Physics 828: Homework Set No. 3

Due date: Friday, January 27, 2011, 1:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 60 points + 10 bonus points

Problem 1 (20 pts.): Exercise 10.3.5 (Shankar, p. 278)

Problem 2 (10 pts.): Exercise 10.3.6 (Shankar, p. 278)

Problem 3 (5 pts.): Exercise 11.2.2 (Shankar, p. 283)

Problem 4 (5 pts.): Exercise 11.4.1 (Shankar, p. 300)

Problem 5 (5 pts.): Exercise 11.4.2 (Shankar, p. 300). If you correctly derive in closed form the explicit expression for $[\hat{P}, \hat{H}]$ you receive **10 bonus points**.

Problem 6 (10 pts.): Exercise 11.4.3 (Shankar, p. 300)

Problem 7 (5 pts.): Exercise 11.4.4 (Shankar, p. 300)

Exercise 10.3.5 (p. 278)

- $\hat{P}_{12}|x_1 x_2\rangle = |x_2 x_1\rangle$ defn. of exchange operator

(1) $\hat{P}_{12}^2 |x_1 x_2\rangle = \hat{P}_{12}|x_2 x_1\rangle = |x_1 x_2\rangle \quad \forall x_1, x_2 \Rightarrow \hat{P}_{12}^2 = \mathbb{1}$

$\Rightarrow P_{12} = (P_{12})^{-1}$ ✓

since $\langle x_1' x_2' | \hat{P}_{12}^+ \hat{P}_{12} | x_1 x_2\rangle = \left(\hat{P}_{12}|x_1' x_2'\rangle\right)^+ \hat{P}_{12}|x_1 x_2\rangle$

$= \langle x_2' x_1' | x_2 x_1\rangle = \delta(x_2 - x_2')\,\delta(x_1 - x_1') \quad \forall x_1, x_2$,

we see that $\hat{P}_{12}^+ \hat{P}_{12} = \hat{\mathbb{1}} \Rightarrow \hat{P}_{12}^+ = (\hat{P}_{12})^{-1}$ (unitary) ✓

$\Rightarrow \hat{P}_{12}^+ \hat{P}_{12} = \hat{P}_{12}^2$ and hence $\hat{P}_{12}^+ = \hat{P}_{12}$ (Hermitean) ✓

and

since $\hat{P}_{12}$ is unitary, its eigenvalues have the form

$\lambda = e^{i\alpha}$ $(\alpha \in \mathbb{R})$

since $\hat{P}_{12}$ is Hermitean, its eigenvalues are real.

since $\hat{P}_{12}^2 = \hat{\mathbb{1}}$, all eigenvalues obey $\lambda^2 = 1$

$\Rightarrow P_{12}$ has eigenvalues $\boxed{\lambda_\pm = \pm 1}$ ✓

(2) $\hat{P}_{12}|\omega_1 \omega_2\rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \left(\hat{P}_{12}|x_1 x_2\rangle\right) \langle x_1|\omega_1\rangle \langle x_2|\omega_2\rangle$

$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, |x_2 x_1\rangle \underbrace{\langle x_2|\omega_2\rangle \langle x_1|\omega_1\rangle}_{= \langle x_2 x_1 | \omega_2 \omega_1\rangle} = |\omega_2 \omega_1\rangle$ ✓

$= \hat{\mathbb{1}}$

$\mathbb{V}_S = \{ |\omega_1 \omega_2, S\rangle \mid \omega_1, \omega_2 \in \{\text{eigenvalues of } \hat{S}^2\}\}$

$|\psi_S\rangle \in \mathbb{V}_S$ can be written as

$$|\psi_S\rangle = \sum_{\omega_1 \le \omega_2} c_{\omega_1\omega_2} |\omega_1\omega_2, S\rangle$$

$$\hat{P}_{12}|\psi_S\rangle = \sum_{\omega_1 \le \omega_2} c_{\omega_1\omega_2} \hat{P}_{12}|\omega_1\omega_2, S\rangle$$

$$= \sum_{\omega_1 \le \omega_2} c_{\omega_1\omega_2} (+1) |\omega_1\omega_2, S\rangle$$

$$= |\psi_S\rangle \checkmark$$

← for $\omega_1 = \omega_2$, $\hat{P}_{12}|\omega\omega, S\rangle = \hat{P}_{12}|\omega\omega\rangle = |\omega\omega\rangle = (+1)|\omega\omega\rangle$

for $\omega_1 \neq \omega_2$, $\hat{P}_{12}|\omega_1\omega_2, S\rangle = \frac{1}{\sqrt{2}}\hat{P}_{12}(|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle) = \frac{1}{\sqrt{2}}(|\omega_2\omega_1\rangle + |\omega_1\omega_2\rangle) = (+1)|\omega_1\omega_2, S\rangle$

