Physics 828: Homework Set No. 2

Due date: Friday, January 20, 2011, 1:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 60 points

Problem 1 (10 pts.): Exercise 10.2.1 (Shankar, p. 259)

Problem 2 (10 pts.): Exercise 10.2.2 (Shankar, p. 259)

Problem 3 (10 pts.): Exercise 10.2.3 (Shankar, p. 259)

Problem 4 (5 pts.): Exercise 10.3.1 (Shankar, p. 277)

Problem 5 (10 pts.): Exercise 10.3.2 (Shankar, p. 277)

Problem 6 (5 pts.): Exercise 10.3.3 (Shankar, p. 277)

Problem 7 (10 pts.): Exercise 10.3.4 (Shankar, p. 278)

2M Honors HW#2 Biao Huang

Total=35

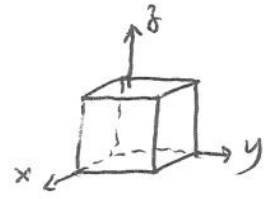
⑤ 10.2.1 Three dimensional box.

Inside the box, $H = H_x + H_y + H_z = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m}$, variables are separat

$\Rightarrow$ The eigenfunctions are ~~then~~ $\psi_E(x,y,z) = \psi_{Ex}(x)\,\psi_{Ey}(y)\,\psi_{Ez}(z)$

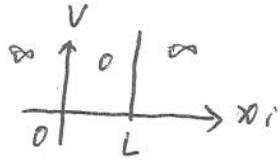
$\Rightarrow H\psi_E = (H_x + H_y + H_z)\,\psi_{Ex}(x)\,\psi_{Ey}(y)\,\psi_{Ez}(z) = (E_x + E_y + E_z)\psi_E$

(*) $\Rightarrow H_i \psi_{Ei}(x_i) = E_i \psi_{Ei}(x_i)$, $i = x, y, z$



The solution to this 1-Dim problem is

$$\begin{cases} \psi_n(x_i) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_i \pi x_i}{L}\right) \\ E_i = \frac{\hbar^2 \pi^2 n_i^2}{2mL^2} \end{cases}$$



Thus,

$$\begin{cases} \psi_E(x,y,z) = \left(\frac{2}{L}\right)^{\frac{3}{2}} \prod_{i=x,y,z} \sin\left(\frac{n_i \pi x_i}{L}\right) \\ E = E_x + E_y + E_z = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right) \end{cases}$$

The state $\psi_E(x,y,z)$ is normalized because

$$\int d^3x\, \psi_E(x,y,z) = \prod_{i=x,y,z} \int dx_i \left|\psi_{Ei}(x_i)\right|^2 = \prod_{i=x,y,z} 1 = 1.$$

(*): This step is because $\frac{H_x \psi_{Ex}(x)}{\psi_{Ex}(x)} = E - \frac{H_y \psi_{Ey}(y)}{\psi_{Ey}(y)} - \frac{H_z \psi_{Ez}(z)}{\psi_{Ez}(z)}$. Since two sides involve different variables independently, they must equal to a constant E_x. So $H_x \psi_{Ex}(x) = E_x \psi_{Ex}(x)$.

(10)

10.2.2 (1) Two dimensional oscillator.

$$H = \left(\frac{P_x^2}{2m} + \frac{1}{2}m\omega_x^2 x^2\right) + \left(\frac{P_y^2}{2m} + \frac{1}{2}m\omega_y^2 y^2\right)$$

$$= H_x + H_y$$

Similar to the last problem, we can separate variables

$$\psi_{n_x n_y}(x,y) = \psi_{n_x}(x)\,\psi_{n_y}(y)$$

$$\Rightarrow (H_x + H_y)\,\psi_{n_x}(x)\,\psi_{n_y}(y) = (E_x + E_y)\,\psi_{n_x}(x)\,\psi_{n_y}(y).$$

