Physics 827: Homework Set No. 7

Due date: Friday, November 18, 2011, 1:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 50 points

Problem 1 (5 pts.): Exercise 7.3.3 (Shankar, p. 196)

Problem 2 (10 pts.): Exercise 7.3.4 (Shankar, p. 196). No MATHEMATICA!

Problem 3 (10 pts.): Exercise 7.3.5 (Shankar, p. 196) No MATHEMATICA!

Problem 4 (15 pts.): Exercise 7.3.6 (Shankar, p. 197)
Additional questions: What is the ground state energy and wavefunction? What is the level spacing between energy eigenvalues? Show that your eigenfunctions are correctly orthonormalized!

Problem 5 (10 pts.): Exercise 7.3.7 (Shankar, p. 202)

Exercise 7.3.3 (p. 196)

Given: $\psi(x) = \psi(-x)$ (even)

$\phi(x) = -\phi(-x)$ (odd)

- Then $\int_{-\infty}^{\infty} dx\, \phi(x)\psi(x) = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) dx\, \phi(x)\psi(x)$

$$\overset{(x=-y)}{=} \int_{y=\infty}^{0} (-dy)\, \underbrace{\phi(-y)}_{-\phi(y)}\psi(-y) + \int_{0}^{\infty} dx\, \phi(x)\psi(x)$$

$$= -\int_{0}^{\infty} dy\, \phi(y)\psi(y) + \int_{0}^{\infty} dx\, \phi(x)\psi(x) = 0$$

- Therefore, with $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$ = even

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar 4}\right)^{1/4} \underbrace{2\left(\frac{m\omega}{\hbar}\right)^{1/2} x}_{H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right)} e^{-\frac{m\omega^2 x^2}{2\hbar}} = \text{odd}$$

being the wavefunctions for the ground and first excited state of the harmonic oscillator,

$$\langle \psi_0 | \psi_1 \rangle = \int_{-\infty}^{\infty} dx\, \psi_0^*(x)\psi_1(x) = \int_{-\infty}^{\infty} dx\, \underbrace{\psi_0(x)}_{\text{even}} \underbrace{\psi_1(x)}_{\text{odd}} = 0 \;\checkmark$$

- Further, with $\psi_2(x) = \left(\frac{m\omega}{\pi\hbar 16\cdot(2!)^2}\right)^{1/4} \left(-2\left(1 - 2\frac{m\omega}{\hbar}x^2\right)\right) e^{-\frac{m\omega^2 x^2}{2\hbar}}$

we find

$$\langle \psi_0 | \psi_2 \rangle = -\sqrt{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dx \left(1 - 2\frac{m\omega x^2}{\hbar}\right) e^{-\frac{m\omega x^2}{\hbar}}$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi\, (1 - 2\xi^2) e^{-\xi^2} = -\frac{1}{\sqrt{2\pi}} \left(\sqrt{\pi} - 2\cdot\frac{1}{2}\sqrt{\pi}\right) = 0 \;\checkmark$$

and $\langle \psi_1 | \psi_2 \rangle = -\sqrt{\frac{m\omega}{2\hbar}} \int_{-\infty}^{\infty} d\xi\, \xi\,(1 - 2\xi^2) e^{-\xi^2} = 0 \;\checkmark$ by symmetric integration

Exercise 7.3.4 (p. 196)

I can write (7.3.26) more symmetrically:

$$\int_{-\infty}^{\infty} d\xi\, H_n(\xi)\, H_{n'}(\xi)\, e^{-\xi^2} = \delta_{nn'} \sqrt{\pi 2^{n+n'} n!\, n'!}$$