$|\psi_A\rangle \in \mathbb{V}_A$ can be written as

$$|\psi_A\rangle = \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} |\omega_1\omega_2, A\rangle$$

$$\hat{P}_{12}|\psi_A\rangle = \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} \hat{P}_{12}|\omega_1\omega_2, A\rangle = \frac{1}{\sqrt{2}} \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} \hat{P}_{12}(|\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle)$$

$$= \frac{1}{\sqrt{2}} \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} (|\omega_2\omega_1\rangle - |\omega_1\omega_2\rangle) = \sum_{\omega_1 < \omega_2} c_{\omega_1\omega_2} (-1) \frac{1}{\sqrt{2}} (|\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle)$$

$$= -|\psi_A\rangle$$

$\Rightarrow \mathbb{V}_S \subset \mathbb{V}_{1\otimes 2}$ is the eigenspace of $\hat{P}_{12}$ with eigenvalue $+1$

$\mathbb{V}_A \subset \mathbb{V}_{1\otimes 2}$ " " " " $\hat{P}_{12}$ " " -1 ✓

(4) For two identical particles

$$\hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2) = \hat{H}(\hat{X}_2, \hat{P}_2; \hat{X}_1, \hat{P}_1)$$ (switching the two particles does not change their total energy)

$$= \hat{P}_{12}\, \hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2)\, \hat{P}_{12}$$

$$\Rightarrow \quad \hat{P}_{12}\, \hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2) = \hat{H}(\hat{X}_1, \hat{P}_1; \hat{X}_2, \hat{P}_2)\, \hat{P}_{12}$$

$$\Rightarrow \underline{[\hat{P}_{12}, \hat{H}] = 0} \quad \checkmark$$

This holds irrespective of whether or not $\hat{H}$ depends on time t.

Since $\hat{U}(t) = \hat{T} e^{-\frac{i}{\hbar}\int_0^t dt' \hat{H}(t')}$ (time-ordered product)

and $[\hat{P}_{12}, \hat{H}(t')] = 0 \quad \forall\, t'$

$$\Rightarrow \underline{[\hat{P}_{12}, \hat{U}(t)] = 0} \quad \checkmark \qquad (*)$$

If $\hat{P}_{12}|\psi(0)\rangle = \lambda|\psi(0)\rangle$, $\lambda = \pm 1$, at $t = 0$,

then

$$\underline{\hat{P}_{12}|\psi(t)\rangle} = \hat{P}_{12}\hat{U}(t)|\psi(0)\rangle \overset{(*)}{=} \hat{U}(t)\hat{P}_{12}|\psi(0)\rangle$$

$$= \hat{U}(t)\lambda|\psi(0)\rangle = \underline{\lambda|\psi(t)\rangle}$$

i.e. $|\psi(t)\rangle$ is an eigenstate of $\hat{P}_{12}$ with eigenvalue λ at at all times.

$\Rightarrow$ Symmetric states always remain in $\mathcal{H}_S$,
Antisymmetric " " " $\mathcal{H}_A$ $\checkmark$

(3) For a general observable $\hat{\Omega}$ with eigenstates $|\omega_i\rangle$ and eigenvalues ω_i we have

$$\hat{P}_{12}\hat{\Omega}_1^{1\otimes 2}\hat{P}_{12}|\omega_1\omega_2\rangle = \hat{P}_{12}(\hat{\Omega}_1\times\hat{\mathbb{1}}_2)|\omega_2\omega_1\rangle = \hat{P}_{12}(\omega_2|\omega_2\omega_1\rangle$$

$$= \omega_2\hat{P}_{12}|\omega_2\omega_1\rangle = \omega_2|\omega_1\omega_2\rangle = \hat{\Omega}_2^{1\otimes 2}|\omega_1\omega_2\rangle \quad \forall\,\omega_1,\omega_2$$

$$\Rightarrow \hat{P}_{12}\hat{\Omega}_1^{1\otimes 2}\hat{P}_{12} = \hat{\Omega}_2^{1\otimes 2}$$

