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PHY 8360
5/9/2012
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50/50

9.7) $a_{D0}(\vec{r}) = i a_0 \vec{k}\cdot\hat{n}' \frac{e^{ikr}}{2\pi r} \int_{S_a} f_0(r') e^{-i\Delta k\cdot r'} dS'$

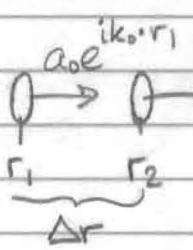
$\Delta\vec{k} = \vec{k} - \vec{k}_0$ & $\vec{k} = k\hat{r}$

$f_0(r)$ ≡ aperture function $f_0 = 1$ in aperture
$f_0 = 0$ outside aperture

for circular aperture $a(r) = i a_0 R^2 (\vec{k}\cdot\hat{n}') \frac{J_1(\Delta k R)}{\Delta k R} \frac{e^{ikr}}{r}$

b/D

a) for a plane wave $a_0 e^{ik_0\cdot r_1}$ coming out of aperture 1
& plane wave 2 $a_0 e^{ik\cdot r_2}$



then at test point P we will see

$a_0 e^{ik\cdot r_1} + a_0 e^{ik_0\cdot r_2}$

$= a_0(e^{ik\cdot r_1} + e^{ik_0\cdot r_2})$

$= a_0(e^{i\Delta k\cdot\Delta r} + 1) e^{ik\cdot r_2}$

$= a_0(e^{\frac{1}{2}\Delta k\cdot\Delta r} + e^{-\frac{1}{2}\Delta k\cdot\Delta r}) e^{ik_0\cdot r_2} e^{\frac{1}{2}\Delta k\cdot\Delta r}$

$= 2\cos(\frac{1}{2}\Delta k\cdot\Delta r)$

so then we have $2a_0\cos(\frac{1}{2}\Delta k\cdot\Delta r)$
& to find the intensity take it & square w/ complex conjugate

$\Rightarrow 4a_0a_0^*\cos^2(\frac{1}{2}\Delta k\cdot\Delta r)$

↳ this should also work generally for two slits side by side

b) $\theta_1 = 1.22\frac{\lambda}{D} \Rightarrow \frac{1}{2}\Delta k \cdot \Delta r = 1.22\frac{\lambda}{D}$ $\quad \Delta k = 1.22$

$D = 100\mu m$ distance of 1m from apertures

$\lambda = 500\mu m$

$\Delta r = 150\mu m$



$\tan\theta = \frac{y}{1m}$

Plot $\frac{R^4 J_1^2(kR\theta)}{r^2(kR\theta)^2} \cdot \cos^2\left(\frac{1}{2}k\Delta r\theta\right)$ $\quad k = \frac{2\pi}{\lambda}$

$$\Rightarrow \frac{(50\mu m)^4 J_1^2\left(\frac{2\pi}{500\mu m}(50\mu m)\theta\right)}{1m^2\left(\frac{2\pi}{500\mu m}(50\mu m)\theta\right)^2} \cdot \cos^2\left(\frac{1}{2}\frac{2\pi}{500\mu m}(150\mu m)\theta\right)$$

Problem 1b)