With this I compute

$$\langle n'|\hat{X}|n\rangle = \sqrt{\frac{m\omega}{\pi\hbar 2^{n+n'} n!\, n'!}} \int_{-\infty}^{\infty} dx\; x\; H_{n'}\left(\sqrt{\tfrac{m\omega}{\hbar}}\,x\right) H_n\left(\sqrt{\tfrac{m\omega}{\hbar}}\,x\right) e^{-\frac{m\omega x^2}{\hbar}}$$

$$= \sqrt{\frac{\hbar}{\pi m\omega}\,\frac{1}{2^{n+n'} n!\, n'!}} \int_{-\infty}^{\infty} d\xi\; \xi\, H_{n'}(\xi)\, H_n(\xi)\, e^{-\xi^2}$$

$$\overset{(7.3.25)}{=} \sqrt{\frac{\hbar}{\pi m\omega}\,\frac{1}{2^{n+n'} n!\, n'!}} \int_{-\infty}^{\infty} d\xi \left(\tfrac{1}{2} H_{n'+1}(\xi) + n' H_{n'-1}(\xi)\right) H_n(\xi)\, e^{-\xi^2}$$

$$\overset{(7.3.26)}{=} \sqrt{\frac{\hbar}{m\omega}} \left\{ \frac{1}{2} \sqrt{\frac{2^{n+n'+1}\, n!\,(n'+1)!}{2^{n+n'} n!\, n'!}}\, \delta_{n'+1,n} + n' \sqrt{\frac{2^{n+n'-1}\, n!\,(n'-1)!}{2^{n+n'} n!\, n'!}}\, \delta_{n'-1,n} \right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sqrt{n'+1}\, \delta_{n'+1,n} + n' \sqrt{\frac{1}{n'}}\, \delta_{n'-1,n} \right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\delta_{n',n+1} \sqrt{n+1} + \delta_{n',n-1} \sqrt{n} \right) \checkmark$$

Next,

$$\langle n'|\hat{P}|n\rangle = \sqrt{\frac{m\omega}{\pi\hbar 2^{n+n'} n!\, n'!}} \int_{-\infty}^{\infty} dx\; H_{n'}\left(\sqrt{\tfrac{m\omega}{\hbar}}\,x\right) e^{-\frac{m\omega x^2}{2\hbar}} \left(-i\hbar \frac{d}{dx}\right) H_n\left(\sqrt{\tfrac{m\omega}{\hbar}}\,x\right) \cdot e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= -i\hbar\sqrt{\frac{m\omega}{\pi\hbar}\frac{1}{2^{n+n'}n!n'!}}\int_{-\infty}^{\infty}d\xi\, H_{n'}(\xi)e^{-\xi^2/2}\frac{d}{d\xi}\left(H_n(\xi)e^{-\xi^2/2}\right)$$

$$= -i\sqrt{\frac{m\omega\hbar}{\pi}}\sqrt{\frac{1}{n!n'!2^{n+n'}}}\int_{-\infty}^{\infty}d\xi\, H_{n'}(\xi)e^{-\xi^2/2}\left[H_n'(\xi)e^{-\xi^2/2} - \xi H_n(\xi)e^{-\xi^2/2}\right]$$

(7.3.24)

$$= \pm i\sqrt{\frac{m\omega\hbar}{\pi}}\sqrt{\frac{1}{n!n'!2^{n+n'}}}\int_{-\infty}^{\infty}d\xi\, e^{-\xi^2}H_{n'}(\xi)\left[2nH_{n-1}(\xi) - \xi H_n(\xi)\right]$$

(7.3.25)

$$= -i\sqrt{\frac{m\omega\hbar}{\pi}}\sqrt{\frac{1}{n!n'!2^{n+n'}}}\int_{-\infty}^{\infty}d\xi\, e^{-\xi^2}H_{n'}(\xi)\left[2nH_{n-1}(\xi) - \frac{1}{2}H_{n+1}(\xi) - nH_{n-1}(\xi)\right]$$

$$= +i\sqrt{\frac{m\omega\hbar}{\pi}}\sqrt{\frac{1}{2^{n+n'}n!n'!}}\int_{-\infty}^{\infty}d\xi\, e^{-\xi^2}\left[\frac{1}{2}H_{n'}(\xi)H_{n+1}(\xi) - nH_{n'}(\xi)H_{n-1}(\xi)\right]$$

$$= i\sqrt{m\omega\hbar}\left\{\frac{1}{2}\sqrt{\frac{2^{n+1+n'}n'!(n+1)!}{2^{n+n'}n!n'!}}\delta_{n',n+1} - n\sqrt{\frac{2^{n-1+n'}n'!(n-1)!}{2^{n+n'}n!n'!}}\delta_{n',n-1}\right\}$$

$$= i\sqrt{\frac{m\omega\hbar}{2}}\left(\delta_{n',n+1}\sqrt{n+1} - \delta_{n',n-1}\sqrt{n}\right)\checkmark$$

