatrix} b \\ c \\ c \end{pmatrix} \cdot \frac{1}{\sqrt{b^2 + 2c^2}}$

$(a, b \in \mathbb{R}$ arbitrary) → This is a 2-d subspace.

(4) Find two orthonormal eigenvectors in the subspace corresponding to $\omega = 1$:

Can start by choosing $\boxed{|\omega=1,1\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ $(b=1, c=0)$

Orthonormalize $|\omega=1,2\rangle$ on their state with Gram-Schmidt procedure:

$$|\omega=1,2\rangle' = \begin{pmatrix} b \\ c \\ c \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1,0,0)\begin{pmatrix} b \\ c \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ c \end{pmatrix}$$

Normalized solution: $\boxed{|\omega=1,2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$

(Many other solutions are possible)

Exercise 1.8.6 (p.42)

(1) Show that for Hermitean $\hat{\Omega}$, $\det \hat{\Omega} = \prod_{i=1}^{n} \omega_i$ where $(\omega_i, i=1,\dots,n)$ are the eigenvalues of $\hat{\Omega}$

Answer: since $\hat{\Omega}$ is Hermitean, a unitary matrix $\hat{u}$ can be constructed such that $\hat{\Omega}' = \hat{u}\hat{\Omega}\hat{u}^{-1} = \begin{pmatrix} \omega_1 & & \\ & \omega_2 & \\ & \cdots & \\ & & \omega_n \end{pmatrix}$ is diagonal, with eigenvalues on the diagonal.

$$\Rightarrow \det \hat{\Omega} = \det \hat{\Omega} \cdot \det(\hat{u}^{-1}\hat{u}) = \det(\hat{\Omega}\hat{u}^{-1}\hat{u})$$

$$= \det(\hat{u}\hat{\Omega}\hat{u}^{-1}) = \det(\Omega') = \prod_{i=1}^{n} \omega_i \quad \checkmark$$

(2) $$\operatorname{tr}\hat{\Omega} = \operatorname{tr}(\hat{\Omega}\hat{u}^{-1}\hat{u}) = \operatorname{tr}(\hat{u}\hat{\Omega}\hat{u}^{-1}) = \operatorname{tr}(\Omega') = \sum_{i=1}^{n} \omega_i$$

$$\operatorname{tr}(\hat{\Omega}\hat{u}^{-1}\hat{u}) = \sum_{i,j,k} \Omega_{ij} (u^{-1})_{jk} u_{ki} = \sum_{kji} u_{ki} \Omega_{ij} (u^{-1})_{jk} = \operatorname{tr}(\hat{u}\hat{\Omega}\hat{u}^{-1})$$

Exercise 1.8.8 (p. 43)

$\hat{M}^i \hat{M}^j + \hat{M}^j \hat{M}^i = 2\delta^{ij} \hat{I}$ $(i,j = 1,\dots,4)$, $\hat{M}^i$ Hermitean

(1) For $i = j$, this gives

$$2(\hat{M}^i)^2 = 2\hat{I} \Rightarrow (\hat{M}^i)^2 = \hat{I} \quad (i = 1,\dots,4)$$

$\Rightarrow$ each $\hat{M}^i$ is its own inverse!

In the eigenbasis of $\hat{M}^i$ (a different basis for each i!) this becomes

$$\begin{pmatrix} (m_1^i)^2 & & & \\ & (m_2^i)^2 & & \\ & & \ddots & \\ & & & (m_n^i)^2 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \Rightarrow (m_j^i)^2 = 1, \begin{pmatrix} i = 1,\dots,4 \\ j = 1,\dots,n \end{pmatrix}$$

Since the $\hat{M}^i$ are Hermitean, all eigenvalues m_j^i are real $\Rightarrow$ $\boxed{m_j^i = \pm 1 \text{ for } i = 1,\dots,4;\ j = 1,\dots,n}$ ✓

(2) For $i \neq j$ we have $\hat{M}^i \hat{M}^j + \hat{M}^j \hat{M}^i = 0$

Multiply this with $\hat{M}^i$ or $\hat{M}^j$, from left or right; for example,

$$(\hat{M}^i)^2 \hat{M}^j + \hat{M}^i \hat{M}^j \hat{M}^i = 0$$

Take trace and use $(\hat{M}^i)^2 = \hat{I}$ from (1) above:

$$\mathrm{tr}((\hat{M}^i)^2 \hat{M}^j) + \mathrm{tr}(\hat{M}^i \hat{M}^j \hat{M}^i) = \mathrm{tr}(\hat{M}^j) + \mathrm{tr}((\hat{M}^i)^2 \hat{M}^j)$$