In collaboration w/ Jessica Brinson

In[1]:= `c = 3*10^8`

Out[1]= 300 000 000

In[2]:= `ω = c k`

Out[2]= 300 000 000 k

In[3]:= `λ = 500 * 10^-9`

Out[3]= $\frac{1}{2\,000\,000}$

In[4]:= `k = 2 π / λ`

Out[4]= 4 000 000 π

In[5]:= `R = 25 * 10^-6`

Out[5]= $\frac{1}{40\,000}$

In[6]:= `Δr = 150*10^-6`

Out[6]= $\frac{3}{20\,000}$

In[7]:= `r = 1`

Out[7]= 1

In[8]:= `θ[y_] := ArcTan[y/r]`

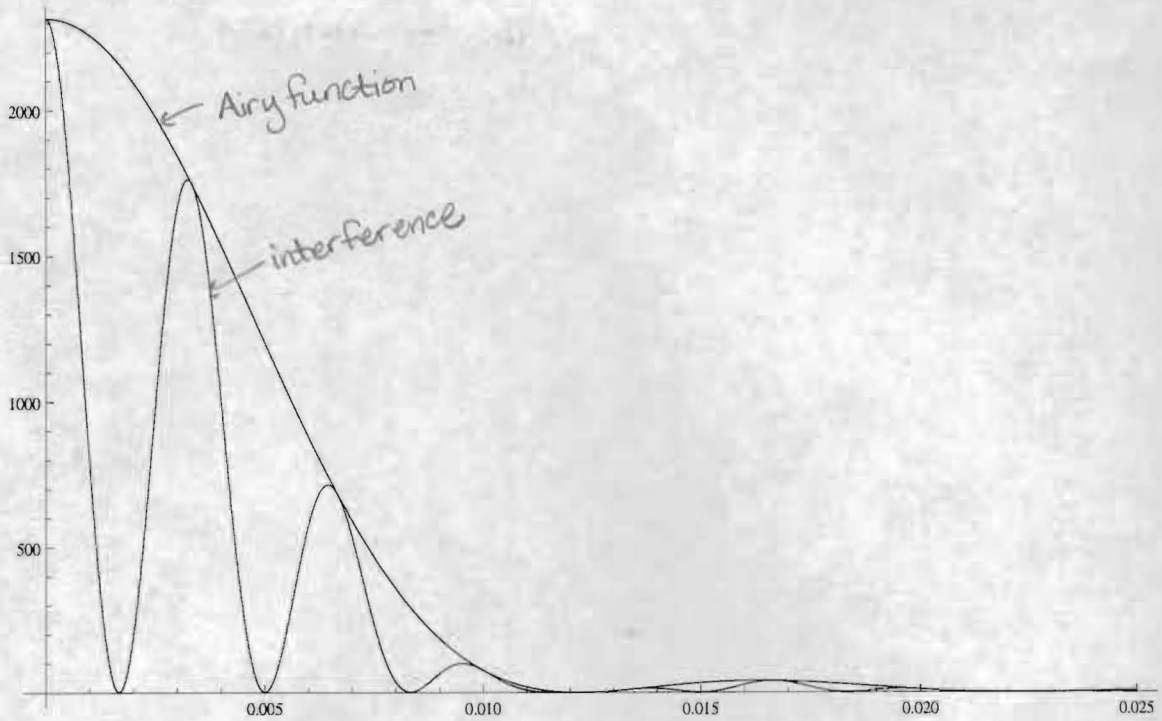
In[9]:= `S[y_] := ω k / 2 R^4 / r^2 BesselJ[1, k R θ[y]]^2 / (k R θ[y])^2 (Cos[1/2 (k θ[y] * Δr)])^2`

In[10]:= `S2[y_] := ω k / 2 R^4 / r^2 BesselJ[1, k R θ[y]]^2 / (k R θ[y])^2`

Printed by Mathematica for Students

In[17]:=
```
Plot[{S[y], S2[y]}, {y, 0, 1000 R}]
```

Out[17]=



Where we have plotted the Airy function envelope in pink and the diffraction pattern observed in blue

Problem 2c)