Exercise 7.3.5 (p. 196)

- $\langle n|\hat{X}|n\rangle = |A_n|^2 \int_{-\infty}^{\infty} dx \underbrace{H_n^2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega x^2}{\hbar}}}_{\text{even}} \cdot \underbrace{x}_{\text{odd}} = 0$ ✓

- $\langle n|\hat{P}|n\rangle = |A_n|^2 \int_{-\infty}^{\infty} dx\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega x^2}{2\hbar^2}}\left(-i\hbar\frac{d}{dx}\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{\frac{m\omega x^2}{2\hbar}}$

$$= |A_n|^2(-i\hbar)\int_{-\infty}^{\infty} d\xi\, H_n(\xi)e^{-\xi^2/2}\left[H_n'(\xi)e^{-\xi^2/2} - \xi H_n(\xi)e^{-\xi^2/2}\right]$$

$$= (-i\hbar)|A_n|^2 \int_{-\infty}^{\infty} d\xi \left[\underbrace{\xi H_n^2(\xi)}_{\text{odd}\ \cdot\ \text{even}} - H_n' H_n\right]\underbrace{e^{-\xi^2}}_{\text{even}}$$

(7.3.25)

$$= +i\hbar|A_n|^2 \int_{-\infty}^{\infty} d\xi\, 2n\, H_{n-1}(\xi)H_n(\xi)e^{-\xi^2}$$

(7.3.26)

$$= \sim \delta_{n,n-1} = 0 \quad ✓$$

- Hence $\langle\hat{X}^2\rangle = (\Delta X)^2 + \langle\hat{X}\rangle^2 = (\Delta X)^2$

and $\langle\hat{P}^2\rangle = (\Delta P)^2 + \langle\hat{P}\rangle^2 = (\Delta P)^2$

- $\langle 1|\hat{X}^2|1\rangle = \sqrt{\frac{m\omega}{\pi\hbar 4}}\int_{-\infty}^{\infty} dx\, x^2 4\left(\frac{m\omega}{\hbar}x^2\right)e^{-\frac{m\omega x^2}{\hbar}}$

$$= \frac{2}{\sqrt{\pi}}\frac{\hbar}{m\omega}\int_{-\infty}^{\infty} d\xi\, \xi^4 e^{-\xi^2} = \frac{2}{\sqrt{\pi}}\frac{\hbar}{m\omega}\left(\frac{d^2}{d\alpha^2}\underbrace{\int_{-\infty}^{\infty} d\xi\, e^{-\alpha\xi^2}}_{\sqrt{\pi/\alpha}}\right)\Big|_{\alpha=1}$$

$$= \frac{2}{\sqrt{\pi}}\frac{\hbar}{m\omega}\underbrace{\left(-\frac{d}{d\alpha}\left(\frac{1}{2}\frac{\sqrt{\pi}}{\alpha^{3/2}}\right)\right)}_{3/4\sqrt{\pi}} = \frac{3}{2}\frac{\hbar}{m\omega}$$ ✓

$$\cdot \langle 1|\hat{P}^2|1\rangle = \sqrt{\frac{m\omega}{4\pi\hbar}} \int_{-\infty}^{\infty} dx \; 4\frac{m\omega}{\hbar}\left(x e^{-\frac{m\omega x^2}{2\hbar}}\right)\left(-\hbar^2\frac{d^2}{dx^2}\right)\left(x e^{-\frac{m\omega x^2}{2\hbar}}\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)(-\hbar^2)\int_{-\infty}^{\infty} d\xi \; \xi e^{-\xi^2/2}\frac{d^2}{d\xi^2}\left(\xi e^{-\xi^2/2}\right)$$