$$\Rightarrow H_i \psi_{n_i} = E_{n_i}\psi_{n_i} \Rightarrow \begin{cases} \psi_{n_i} = A_{n_i} H_{n_i}\left(\sqrt{\frac{m\omega}{\hbar}}\,x_i\right) e^{-\frac{m\omega x_i^2}{2\hbar}} \\ E_{n_i} = \hbar\omega_i\left(n_i + \frac{1}{2}\right),\ n_i = 0,1,2\cdots \end{cases}$$

$$\Rightarrow E = E_{n_x} + E_{n_y} = \hbar\omega_x\left(n_x + \frac{1}{2}\right) + \hbar\omega_y\left(n_y + \frac{1}{2}\right)$$

(2) $$\psi_{n_x n_y} = A_{n_x} A_{n_y} H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}}x\right) H_{n_y}\left(\sqrt{\frac{m\omega}{\hbar}}y\right) e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

where $A_{n_i}^2 = \sqrt{\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}}$ is the normalization factor

$$\Rightarrow \psi_{n_x n_y}(-x,-y) = \left[A_{n_x} A_{n_y} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}\right] H_{n_x}\left[-\sqrt{\frac{m\omega}{\hbar}}x\right] H_{n_y}\left[-\sqrt{\frac{m\omega}{\hbar}}y\right]$$

Since $H_{n_i}\left(-\sqrt{\frac{m\omega}{\hbar}}x_i\right) = (-1)^{n_i} H_{n_i}\left[\sqrt{\frac{m\omega}{\hbar}}x_i\right]$,

$\psi_{n_x n_y}(-x,-y) = (-1)^{n_x+n_y}\psi_E(x,y)$, so the parity depends on $n \equiv n_x + n_y$.

(3) When $\omega_x = \omega_y$, $E = \hbar\omega(n_x + n_y + 1) \equiv \hbar\omega(n+1)$

$\Rightarrow$ First three states are

$n=0$: $n_x = n_y = 0$. $\quad \psi_{00} = \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$

$n=1$:
- $n_x=0, n_y=1$: $\quad \psi_{01}(x,y) = \sqrt{\frac{m\omega}{2\pi\hbar}} \cdot \left(2\sqrt{\frac{m\omega}{\hbar}}\, y\right) e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$
$= \sqrt{\frac{2}{\pi}}\, \frac{m\omega}{\hbar}\, y\, e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$
- $n_x=1, n_y=0$: $\quad \psi_{10}(x,y) = \sqrt{\frac{2}{\pi}}\, \frac{m\omega}{\hbar}\, x\, e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$

In polar coordinates,

$$\begin{cases} \psi_{00}(\rho,\varphi) = \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega\rho^2}{2\hbar}} \\ \psi_{01}(\rho,\varphi) = \sqrt{\frac{2}{\pi}}\, \frac{m\omega}{\hbar}\, \rho\sin\varphi\, e^{-\frac{m\omega\rho^2}{2\hbar}} \\ \psi_{10}(\rho,\varphi) = \sqrt{\frac{2}{\pi}}\, \frac{m\omega}{\hbar}\, \rho\cos\varphi\, e^{-\frac{m\omega\rho^2}{2\hbar}} \end{cases}$$

Degeneracy: for a fixed n, $\begin{cases} n_x = 0, 1, \cdots, n, \\ \Rightarrow n_y = n, n-1, \cdots, 0. \end{cases}$

$\Rightarrow$ ~~$D(n) = \sum_{m=0}^{n} m = \frac{n+1}{2}$~~

$\Rightarrow$ $D(n) = n+1$ as n_x can take $(n+1)$ values, each corresponding to one value for n_y.

10.2.3 3-D istropic harmonic oscillator

$$H = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

1) We can use the standard "separation of variables" method to reduce the 3-D problem to independent 3 1-D problems.