(proved in 1.8.6(2))

$$= 2 + \mathrm{tr}(\hat{M}^j) = 0, \quad j = 1,\dots,4$$

$$\Rightarrow \boxed{\mathrm{tr}(\hat{M}^i) = 0, \quad i = 1,\dots,4}$$ ✓

(3) From 1.8.6 (2) we know that

$$0 = \mathrm{tr}(\hat{M}^i) = \sum_{j=1}^{n} m_j^i = \sum_{j=1}^{n} (\pm 1)$$

This can only be achieved if the sum has an equal number of +1 and -1 eigenvalues, which requires an even number of terms $\Rightarrow$ $\boxed{n = \text{even}}$

The matrices must be even-dimensional

Exercise 1.9.2 (p.55)

If $\hat{H} = \hat{H}^+$, $\hat{U}^+ = \left(e^{i\hat{H}}\right)^+ = \sum_{n=0}^{\infty} \frac{\left[(i\hat{H})^n\right]^+}{n!} = \sum_{n=0}^{\infty} \frac{(-i\hat{H}^+)^n}{n!} \overset{(\hat{H}=\hat{H}^+)}{=} \sum_{n=0}^{\infty} \frac{(-i\hat{H})^n}{n!}$

$= e^{-i\hat{H}} = \hat{U}^{-1}$ since $\hat{U}^+\hat{U} = e^{i\hat{H}} e^{-i\hat{H}} = e^{i\hat{H}-i\hat{H}} = \hat{I}$

since $[\hat{H},\hat{H}]=0$

$\Rightarrow$ $\hat{U} = e^{i\hat{H}}$ is unitary if $\hat{H}$ is Hermitean ✓

Exercise 1.9.3 (p.55)

Since $\hat{H}$ is Hermitean, it can be diagonalised by a unitary transformation to its eigenvalue basis:

$$\hat{H}' = \hat{V}\hat{H}\hat{V}^{-1} = \begin{pmatrix} E_1 & & & \\ & E_2 & & \\ & & \ddots & \\ & & & E_n \end{pmatrix} \qquad (\hat{V} \text{ unitary})$$

Under this transformation

$$\hat{U} \mapsto \hat{U}' = e^{i\hat{H}'} = e^{i\hat{V}\hat{H}\hat{V}^{-1}} = \sum_{n=0}^{\infty} \frac{(i\hat{V}\hat{H}\hat{V}^{-1})^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\overbrace{i\hat{V}\hat{H}\hat{V}^{-1} i\hat{V}\hat{H}\hat{V}^{-1} \dots i\hat{V}\hat{H}\hat{V}^{-1}}^{n\text{-times}}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\hat{V}\overbrace{(i\hat{H}\, i\hat{H} \dots i\hat{H})}^{n\text{times}}\hat{V}^{-1}}{n!} \qquad (\text{using } \hat{V}^{-1}\hat{V} = \hat{I})$$

$$= \hat{V}\left(\sum_{n=0}^{\infty} \frac{(i\hat{H})^n}{n!}\right)\hat{V}^{-1} = \hat{V}e^{i\hat{H}}\hat{V}^{-1} = \hat{V}\hat{U}\hat{V}^{-1}$$

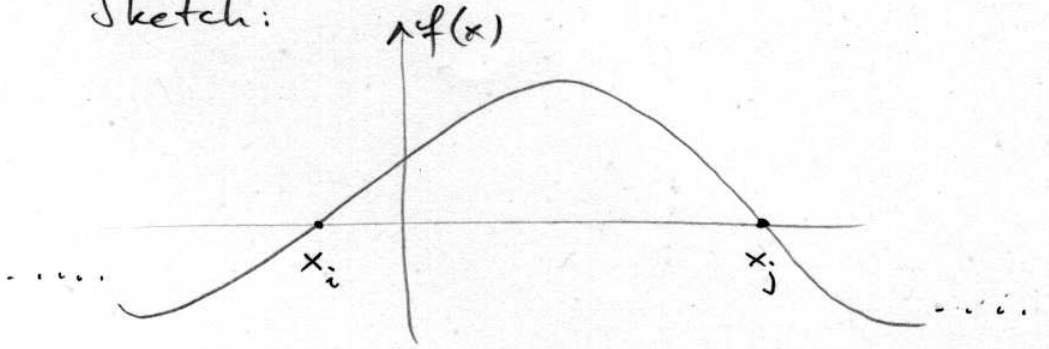
Hence $\det \hat{U} = \det(\hat{V}^{-1}\hat{U}'\hat{V}) = \det(\hat{V}\hat{V}^{-1}\hat{U}') = \det \hat{U}'$