$$= -\frac{2}{\sqrt{\pi}}(\hbar m\omega)\int_{-\infty}^{\infty} d\xi \; \xi e^{-\xi^2/2}\frac{d}{d\xi}\left(e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2}\right)$$

$$= -\frac{2}{\sqrt{\pi}}(\hbar m\omega)\int_{-\infty}^{\infty} d\xi \; \xi e^{-\xi^2/2}\left(-\xi e^{-\xi^2/2} - 2\xi e^{-\xi^2/2} + \xi^3 e^{-\xi^2/2}\right)$$

$$= -\frac{2}{\sqrt{\pi}}(\hbar m\omega)\int_{-\infty}^{\infty} d\xi \; e^{-\xi^2}\left(-3\xi^2 + \xi^4\right) = -\frac{2}{\sqrt{2}}(\hbar m\omega)\left(-3\tfrac{1}{2}\sqrt{\pi} + \tfrac{3}{4}\sqrt{\pi}\right)$$

$$= +\frac{3}{2}m\omega\hbar \;\checkmark \qquad \Rightarrow (\Delta X\cdot\Delta P)_{(1)} = \frac{3}{2}\hbar > \frac{\hbar}{2}$$

uncertainty product.

Finally,

$$\cdot \langle 0|\hat{X}^2|0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{m\omega x^2}{\hbar}} = \frac{1}{\sqrt{\pi}}\frac{\hbar}{m\omega}\int_{-\infty}^{\infty} d\xi\, \xi^2 e^{-\xi^2} = \frac{\hbar}{2m\omega}$$

$$\cdot \langle 0|\hat{P}^2|0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-\frac{m\omega x^2}{2\hbar}}\left(-\hbar^2\frac{d^2}{dx^2}\right)e^{-\frac{m\omega x^2}{2\hbar}}$$

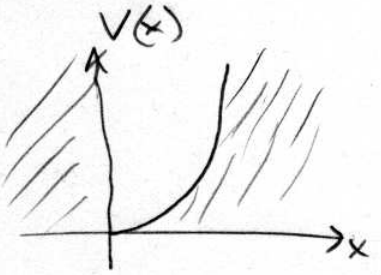
$$= -\frac{\hbar^2}{\sqrt{\pi}}\frac{m\omega}{\hbar}\int_{-\infty}^{\infty} d\xi\, e^{-\xi^2/2}\underbrace{\frac{d^2}{d\xi^2}\left(e^{-\xi^2/2}\right)}_{d/d\xi\left(-\xi e^{-\xi^2/2}\right)}$$

$$= -\frac{m\omega\hbar}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\xi\, e^{-\xi^2/2}\left(-e^{-\xi^2/2} + \xi^2 e^{-\xi^2/2}\right) = \frac{m\omega\hbar}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} d\xi\,(1-\xi^2)e^{-\xi^2}}_{\sqrt{\pi}\left(1-\frac{1}{2}\right)}$$

$$= \frac{m\omega\hbar}{2}$$

$$\Rightarrow (\Delta X\cdot\Delta P)_{(0)} = \frac{\hbar}{2} \;\checkmark$$ Saturates the uncertainty relation

Exercise 7.3.6 (p. 197)



$$V(x) = \begin{cases} \frac{m\omega^2}{2} x^2 & x > 0 \\ \infty & x \leq 0 \end{cases}$$

The infinite potential at $x \leq 0$ enforces the vanishing of all eigenfunctions at $x = 0$:

$$\psi_n(0) = 0$$

In the region $x > 0$, the eigenfunctions solve the same differential equation as for the harmonic oscillator:

$$\psi''(\xi) + (2\varepsilon - \xi^2)\psi(\xi) = 0 \qquad \left(\xi = x\sqrt{\frac{m\omega}{\hbar}}\right)$$

This is solved by the ansatz

$$\psi(\xi) = u_n(\xi)\, e^{-\xi^2/2}$$

as before because the asymptotic behavior at $x \to 0^+$ and $x \to +\infty$ of the differential equation remains the same. The $u_n(\xi)$ satisfy the same power expansion

$$u_n(\xi) = \sum_{m=0}^{\infty} c_m \xi^m$$

with c_m satisfying the same recursion relation

$$c_{m+2} = c_m \frac{2m+1-2\varepsilon}{(m+2)(m+1)}$$

as before, since this relation follows from the structure of the differential equation, not from the boundary conditions.