In[15]:=
```
Sum[x^n, {n, N}]
```

Out[15]=
$$\frac{x\left(-1+x^{N}\right)}{-1+x}$$

10/10

2) a) $a_0 = i a_0 \vec{k}\cdot\hat{n}' \frac{e^{ikr}}{2\pi r} \int f_n(r') \sum_{n=1}^{N} e^{-i\Delta k\cdot(r'-r_n)} dS'$

$$\hat{f}(\Delta k) = \frac{1}{2\pi}\int dS' f_n(r') e^{-i\Delta k\cdot r'}$$

$\therefore$ we get $a_0 = i a_0 \vec{k}\cdot\hat{n}' \frac{e^{ikr}}{r} \underbrace{\frac{1}{2\pi}\int dS' f_n(r') e^{-i\Delta k\cdot r'}}_{\hat{f}(\Delta k)} \sum_{n=1}^{N} e^{i\Delta k\cdot r_n}$

$$\Rightarrow a_0 = i a_0 \vec{k}\cdot\hat{n}' \frac{e^{ikr}}{r} \hat{f}(\Delta k) \sum_{n=1}^{N} e^{i\Delta k\cdot r_n}$$

2) b) $\hat{f}(\Delta k) = \frac{1}{2\pi}\int_{S'} e^{i\Delta k\cdot r'} f(r')\,dS'$

$= \frac{1}{2\pi}\int_{S'} e^{i\Delta k_\ell x} e^{i\Delta k_w y}\,dx\,dy = \frac{1}{2\pi}\int_0^\ell\int_0^w e^{i\Delta k_\ell x} e^{i\Delta k_w y}\,dx\,dy$

$= \frac{1}{2\pi}\, e^{i\Delta k_\ell x}\Big|_0^\ell \frac{e^{i\Delta k_w y}}{i\Delta k_w}\Big|_0^w$

$= -\frac{1}{2\pi}\frac{(e^{i\Delta k_\ell \ell}-1)(e^{i\Delta k_w w}-1)}{\Delta k_\ell \Delta k_w}$

$(e^{i\Delta k_\ell \ell} - 1) = (e^{\frac{i}{2}\Delta k_\ell \ell} - e^{-\frac{i}{2}\Delta k_\ell \ell})\, e^{-\frac{i}{2}\Delta k_\ell \ell}$

$= 2i\sin(\tfrac{1}{2}\Delta k_\ell \ell)$

$\Rightarrow \frac{4}{2\pi}\frac{\sin(\frac{1}{2}\Delta k_\ell \ell)\sin(\frac{1}{2}\Delta k_w w)}{\Delta k_\ell \Delta k_w}$

$= \frac{2}{\pi}\frac{\sin(\frac{1}{2}\Delta k_\ell \ell)\sin(\frac{1}{2}\Delta k_w w)}{\Delta k_\ell \Delta k_w}$ ✓

c) By mathematica $\sum_{n=1}^{N} = \frac{x(x^N-1)}{x-1}$ ✓ → See earlier mathematica print out

Show that $\sum_{n=1}^{N} x^n = e^{\frac{i}{2}(N+1)\phi}\frac{\sin(\frac{1}{2}N\phi)}{\sin(\frac{1}{2}\phi)}$

$$\sum_{n=1}^{N} x^n = \frac{x(x^N-1)}{x-1} \quad \text{for } x = e^{i\phi}$$

$$\Rightarrow \frac{e^{i\phi}(e^{iN\phi}-1)}{e^{i\phi}-1} \qquad e^{i\phi}-1 = (e^{i\phi/2}-e^{-i\phi/2})e^{i\phi/2} = 2i\sin(\tfrac{1}{2}\phi)e^{i\phi/2}$$

$$= \frac{e^{i\phi}(e^{iN\phi}-1)}{2i\sin(\frac{1}{2}\phi)e^{i\phi/2}} \qquad (e^{iN\phi}-1) = 2i\sin(\tfrac{1}{2}N\phi)e^{iN\phi/2}$$