$= \det(e^{i\hat{H}'}) = \det \begin{pmatrix} e^{iE_1} & & & \\ & e^{iE_2} & & \\ & & \ddots & \\ & & & e^{iE_n} \end{pmatrix} = e^{i\sum_{j=1}^{n} E_j} = e^{i\,\mathrm{tr}\hat{H}'} = e^{i\,\mathrm{tr}(\hat{V}\hat{H}\hat{V}^{-1})} = e^{i\,\mathrm{tr}\hat{H}}$ ✓

Exercise 1.10.2 (p.63)

Show $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\right|_{x=x_i}}$ where x_i are the zeros of $f(x)$.

Sketch:



x_i, x_j are two (of possibly many more) zeros of $f(x)$

Near the zeros, $f(x)$ can be approximated by a linear function: $f(x) \approx \left.\frac{df}{dx}\right|_{x_i} (x-x_i)$ (*)

$\left.\frac{df}{dx}\right|_{x=x_i}$ is a (positive or negative) real number.

At every place $f(x)=0$, $\delta(f(x))$ has a singularity. Everywhere else it is zero. So near each zero we can use the linear approximation (*); it becomes inaccurate as we move away from the zero x_i, but that doesn't matter because $\delta(f(x))$ is there zero, irrespective of whether we use the exact form for $f(x)$ or the linear approximation (*). So we can write

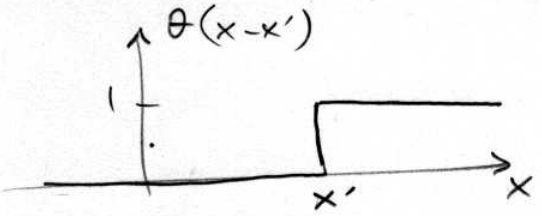
$$\delta(f(x)) = \sum_{\substack{i \\ f(x_i)=0}} \delta\left(\left.\frac{df}{dx}\right|_{x_i} (x-x_i)\right)$$

Using $\delta(ax) = \frac{1}{|a|}\delta(x)$ we arrive at the desired result:

$$\boxed{\delta(f(x)) = \sum_{\substack{i \\ f(x_i)=0}} \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\right|_{x_i}}} \checkmark$$

Exercise 1.10.3 (p. 63)

Show $\delta(x-x') = \frac{d}{dx}\theta(x-x')$



Obviously $\frac{d}{dx}\theta(x-x') = 0$ for $x > x'$ and $x < x'$, i.e. for $x \neq x'$

Also obviously $\theta(x-x')$ has infinite slope at $x = x'$

$\Rightarrow \frac{d}{dx}\theta(x-x') = A\,\delta(x-x')$ with unknown normalization A.

Check

$$\int_{-\infty}^{\infty} dx \frac{d}{dx}\theta(x-x') = \theta(x-x')\Big|_{x=-\infty}^{x=\infty} = \theta(\infty) - \theta(-\infty) = 1 \quad \text{(indep. of } x'\text{)}$$

$$= \int_{-\infty}^{\infty} dx\, A\,\delta(x-x') = A \qquad \Rightarrow A = 1 \checkmark$$

Problem 8 ($\hat{H}$ Hermitean ($\hat{H}^+=\hat{H}$); $\hat{U},\hat{V}$ unitary ($\hat{U}^+=\hat{U}^{-1}$, $\hat{V}^+=\hat{V}^{-1}$))

(i) $(\hat{U}\hat{H}\hat{U}^{-1})^+ = (\hat{U}^{-1})^+\hat{H}^+\hat{U}^+ = \hat{U}\hat{H}\hat{U}^{-1}$ Hermitean. ◊

(ii) $(\hat{U}^n)^+\cdot\hat{U}^n = (\underbrace{\hat{U}\cdot\hat{U}\cdots\hat{U}}_{n})^+(\underbrace{\hat{U}\cdot\hat{U}\cdots\hat{U}}_{n})$

$= (\underbrace{\hat{U}^+\hat{U}^+\cdots\hat{U}^+}_{n})(\underbrace{\hat{U}\cdot\hat{U}\cdots\hat{U}}_{n}) = \hat{I}$