Normalizability of the solution still enforces the truncation of the power series at finite $n \in \mathbb{N}_0$, because otherwise the solution $u_n(\xi)$ behaves like $e^{+\xi^2}$ at $\xi \to +\infty$. This yields the quantized energy eigenvalues

$$\varepsilon_n = \frac{2n+1}{2} \quad \longleftrightarrow \quad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \,, \quad n = 0, 1, 2, \ldots$$

as before, with corresponding solutions

$$\psi_n(\xi) = u_n(\xi)\, e^{-\xi^2/2} = \begin{cases} \left(c_0 + c_2\xi^2 + c_4\xi^4 + \ldots + c_n\xi^n\right) e^{-\xi^2/2} & n \text{ even} \\ \left(c_1\xi + c_3\xi^3 + \ldots + c_n\xi^n\right) e^{-\xi^2/2} & n \text{ odd.} \end{cases}$$

However, the boundary condition $\psi_n(\xi = 0)$ can now only be fulfilled if n is odd ($u_n(\xi) \xrightarrow[\xi\to 0]{} c_1\xi + O(\xi^3) \to 0$) but not for even n where $u_n(\xi) \xrightarrow[\xi\to 0]{} c_0 \neq 0$.

(For $c_0 = 0$, all $c_{n=2k}$ vanish, so c_0 must be $\neq 0$.)

So the boundary condition $\psi_n(0) = 0$ eliminates all solutions with even n, and the eigenvalue

Spectrum reduces to $(n = 2k+1,\ k = 0, 1, 2, \ldots)$

$$\boxed{E_k = \left(2k + \tfrac{3}{2}\right)\hbar\omega = \frac{4k+3}{2}\hbar\omega \qquad k = 0, 1, 2, \ldots}$$

with eigenfunctions

$$\boxed{\psi_k(x) = \underbrace{\left(\frac{m\omega}{\pi\hbar 2^{4k}(2k+1)!}\right)^{1/4}}_{B_k = \sqrt{2}A_{2k+1}} H_{2k+1}\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) e^{-\frac{m\omega x^2}{2\hbar}}\theta(x) \qquad k = 0, 1, 2, \ldots} \quad *)$$

The ground state has energy

$$E_0 = \frac{3}{2}\hbar\omega$$

and the eigenfunction

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2\sqrt{\frac{m\omega}{\hbar}}\,x\, e^{-\frac{m\omega x^2}{2\hbar}} = 2\left(\frac{m\omega}{\hbar}\right)^{3/4}\left(\frac{1}{\pi}\right)^{1/4} x\, e^{-\frac{m\omega x^2}{2\hbar}}$$

The energy splitting between eigenstates is

$$\Delta E = E_{k+1} - E_k = 2\hbar\omega$$

independent of k, so the energy spectrum has constant spacing of $2\hbar\omega$ (instead of $\hbar\omega$ for the harmonic oscillator).

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*) These eigenfunctions are orthonormal on $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} \psi_k^*(x)\psi_{k'}(x)\,dx = \int_0^{\infty} \psi_k^*(x)\psi_{k'}(x)\,dx =$$

$$= B_k B_{k'} \int_0^\infty H_{2k+1}\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right) H_{2k'+1}\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{\hbar}} dx$$

$$= \frac{1}{2} B_k B_{k'} \int_0^\infty H_{2k+1}\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right) H_{2k'+1}\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{\hbar}} dx$$

$$(y=-x) + \frac{1}{2} B_k B_{k'} \int_0^{-\infty} (-dy)\, \underbrace{H_{2k+1}\left(-\sqrt{\tfrac{m\omega}{\hbar}}y\right)}_{=-H_{2k+1}\left(\sqrt{\frac{m\omega}{\hbar}}y\right)} \underbrace{H_{2k'+1}\left(-\sqrt{\tfrac{m\omega}{\hbar}}y\right)}_{=-H_{2k'+1}\left(\sqrt{\frac{m\omega}{\hbar}}y\right)} e^{\frac{m\omega y^2}{\hbar}}$$