Assume the solution takes the form $\psi(x,y,z) = X(x)Y(y)Z(z)$, where X, Y, Z are functions of x, y, z respectively. Then the Schrödinger equation can be written as

$$\frac{1}{X(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)X(x) = -\frac{1}{Y(y)}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2y^2\right)Y(y)$$

$$-\frac{1}{Z(z)}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + \frac{1}{2}m\omega^2z^2\right)Z(z) + E \qquad (*)$$

Since the LHS & RHS of the equation depends on independent variables, they must equal to a constant irrelavant to x, y, z. Let the constant be E_x.

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)X(x) = E_x X(x)$$

which is a 1-D harmonic oscillator with energy $E_x = (n_x + \frac{1}{2})\hbar\omega$

Similarly split y, z on the RHS of equation (*), we obtain

$$E = E_x + E_y + E_z = (n_x + \tfrac{1}{2})\hbar\omega + (n_y + \tfrac{1}{2})\hbar\omega + (n_z + \tfrac{1}{2})\hbar\omega$$

$$= (n_x + n_y + n_z + \tfrac{3}{2})\hbar\omega, \quad (n_i = 0, 1, 2, \cdots)$$

2) As stated in 1), the eigenfunctions are

$$\psi_n(x,y,z) = X_{n_x}(x)\, Y_{n_y}(y)\, Z_{n_z}(z),$$

where $(X_{n_x}, Y_{n_y}, Z_{n_z})$ are eigenfunctions to the 1-D harmonic oscillator problem (Shankar (7.3.22)).

We know the eigenfunction with quantum number n_i ($i=x,y,z$) has parity $(-1)^{n_i}$ (due to the Hermite polynomial H_{n_i}). Thus, the parity of the total wavefn. is $(-1)^{n_x+n_y+n_z} = (-1)^n$.

The first four states are

$$\psi_0 = X_0 Y_0 Z_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)} \xrightarrow{(r,\theta,\varphi)} \boxed{\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} e^{-\frac{m\omega}{2\hbar}r^2} \equiv \psi_0(r)}$$

$$\psi_1^{(x)} = X_1 Y_0 Z_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)} \; 2\sqrt{\frac{m\omega}{\hbar}}\,x \rightarrow \boxed{\psi_0(r)\,\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}\, r\sin\theta\cos\varphi}$$

$$\psi_1^{(y)} = X_0 Y_1 Z_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)} \; 2\sqrt{\frac{m\omega}{\hbar}}\,y \rightarrow \boxed{\psi_0(r)\,\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}\, r\sin\theta\sin\varphi}$$

$$\psi_1^{(z)} = X_0 Y_0 Z_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)} \; 2\sqrt{\frac{m\omega}{\hbar}}\,z \rightarrow \boxed{\psi_0(r)\,\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}\, r\cos\theta}$$

The degeneracy of a level $E_n = (n+\frac{3}{2})\hbar\omega$ results from the multiple equivalent combinations of $n = n_x+n_y+n_z$, $n_i = 0, 1, 2, \cdots$, the number of which amounts to $\frac{(n+1)(n+2)}{2}$ as shown below:

$n_x = 0$	$n_y = 0, \cdots, n$	$(n+1)$ choices
$n_x = 1$	$n_y = 0, \cdots, n-1$	n choices
$n_x = 2$	$n_y = 0, \cdots, n-2$	$n-1$ choices
$\vdots$	$\vdots$	$\vdots$
$n_x = n$	$n_y = 0$	1 choice

(n_z fixed by (n_x, n_y))

$$\Rightarrow \text{Degeneracy} = 1+2+3+\cdots+(n+1)$$

$$= \frac{n+1}{2}(1+(n+1))$$

$$= \frac{(n+1)(n+2)}{2}$$

③ 10.3.1 When these two bosons form a system,

$$|\lambda\rangle = A\left(|4\phi\rangle + |\phi 4\rangle\right).$$

Here $|4|^2 = 1$, $|\phi|^2 = 1$, but $\langle 4|\phi\rangle \neq 0$, so.