Similarly

$$\hat{P}_{12}\hat{\Omega}_2^{1\otimes 2}\hat{P}_{12}|\omega_1\omega_2\rangle = \hat{P}_{12}(\hat{\mathbb{1}}_1\otimes\hat{\Omega}_2)|\omega_2\omega_1\rangle = \omega_1\hat{P}_{12}|\omega_2\omega_1\rangle$$

$$= \omega_1|\omega_1\omega_2\rangle = \hat{\Omega}_1^{1\otimes 2}|\omega_1\omega_2\rangle \quad \forall\,\omega_1,\omega_2$$

$$\Rightarrow \hat{P}_{12}\hat{\Omega}_2^{1\otimes 2}\hat{P}_{12} = \hat{\Omega}_1^{1\otimes 2}$$

These hold in particular for $\hat{\Omega}=\hat{x}$ and $\hat{\Omega}=\hat{p}$ ✓

Now consider

$$\hat{P}_{12}\hat{\Omega}(\hat{X}_1,\hat{P}_1;\hat{X}_2,\hat{P}_2)\hat{P}_{12} = \hat{P}_{12}\left(\sum_{n_1,\dots n_4=0}^{\infty} C_{n_1n_2n_3n_4}\hat{X}_1^{n_1}\hat{P}_1^{n_2}\hat{X}_2^{n_3}\hat{P}_2^{n_4}\right)\hat{P}_{12}$$

$$= \sum_{n_1\dots n_4=0}^{\infty} C_{n_1n_2n_3n_4}\,\hat{P}_{12}\hat{X}_1^{n_1}\underbrace{\hat{P}_{12}\hat{P}_{12}}_{\hat{\mathbb{1}}}\hat{P}_1^{n_2}\hat{P}_{12}\underbrace{\hat{P}_{12}\hat{X}_2^{n_3}\hat{P}_{12}}_{=\hat{X}_1^{n_3}}\underbrace{\hat{P}_{12}\hat{P}_2^{n_4}\hat{P}_{12}}_{=\hat{P}_1^{n_4}}$$

$$= \hat{P}_{12}\hat{X}_1\hat{P}_{12}\hat{P}_{12}\hat{X}_1\hat{P}_{12}\cdots\hat{P}_{12}\hat{X}_1\hat{P}_{12} = \hat{X}_2^{n_1}$$

(n_1 factors)

$$= \sum_{n_1\dots n_4=0}^{\infty} C_{n_1n_2n_3n_4}\hat{X}_2^{n_1}\hat{P}_2^{n_2}\hat{X}_1^{n_3}\hat{P}_1^{n_4} = \hat{\Omega}(\hat{X}_2,\hat{P}_2;\hat{X}_1,\hat{P}_1)$$ ✓

Exercise 10.3.6 (p. 278)

Consider a state consisting of two identical particles P positioned at x_1^P and x_2^P:

$$|\psi\rangle = |x_1^P x_2^P\rangle$$

If each particle consist of N fermions, the position eigenstates for the particle P can be expanded into a complete set of position eigenstates for each fermion:

$$|x^P\rangle = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \, C(y_1, \ldots, y_N) \, |y_1 y_2 \cdots y_N\rangle$$

We don't know to what extent the states $|y_1, y_2, \ldots, y_N\rangle$ must be antisymmetrized, since we are not told whether the N fermions are all distinguishable species or some of them are identical. But our answer will not depend on this.

With this expansion the state $|\psi\rangle$ can be written as

$$|\psi\rangle = \int_{-\infty}^{\infty} dy_1 \, dy_2 \cdots dy_N \int_{-\infty}^{\infty} dz_1 \, dz_2 \cdots dz_N \, C_1(y_1 \cdots y_N) \, C_2(z_1 \cdots z_N) \cdot$$

$$\cdot \, |y_1 y_2 \cdots y_N\rangle \otimes |z_1 z_2 \cdots z_N\rangle$$

$$= \left(|y_1\rangle \otimes |y_2\rangle \otimes \cdots \otimes |y_n\rangle\right) \otimes \left(|z_1\rangle \otimes |z_2\rangle \otimes \cdots \otimes |z_N\rangle\right)$$