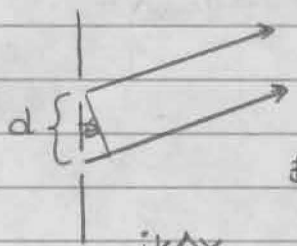
$$= e^{i\phi}\frac{2i\sin(\frac{1}{2}N\phi)e^{iN\phi/2}e^{-i\phi/2}}{2i\sin(\frac{1}{2}\phi)} \quad \checkmark$$

$$= e^{i\phi/2}e^{iN\phi/2}\frac{\sin(\frac{1}{2}N\phi)}{\sin(\frac{1}{2}\phi)} = e^{\frac{i}{2}(N+1)\phi}\frac{\sin(\frac{1}{2}N\phi)}{\sin(\frac{1}{2}\phi)}$$

So show for N apertures $\phi = \Delta k w d$

from $$\sum_{n=1}^{N} e^{i\Delta\vec{k}\cdot\vec{r}_n} = e^{i(\frac{1}{2})(N+1)\Delta k w d}\frac{\sin(\frac{1}{2}N\Delta k w d)}{\sin(\frac{1}{2}\Delta k w d)}$$

$\Delta k \cdot r_n$ which we know by the geometry of the problem the path difference is



$$\sin\theta = \frac{\Delta x}{d} \Rightarrow \Delta x = d\sin\theta$$

$$\Delta x = d\theta$$

\$ For small θ we get

then for $e^{ik\Delta x} = e^{ikd\theta}$ \$ by the Fraunhofer approx. $\Delta k = k\theta$

$\Rightarrow$ our phase factor $i\Delta k w d$ $\therefore \phi = \Delta k w d$ \$ we get ☺

$$\sum_{n=1}^{N} e^{i\Delta\vec{k}\cdot\vec{r}_n} = e^{i(\frac{1}{2})(N+1)\Delta k w d}\frac{\sin(\frac{1}{2}N\Delta k w d)}{\sin(\frac{1}{2}\Delta k w d)} \quad \checkmark$$

2)d) Show that the peaks occur at $\Delta k_{wd} = m2\pi$

$$\frac{\sin\left(\frac{1}{2}N\Delta k_{wd}\right)}{\sin\left(\frac{1}{2}\Delta k_{wd}\right)}$$

where $\sin\left(\frac{1}{2}\Delta k_{wd}\right) \Rightarrow 0, \pi, 2\pi, \ldots \Rightarrow m\pi$

$\frac{1}{2}\Delta k_{wd} = m\pi \Rightarrow \Delta k_{wd} = 2m\pi$

2) e) as we showed in 2d) we have maxima every $2m\pi$
& we want to find the minima about these maxima

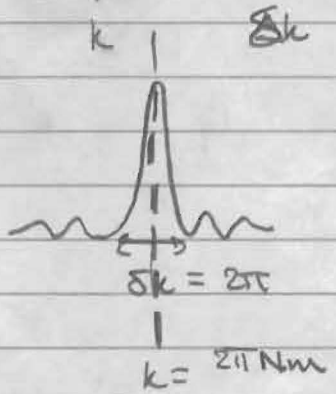
$$\Rightarrow \frac{\sin(Nx)}{\sin(x)} = 0 \Rightarrow x = \frac{m\pi}{N}$$ & we want the first minima $\Rightarrow m = 1$

$$\frac{1}{2}\Delta kd = \frac{m\pi}{N} \Rightarrow \Delta kd = \frac{2\pi}{N}$$ $\therefore$ these minima occur when

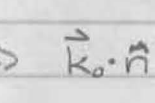
$$\Delta kd = 2m\pi \pm \frac{2\pi}{N} \Rightarrow N\Delta kd = (Nm \pm 1)2\pi$$

so then for the resolution we can resolve

$$\underbrace{Nm(2\pi)}_{k} \pm \underbrace{2\pi}_{\delta k} \Rightarrow \frac{\delta k}{k} = \frac{2\pi}{Nm(2\pi)} = \frac{1}{Nm}$$ ✓