Similarly $\hat{U}^n(\hat{U}^n)^+=\hat{I}$ $\Rightarrow$ $\hat{U}^n$ is unitary. ◊

(iii) $(\hat{U}\hat{V})^+ = \hat{V}^+\hat{U}^+ = \hat{V}^{-1}\hat{U}^{-1} = (\hat{U}\hat{V})^{-1}$ unitary. ◊

(iv) Since $f(x)$ has no singularities, it can be Taylor expanded around any point $x_0\in\mathbb{R}$ with a finite (non-zero) convergence radius:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf}{dx^n}\Big|_{x_0}(x-x_0)^n$$

$$\Rightarrow f(\hat{H}) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf}{dx^n}\Big|_{x_0}(\hat{H}-x_0\hat{I})^n$$

$$\Rightarrow [f(\hat{H})]^+ = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d^nf}{dx^n}\right)^*\Big|_{x_0}\left[(\hat{H}-x_0\hat{I})^n\right]^+ = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf}{dx^n}\Big|_{x_0}(\underbrace{\hat{H}^+}_{=\hat{H}}-x_0\hat{I})^n = f(\hat{H}^+)$$

(f is real)

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf}{dx^n}\Big|_{x_0}(\hat{H}-x_0\hat{I})^n = f(\hat{H})$$

So if $\hat{H}=\hat{H}^+$, $f(\hat{H}) = f(\hat{H}^+) = [f(H)]^+$ ◊

The proof works for any x_0, so for any eigenstate of $\hat{H}$ I can expand around the corresponding eigenvalue, proving Hermiticity on that eigenstate. If $\hat{H}$ is Hermitean on all eigenstates, it is Hermitean in the entire Hilbert space.

(v) $i\dfrac{\hat{U}-\hat{I}}{\hat{U}+\hat{I}} = i(\hat{U}-\hat{I})(\hat{U}+\hat{I})^{-1} = i(\hat{U}+\hat{I})^{-1}(\hat{U}-\hat{I})$

since $(\hat{U}+\hat{I})^{-1}$ commutes with $\hat{U}-\hat{I}$:

$$(\hat{U}+\hat{I})(\hat{U}-\hat{I})(\hat{U}+\hat{I})^{-1} = (\hat{U}+\hat{I})\hat{U}(\hat{U}+\hat{I})^{-1} - (\hat{U}+\hat{I})\hat{I}(\hat{U}+\hat{I})^{-1}$$
$$= \hat{U}(\hat{U}+\hat{I})(\hat{U}+\hat{I})^{-1} - \hat{I} = \hat{U}-\hat{I}$$

Hence $\left(i\dfrac{\hat{U}-\hat{I}}{\hat{U}+\hat{I}}\right)^{+} = \left(i(\hat{U}-\hat{I})(\hat{U}+\hat{I})^{-1}\right)^{+} = -i\left[(\hat{U}+\hat{I})^{-1}\right]^{+}(\hat{U}^{+}-\hat{I})$

Now $(\hat{U}+\hat{I})^{-1}$ exists as long as none of $\hat{U}$'s eigenvalues equals -1. Then we can use (iv) and write

$$\left[(\hat{U}+\hat{I})^{-1}\right]^{+} = \left[f(\hat{U})\right]^{+} = f(\hat{U}^{+}) = f(\hat{U}^{-1}) = (\hat{U}^{-1}-\hat{I})^{-1}$$

$$\Rightarrow \underbrace{\left(i\frac{\hat{U}-\hat{I}}{\hat{U}+\hat{I}}\right)^{+}} = -i(\hat{U}^{-1}+\hat{I})^{-1}(\hat{U}^{-1}-\hat{I}) = -i(\hat{U}^{-1}+\hat{I})^{-1}(\hat{I}-\hat{U})\hat{U}^{-1}$$
$$= i(\hat{U}^{-1}+\hat{I})^{-1}(\hat{U}-\hat{I})\hat{U}^{-1} = i(\hat{U}-\hat{I})(\hat{U}^{-1}+\hat{I})^{-1}\hat{U}^{-1}$$
$$= i(\hat{U}-\hat{I})\left(\hat{U}(\hat{U}^{-1}+\hat{I})\right)^{-1} = i(\hat{U}-\hat{I})(\hat{I}+\hat{U})^{-1}$$
$$= \underbrace{i\frac{\hat{U}-\hat{I}}{\hat{U}+\hat{I}}} \quad \text{Hermitian}$$