$|y_1\rangle \in \mathbb{V}_1$ for fermion 1; $|y_n\rangle \in \mathbb{V}_1$ for fermion N; $|z_1\rangle \in \mathbb{V}_2$ for fermion 1; $|z_N\rangle \in \mathbb{V}_2$ for fermion N

Let us reorder the factors on the right hand side as

$$\left(|y_1\rangle \otimes |z_1\rangle\right) \otimes \left(|y_2\rangle \otimes |z_2\rangle\right) \otimes \ldots \otimes \left(|y_N\rangle \otimes |z_N\rangle\right) \equiv$$
$$\equiv |y_1 z_1\rangle |y_2 z_2\rangle \ldots |y_N z_N\rangle$$

and write the expansion in the form

$$|\psi\rangle = \int_{-\infty}^{\infty} dy_1 dz_1 \int_{-\infty}^{\infty} dy_2 dz_2 \int \ldots \int_{-\infty}^{\infty} dy_N dz_N \; \tilde{C}(y_1 z_1; y_2 z_2; \ldots; y_N z_N) \cdot$$
$$\cdot |y_1 z_1\rangle |y_2 z_2\rangle \ldots |y_N z_N\rangle \qquad (*)$$

Under interchange of the two composite particles the states $|y_1 z_1\rangle |y_2 z_2\rangle \ldots |y_N z_N\rangle$ go into $|z_1 y_1\rangle |z_2 y_2\rangle \ldots |z_N y_N\rangle$ since all of the constituent fermions get interchanged. Since all constituents are fermions, each of the states $|y_k z_k\rangle$ $(k = 1, \ldots, N)$ must really be antisymmetrized. For this to be true, the expansion coefficients $\tilde{C}(y_1 z_1; \ldots; y_N z_N)$ must have appropriate symmetries under the exchange $y_k \leftrightarrow z_k$ such that the expansion $(*)$ takes the form

$$|\psi\rangle = \int_{y_1 < z_1} dy_1 dz_1 \int_{y_2 < z_2} dy_2 dz_2 \ldots \int_{y_N < z_N} dy_N dz_N \; \bar{C}(y_1 z_1; y_2 z_2; \ldots; y_N z_N) \cdot$$
$$\cdot |y_1 z_1, A\rangle |y_2 z_2, A\rangle \ldots |y_N z_N, A\rangle$$

What matters is the antisymmetry between the fermions in particle 1 and their identical partners in particle 2, not the antisymmetry between possibly

identical constituents inside each of the composite particles.

The state in which the two composite particles are interchanged is given by ($\hat{P}_{12}^{(k)}$ = exchange operator for two particles of type k)

$$\hat{P}_{12}|\psi\rangle = \int_{y_1<z_1} dy_1 dz_1 \cdots \int_{y_N<z_N} dy_N dz_N \, C(y_1,z_1;\cdots;y_N,z_N) \cdot$$

$$\cdot \underbrace{\left(\hat{P}_{12}^{(1)}|y_1 z_1, A\rangle\right)}_{=|z_1 y_1, A\rangle = -|y_1 z_1; A\rangle} \left(\hat{P}_{12}^{(2)}|y_2 z_2, A\rangle\right) \cdots \underbrace{\left(\hat{P}_{12}^{(N)}|y_N z_N, A\rangle\right)}_{= -|y_N z_N, A\rangle}$$

$$= (-1)^N |\psi\rangle$$

So $|\psi\rangle$ is an eigenstate of $\hat{P}_{12}$ with eigenvalue

$$(-1)^N = \begin{cases} +1 & N \text{ even}, \\ -1 & N \text{ odd}. \end{cases}$$

Hence a composite particle consisting of an even/odd number of fermions behaves like a boson/fermion ✓

In particular, hydrogen atoms behave like bosons, deuterium behaves like a fermion, tritium like a boson but ^{3}He like a fermion. ^{6}Li atoms are fermions.