$$1 = \langle\lambda|\lambda\rangle = |A|^2\left[\langle 4\phi|4\phi\rangle + \langle 4\phi|\phi 4\rangle + \langle\phi 4|4\phi\rangle + \langle\phi 4|\phi 4\rangle\right]$$

$$= |A|^2\left[\langle 4|4\rangle\langle\phi|\phi\rangle + \langle 4|\phi\rangle\langle\phi|4\rangle + \langle\phi|4\rangle\langle 4|\phi\rangle + \langle\phi|\phi\rangle\langle 4|4\rangle\right]$$

$$= |A|^2\left[1 + 2|\langle\phi|4\rangle|^2 + 1\right]$$

$$\Rightarrow |A| = \frac{1}{\sqrt{2(1+|\langle\phi|4\rangle|^2}}, \text{ we can choose the phase } 0$$

$$\Rightarrow |\lambda\rangle = \frac{|4\phi\rangle + |\phi 4\rangle}{\sqrt{2(1+|\langle\phi|4\rangle|^2)}}$$

③ 10.3.2. Here we have $|3\rangle$, $|3\rangle$, $|4\rangle$, where

$$\langle 3|3\rangle = \langle 4|4\rangle = 1, \quad \langle 3|4\rangle = 0.$$

$$\Rightarrow |\lambda\rangle = \frac{1}{\sqrt{3}}\left(|3,3,4\rangle + |3,4,3\rangle + |4,3,3\rangle\right)$$

$\langle\lambda|\lambda\rangle = 1$, and $|\lambda\rangle$ is invariant under permutations of any two particles.

10.3.3

1) Distinguishable particles: Each level can support 0-3 particles. So total configurations are $3^3 = 27$ because each distinguishable particle can choose one of these three levels.

2) bosons: $C_3^1 + (C_3^1 \cdot C_2^1) + C_3^3 = \frac{3!}{(3-1)!1!} + \frac{3!}{(3-1)!1!}\frac{2!}{(2-1)!1!} + \frac{3!}{(3-3)!3!} = 3 + 6 + 1 = 10$

Here C_n^m is the binomial coefficients $\frac{n!}{(n-m)!m!}$. This is also written as $\binom{n}{m}$.

3) fermion: Each level can support 0-1 particle, so three particles can only be in different levels, and there is only 1 possibility.

10.3.4

Energy levels for this system are $E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2, (n = 1,2, \dots)$. If we fill two particles into this system, the total energy will be

$$E_{n_1 n_2} = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2), \qquad (n_1, n_2 = 1,2,3 \dots)$$

The first measurement tells $E_{n_1 n_2} = \frac{\hbar^2\pi^2}{2mL^2} \times 2$, so both particles are in the ground state, namely $n_1 = n_2 = 1$. Now because identical particles have to be either fermions or bosons, we see that in this case they should be bosons as two fermions could not crowd into the same state $n = 1$ due to Pauli's exclusion principle. Explicitly, the state for the whole boson system is

$$|\psi\rangle = |1_1\rangle \otimes |1_2\rangle \equiv |11\rangle$$

The second measurement gives $E_{n_1 n_2} = \frac{\hbar^2\pi^2}{2mL^2} \times 5$, which means $n_1^2 + n_2^2 = 5$, giving $n_1 = 1, n_2 = 2$ or $n_1 = 2, n_2 = 1$. Since these two particles are bosons, the state should be symmetric under exchange of the labels (1,2) for the two particles. Thus, the state for the system reads

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\,|12\rangle\, + |21\rangle)$$

*The above solution understands the problem in the way that we prepare certain state $|\psi\rangle$ which is a superposition of many energy eigenstates, and measure it, and then prepare the state $|\psi\rangle$ again, and measure it again. So the first measurement tells the system under consideration is a boson system and therefore the second state could only be symmetric $\sim|12\rangle + |21\rangle$. If these two measurements refer to independent systems, the second measurement could fairly imply a fermion system as well, so it should be $\sim|12\rangle \pm |21\rangle$, with negative sign referring to a fermion system.

□