(odd-order Hermite polynomials are odd fcts.)

$$= \underbrace{\frac{1}{2} B_k B_{k'}} \int_{-\infty}^{\infty} dx\, H_{2k+1}\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right) H_{2k'+1}\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{\hbar}}$$

$$= A_{2k+1} A_{2k'+1}$$

$$= \delta_{2k+1,\,2k'+1} = \delta_{kk'}$$

$$\Rightarrow \boxed{\langle \psi_k | \psi_{k'} \rangle = \delta_{kk'}} \quad \checkmark$$

Exercise 7.3.7 (p. 202)

$$\left(\frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\hat{X}\right)|E\rangle = E|E\rangle$$

$$\int_{-\infty}^{\infty} \langle p|\frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\hat{X}^2|p'\rangle\langle p'|E\rangle dp' = E\langle p|E\rangle$$

$$\Rightarrow \frac{p^2}{2m} + \frac{m\omega^2}{2}(i\hbar)^2\frac{d^2}{dp^2}\psi_E(p) = E\,\psi_E(p)$$

$$\Rightarrow \frac{d^2}{dp^2}\psi_E(p) + \frac{2}{m(\hbar\omega)^2}\left(E - \frac{p^2}{2m}\right)\psi_E(p) = 0$$

Introduce dimensionless momentum variable $\mathcal{P} \equiv p/p_0$:

$$\frac{d^2}{d\mathcal{P}^2}\psi(\mathcal{P}) + \frac{2Ep_0^2}{m(\hbar\omega)^2}\psi - \frac{p_0^4}{(\hbar m\omega)^2}\mathcal{P}^2\psi = 0$$

Choose $p_0 = \sqrt{\hbar m\omega}$: $\Rightarrow$ $\boxed{\mathcal{P} = \frac{p}{\sqrt{\hbar m\omega}}}$

$$\frac{d^2}{d\mathcal{P}^2}\psi(\mathcal{P}) + \frac{2E\hbar m\omega}{m\hbar^2\omega^2}\psi - \mathcal{P}^2\psi = 0$$

Introduce $\varepsilon = \frac{E}{\hbar\omega}$ as before:

$$\boxed{\frac{d^2}{d\mathcal{P}^2}\psi(\mathcal{P}) + (2\varepsilon - \mathcal{P}^2)\psi(\mathcal{P}) = 0}$$

This is the same differential equation as before (7.3.8), with solutions

$$\boxed{\psi_n(\mathcal{P}) = u_n(\mathcal{P})e^{-\mathcal{P}^2/2} = B_n H_n(\mathcal{P})\,e^{-\mathcal{P}^2/2}}$$

$\mathcal{P} = \frac{p}{\sqrt{m\omega\hbar}}$ differs from $\xi = x\sqrt{\frac{m\omega}{\hbar}}$ by the formal replacement $m\omega \to \frac{1}{m\omega}$ as we replace $x \to p$.

So all p-space solutions are obtained from the x-space solutions by replacing $\xi \to \frac{p}{\sqrt{\hbar m\omega}}$ instead of $\xi \to x\sqrt{\frac{m\omega}{\hbar}}$ and $A_n = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4}$ by $B_n = \left(\pi m\omega\hbar 2^{2n}(n!)^2\right)^{-1/4}$.

$$\Rightarrow \psi_0(p) = \left(\frac{1}{\pi m\omega\hbar}\right)^{1/4} e^{-\frac{p^2}{2m(\hbar\omega)}} \quad \checkmark$$