Exercise 11.2.2 (p.283)

$$\hat{T}^{+}(\varepsilon)\hat{T}(\varepsilon) = \left(\mathbb{1} + \frac{i\varepsilon}{\hbar}\hat{G}^{+} + O(\varepsilon^2)\right)\left(\mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{G} + O(\varepsilon^2)\right)$$

$$= \mathbb{1} + \frac{i\varepsilon}{\hbar}\left(\hat{G}^{+} - \hat{G}\right) + O(\varepsilon^2)$$

Since $\hat{T}(\varepsilon)|x\rangle = |x+\varepsilon\rangle$ is normalized,

$$\langle x'|x\rangle = \delta(x-x') = \langle x'+\varepsilon|x+\varepsilon\rangle = \langle x'|\hat{T}^{+}(\varepsilon)\hat{T}(\varepsilon)|x\rangle \quad \forall\, x, x',$$

we have $\hat{T}(\varepsilon)\hat{T}(\varepsilon) = \hat{\mathbb{1}}$ exactly $(\forall \varepsilon)$.

$$\Rightarrow \hat{G}^{+} - \hat{G} = 0 \quad \Rightarrow \boxed{\hat{G}^{+} = \hat{G}}$$

Exercise 11.4.1 (p. 300)

At time $t=0$ we have by assumption

$$\hat{\Pi}|\psi(0)\rangle = \lambda|\psi(0)\rangle \qquad \text{with } \lambda = \pm 1.$$

$$\Rightarrow \hat{\Pi}|\psi(t)\rangle = \hat{\Pi}\,\hat{u}(t)|\psi(0)\rangle$$

Since $[\hat{\Pi}, \hat{H}] = 0 \;\; \forall t$ by assumption,

$$[\hat{\Pi}, \hat{u}(t)] = [\hat{\Pi}, \hat{T}e^{-i/\hbar \int_0^t dt' \hat{H}(t')}] = 0$$

$$\Rightarrow \hat{\Pi}\,\hat{u}(t) = \hat{u}(t)\hat{\Pi} \quad \text{and}$$

$$\hat{\Pi}|\psi(t)\rangle = \hat{u}(t)\,\hat{\Pi}|\psi(0)\rangle = \hat{u}(t)\left(\lambda|\psi(0)\rangle\right)$$

$$= \lambda|\psi(t)\rangle$$

$\Rightarrow |\psi(t)\rangle$ remains a parity eigenstate with the same eigenvalue at all times.

Exercise 11.4.2 (p. 300)

$$\hat{H} = \frac{\hat{P}^2}{2m} + V_0 \sin\left(\frac{2\pi}{a}\hat{X}\right).$$

For momentum conservation to hold,

$$\frac{d}{dt}\langle \hat{P} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{P}] \rangle = 0 \qquad \forall \text{ states}$$

we need $[\hat{H}, \hat{P}] = 0$.

However, here

$$[\hat{H}, \hat{P}] = \left[\frac{\hat{P}^2}{2m} + V_0 \sin\left(\frac{2\pi}{a}\hat{X}\right), \hat{P}\right] = V_0 \left[\sin\left(\frac{2\pi}{a}\hat{X}\right), \hat{P}\right] \neq 0$$

hence momentum is not conserved.

(
$$\sin\left(\frac{2\pi}{a}\hat{X}\right) = \frac{2\pi}{a}\hat{X} - \frac{1}{3!}\left(\frac{2\pi}{a}\right)^3 \hat{X}^3 + \frac{1}{5!}\left(\frac{2\pi}{a}\right)^5 \hat{X}^5$$

$$[\hat{X}^n, \hat{P}] = \hat{X}[\hat{X}^{n-1}, \hat{P}] + [\hat{X}, \hat{P}]\hat{X}^{n-1} = \hat{X}[\hat{X}^{n-1}, \hat{P}] + i\hbar\hat{X}^{n-1}$$

$$[\hat{X}^2, \hat{P}] = 2i\hbar\hat{X}$$

$$[\hat{X}^3, \hat{P}] = \hat{X}(2i\hbar\hat{X}) + i\hbar\hat{X}^2 = 3i\hbar\hat{X}^2$$

$$\Rightarrow [\hat{X}^n, \hat{P}] = \hat{X}(n-1)i\hbar X^{n-2} + i\hbar\hat{X}^{n-1} = ni\hbar X^{n-1}$$

$$\Rightarrow \left[\sin\left(\frac{2\pi}{a}\hat{X}\right), \hat{P}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{2\pi}{a}\right)^{2n+1} [\hat{X}^{2n+1}, \hat{P}]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{2\pi}{a}\right)^{2n+1} (2n+1) i\hbar \hat{X}^{2n}$$