3) 9.175) $a_N(\vec{r}) = ia_0 \vec{k}_0 \cdot \hat{n}' \frac{e^{ikr}}{2\pi r} \int_{Sa} e^{-i\Delta\vec{k}\cdot\vec{r}'} \, dS'$

normal incidence $\Rightarrow \vec{k}_0 \cdot \hat{n}' =$  $\Rightarrow$ we get $-k$

plane aperture $\Rightarrow dS' = dx'dy'$

$r \to L$

10/10 — so now we have $a_N(\vec{r}) = -ia_0 k \frac{e^{ikr}}{2\pi L} \int e^{-i\Delta\vec{k}\cdot\vec{r}'} \, dx'dy'$

$\rightarrow$ pull $e^{ik\cdot r}$ into the integral to get

$e^{ik\cdot r} e^{-i\Delta\vec{k}\cdot\vec{r}'}$ $\quad \Delta\vec{k}\cdot\vec{r}' = (\vec{k} - \vec{k}_0)\vec{r}'$ where $\vec{k}_0 \perp \vec{r}' \Rightarrow \vec{k}_0 \cdot \vec{r}' =$

$= e^{ik\cdot r} e^{-ik\cdot r'} = e^{ik\cdot\Delta r}$ where in 9.185 we get

$\Delta r \cdot \hat{n} L + \frac{r''^2}{2L} + \frac{r'^2}{2L} - \frac{r''\cdot r'}{L}$ ($\to 0$ b/c its really small square)

then the exponential gives $e^{ikL} e^{\frac{ik}{2L}(x^2+y^2)} e^{-\frac{ik}{L}(xx'+yy')}$

$\therefore a_N(\vec{r}) = -ia_0 k \frac{e^{ikL} e^{\frac{ik}{2L}(x^2+y^2)}}{2\pi L} \int_{Sa} e^{-\frac{ik}{L}(xx'+yy')} \, dx'dy'$ ✓

9.210 $$a(x,y) = -ia_0 k \frac{e^{ikL}}{2\pi L} \int_{S_a} dx'dy' e^{\frac{ik}{2L}(x'-x)^2} e^{i(k/2L)(y'-y)}$$

$(x'-x)^2 = x'^2 + x^2 - 2x'x$ $\quad x'^2$ & y'^2 are really small

$(y'-y)^2 = y'^2 + y^2 - 2y'y$

$$a(x,y) = -ia_0 k \frac{e^{ikL}}{2\pi L} e^{\frac{ik}{2L}(x^2+y^2)} \int_{S_a} dx'dy' e^{-\frac{ik}{L}(xx'+yy')}$$ ✓

circular aperture

a) R @ normal incidence by plane wave w/ amplitude a_0 & wave vector k on axis

$$a^*a = 4a_0^*a_0\sin^2\left(\frac{kR^2}{4L}\right) \xrightarrow{L\to\infty} a_0^*a_0\left(\frac{k^2R^4}{4L^2}\right)$$

$$a(x,y) = -ia_0\frac{kL}{2\pi}\int_{S_a} dx'dy' \frac{e^{ik|\vec{r}'-\vec{r}|}}{|\vec{r}'-\vec{r}|^2}$$

$$|\vec{r}'-\vec{r}| = L^2 + (x'-x)^2 + (y'-y)^2 \qquad x=0 \quad y=0$$
$$= L^2 + x'^2 + y'^2 \approx L + \frac{1}{2L}x'^2 + \frac{1}{2L}y'^2$$

$$-ia_0\frac{kL}{2\pi}\int_{S_a} dx'dy' \frac{e^{ikL}\, e^{+\frac{ik}{2L}(x'^2+y'^2)}}{L^2-x'^2-y'^2}$$

$$= -\frac{ia_0kL e^{ikL}}{2\pi L^2}\int_0^R re^{+\frac{ikr^2}{2L}}\,dr\,d\theta = -\frac{ia_0ke^{ikL}}{L}\int_0^R re^{+\frac{ikr^2}{2L}}dr$$