(vi) The inverse of $\hat{I}+i\hat{H}$ exists since all eigenvalues of $\hat{H}$ are real. Like in (v) you show that $(\hat{I}+i\hat{H})^{-1}$ commutes with $\hat{I}-i\hat{H}$.

$$\underbrace{\left(\frac{\hat{I}-i\hat{H}}{\hat{I}+i\hat{H}}\right)^{+}} = \left[(\hat{I}-i\hat{H})(\hat{I}+i\hat{H})^{-1}\right]^{+} = \underbrace{\left[(\hat{I}+i\hat{H})^{-1}\right]^{+}}_{f(i\hat{H})}(\hat{I}-i\hat{H})^{+}$$
$$= \underbrace{(\hat{I}-i\hat{H})^{-1}}_{f((iH)^{+}) = f(-i\hat{H})}(\hat{I}+i\hat{H}) = \left[(\hat{I}+i\hat{H})^{-1}(\hat{I}-i\hat{H})\right]^{-1}$$
$$= \underbrace{\left(\frac{\hat{I}-i\hat{H}}{\hat{I}+i\hat{H}}\right)^{-1}} \quad \text{unitary}$$

Problem 9

Prove $e^{\hat{L}} \hat{M} e^{-\hat{L}} = \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{L}, \hat{M}]_{(n)}$ where $[\hat{L}, \hat{M}]_{(0)} = \hat{M}$,

$[\hat{L}, \hat{M}]_{(n)} = [\hat{L}, [\hat{L}, \hat{M}]_{(n-1)}]$.

Answer: Consider $\hat{F}(\alpha) = e^{\alpha \hat{L}} \hat{M} e^{-\alpha \hat{L}}$ $(\alpha \in \mathbb{R})$

Expand $\hat{F}(\alpha)$ into a Taylor series around $\alpha = 0$:

$$\hat{F}(\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n \hat{F}}{d\alpha^n} \right)_{\alpha=0} \alpha^n \qquad \left(\hat{F}(0) = \hat{M} = [\hat{L}, \hat{M}]_{(0)} \right)$$

Now $\frac{d}{d\alpha} \left(e^{\alpha \hat{L}} \right) = e^{\alpha \hat{L}} \hat{L}$ and $\frac{d}{d\alpha} e^{-\alpha \hat{L}} = -\alpha \hat{L} e^{-\alpha \hat{L}}$

Hence

$$\frac{d}{d\alpha} \hat{F}(\alpha) = \frac{d}{d\alpha} \left(e^{\alpha \hat{L}} \hat{M} e^{-\alpha \hat{L}} \right) = \left(e^{\alpha \hat{L}} \hat{L} \right) \hat{M} e^{-\alpha \hat{L}} + e^{\alpha \hat{L}} \hat{M} \left(-\hat{L} e^{-\alpha \hat{L}} \right)$$

$$= e^{\alpha \hat{L}} [\hat{L}, \hat{M}] e^{-\alpha \hat{L}} \xrightarrow[\alpha=0]{} [\hat{L}, \hat{M}] = [\hat{L}, [\hat{L}, \hat{M}]_{(0)}] = [\hat{L}, \hat{M}]_{(1)}$$

$$\frac{d^2}{d\alpha^2} \hat{F}(\alpha) = \frac{d}{d\alpha} \left(e^{\alpha \hat{L}} [\hat{L}, \hat{M}]_{(1)} e^{-\alpha \hat{L}} \right) = \left(e^{\alpha \hat{L}} \hat{L} \right) [\hat{L}, \hat{M}]_{(1)} e^{-\alpha \hat{L}}$$

$$- e^{\alpha \hat{L}} [\hat{L}, \hat{M}]_{(1)} \hat{L} e^{-\alpha \hat{L}}$$

$$= e^{\alpha \hat{L}} [\hat{L}, [\hat{L}, \hat{M}]_{(1)}] e^{-\alpha \hat{L}} = e^{\alpha \hat{L}} [\hat{L}, \hat{M}]_{(2)} e^{-\alpha \hat{L}} \xrightarrow[\alpha=0]{} [\hat{L}, \hat{M}]_{(2)}$$

etc. for higher derivatives

$$\Rightarrow \left. \frac{d^n F}{d\alpha^n} \right|_{\alpha=0} = [\hat{L}, \hat{M}]_{(n)} \Rightarrow \hat{F}(\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{L}, \hat{M}]_{(n)} \alpha^n$$

$$\Rightarrow \boxed{e^{\hat{L}} \hat{M} e^{-\hat{L}} = \hat{F}(1) = \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{L}, \hat{M}]_{(n)}}$$

□