$$= i\hbar \frac{2\pi}{a} \sum_{n=0}^{\infty} \frac{(-)^n}{2n!}\left(\frac{2\pi\hat{X}}{a}\right)^n = i\hbar \frac{2\pi}{a}\cos\left(\frac{2\pi}{a}\hat{X}\right)$$

$$\Rightarrow [\hat{P}, \hat{H}] = -[\hat{H}, \hat{P}] = -i\hbar \frac{2\pi V_0}{a}\cos\left(\frac{2\pi}{a}\hat{X}\right) = -i\hbar \frac{\partial \hat{V}(\hat{X})}{\partial \hat{X}}.$$

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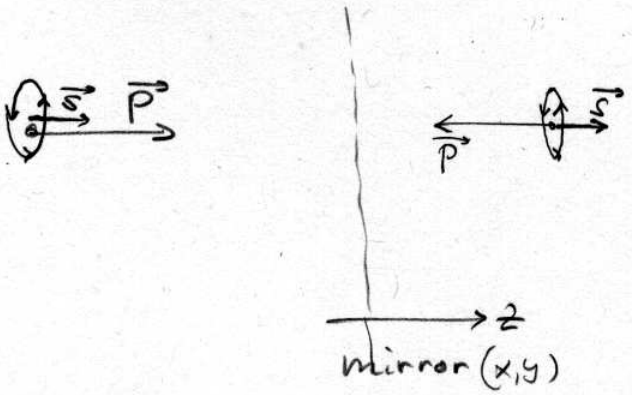
Exercise 11.4.3 (p. 300)

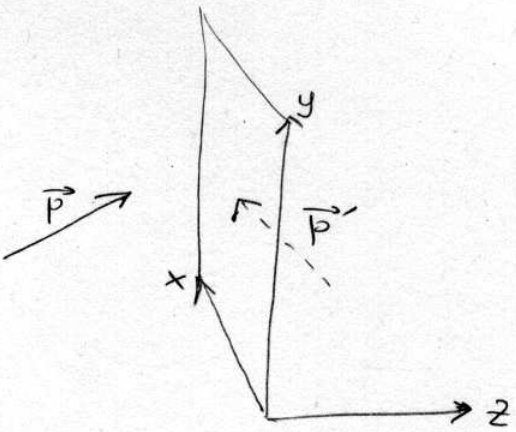
We are told that in a certain reaction the electron always comes out with its spin pointing in its direction of motion, $\vec{s} \parallel \vec{p}$. Let us imagine putting a mirror in the plane $\perp \vec{p}$:



If we imagine the spin of the electron (which is associated with a magnetic moment $\vec{\mu} \parallel \vec{s}$) as being caused by some imaginary electric ring current, we see that in the mirror that current remains unchanged, so the spin of the electron in the mirror still points in the $+z$ direction although it moves in the $-z$ direction. So the electron in the mirror represents an impossible outcome of the reaction under consideration, so the mechanism causing this reaction is not reflection symmetric. It therefore violates parity.

Exercise 11.4.4 (p. 300)

Verify that in 3-d reflection on a mirror in the x-y plane equals a parity transformation followed by a rotation by π about the z-axis.



Consider a polar vector $\vec{p} = (p_x, p_y, p_z)$

Under mirror reflection it turns into

$$\vec{p}\,' = (p_x, p_y, -p_z)$$

Under a parity transformation $\vec{p}$ turns into

$$\hat{\pi}\,\vec{p} = -\vec{p} = (-p_x, -p_y, -p_z)$$

A rotation R_z^z by an angle φ around the z-axis turns the complex number $p_\perp = p_x + i p_y$ into

$$e^{i\varphi} p_\perp = (\cos\varphi + i\sin\varphi)(p_x + i p_y) = (p_x\cos\varphi - p_y\sin\varphi) + i(p_x\sin\varphi + p_y\cos\varphi)$$

For $\varphi = \pi$ this means $p_\perp = (p_x, p_y) \longrightarrow e^{i\pi} p_\perp = -p_\perp = -(p_x, p_y)$

Hence $\vec{p}\,' = (p_x, p_y, -p_z) \xrightarrow{R_\pi^z} (-p_x, -p_y, -p_z) = -\vec{p} = \hat{\pi}\,\vec{p}$ ✓