(b)/(D)

$$= +a_0\frac{k}{k}\left(\frac{k}{k}\right)e^{ikL}e^{+\frac{ikr^2}{2L}}\Big|_0^R = a_0e^{ikL}\left[e^{+\frac{ikR^2}{2L}}-1\right]$$

$$= a_0e^{ikL}\left(e^{\frac{ikR^2}{4L}} - e^{-\frac{ikR^2}{4L}}\right)e^{\frac{ikR^2}{4L}}$$
$$= 2ie^{ikL}e^{ikR^2/4L}\sin\left(\frac{kR^2}{4L}\right)$$

$$\Rightarrow aa^* = 4a_0^2\sin^2\left(\frac{kR^2}{4L}\right)$$

by small angle approx as $L\to\infty$ we get

$$4a_0^2\left(\frac{k^2R^4}{16L^2}\right) = \frac{a_0^2k^2R^4}{4L^2}$$

4)b) $a_0(\vec{r}) = a_0 e^{ikL}$ $\quad a_a = a_0 e^{ikL}\left[e^{ikR^2/2L} - 1\right]$

$a_s = a_0 - a_a$ ← by Babinet's principle

$a_s a_s^* = a_0 e^{ikL}\left[1 - e^{ikR^2/2L} + 1\right] a_0^* e^{-ikL}\left[1 - e^{-ikR^2/2L} + 1\right]$

$= a_0 a_0^*$

$\therefore a a^* = a_0 a_0^*$

a) Show that $a^* a = 4N^2 a_0^* a_0$

we showed in 9.10 that for a circular aperture that $a_0 = a_0 2i e^{ikL} \sin\left(\frac{kR^2}{4L}\right)$

10/10 and so for the zone plate we wish to maximize the amplitude so if we maximize $\sin\left(\frac{kR^2}{4L}\right) = 1$ for a particular value of R, k, & L

we get $a = a_0 2i e^{ikL}$ & summing over all these maximized circles all of the same amplitude we get

$a_{total} = 2a_0 iN e^{ikL}$ & thus the intensity is

$a_+ a_+^* = 4a_0 a_0^* N^2$

b) for a zone plate of radius R show that $N \cong \frac{R^2}{2L\lambda}$

again we want $\sin^2\left(\frac{kR^2}{4L}\right)$ to be a max

$\Rightarrow \frac{kR^2}{4L} = (2n+1)\frac{\pi}{2} \Rightarrow \frac{2\pi R^2}{4\lambda L} = (2n+1)\frac{\pi}{2}$

$\Rightarrow \frac{\pi R^2}{\lambda L} = (2n+1)\pi$ so for large R we'll have a lot of n

$\Rightarrow n \cong \frac{R^2}{2\lambda L}$

c) $$a(\vec{r}) = \frac{-ik}{2\pi r} e^{ik[L+(n-1)t_0+(r''^2/2L)]} \int_{S_a} a_i(\vec{r}') e^{-i\vec{k}\cdot\vec{r}'} dS'$$

if we are far away $r \to L$ as in 9.9

$$= \frac{-ik}{2\pi L} e^{i\Phi} \int_{S_a} a_i(\vec{r}') e^{-i\vec{k}\cdot\vec{r}'} dS' \qquad a_i(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} a_0$$

$$\Rightarrow \frac{-ika_0}{2\pi L} e^{i\Phi} \int_{S_a} e^{i\vec{k}\cdot\Delta\vec{r}} dS'$$ which is the same integral we evaluated in #9.10

$$\Rightarrow aa^* = 4a_0^* a_0 \sin^2\left(\frac{kR^2}{4L}\right) \text{ as } L \to \infty$$

we get $$a_0^* a_0 \frac{k^2 R^4}{4L^2} = a_0^* a \frac{4\pi^2 R^4}{\lambda^2 4L^2} = a_0^* a \frac{\pi^2 R^4}{\lambda^2 L^2}$$