PHYS 235 HW 8
SOLUTIONS

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8.4.2

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi$$

$$B = \nabla \times A$$

now, take $\begin{cases} \phi \to \phi' = \phi + \frac{1}{c}\partial_t \Lambda \\ A \to A' = A - \nabla\Lambda \end{cases}$

$$\Rightarrow E' = -\frac{1}{c}\partial_t A' - \nabla\phi'$$

$$= -\frac{1}{c}\partial_t A + \frac{1}{c}\partial_t \nabla\Lambda - \nabla\phi - \frac{1}{c}\nabla\partial_t\Lambda$$

$$= -\frac{1}{c}\partial_t A - \nabla\phi$$

and $B' = \nabla \times A'$

$$= \nabla \times (A - \nabla\Lambda)$$

$$= \nabla \times A - \underbrace{(\nabla \times \nabla)\Lambda}_{0 \text{ by vector identity}}$$

$$= \nabla \times A$$

So the fields computed using $(\phi, \vec{A})$ and $(\phi', \vec{A}')$ are identical.

(1) $\phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}$

$= \phi + \frac{1}{c}\partial_t\left(-c\int_{-\infty}^{t}\phi(r,t')dt'\right)$ $\left(\text{recall: } \partial_x \int_a^x f(x')dx' = f(x)\right)$

$= \phi - (\phi(r,t))$

$= 0$

$A' = A - \nabla\Lambda = A + c\int_{-\infty}^{t}\nabla\phi(r,t')dt'$

$\nabla\cdot A' = \nabla\cdot A + c\int_{-\infty}^{t}\nabla^2\phi(r,t')dt'$

$\neq 0$ in general.

5.1 If we had left in the $e^{i\vec{k}\cdot\vec{r}}$, Eqn 18.5.19 would read

$$H'_{fi} = N\int e^{-i\vec{p}_f\cdot\vec{r}/\hbar}\, A_0\cdot(-i\hbar\nabla)\, e^{-r/a_0}\, d^3r\, e^{i\vec{k}\cdot\vec{r}}$$

$$= N\int e^{-i(\vec{p}_f-\hbar\vec{k})\cdot\vec{r}/\hbar}\, A_0\cdot(-i\hbar\nabla) e^{-r/a_0}\, d^3r$$

Integrating by parts, 18.5.21 becomes

$$H'_{fi} = N A_0\cdot(p_f-\hbar k)\int e^{-i(\vec{p}_f-\hbar\vec{k})\cdot\vec{r}/\hbar}\, e^{-r/a_0}\, d^3r \qquad \text{(choose } \hat{z} \text{ along } \vec{p}_f$$

$$= \frac{8\pi/a_0}{\left(\left(\frac{1}{a_0}\right)^2 + \left|\frac{\vec{p}_f-\hbar\vec{k}}{\hbar}\right|^2\right)^2} \qquad (18.5.22)$$

$$\Rightarrow R_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{1}{8\pi^3\hbar^3}\,\frac{1}{\pi a_0^3}\,\frac{|\vec{A}_0\cdot(\vec{p}_f-\hbar\vec{k})|^2\, 64\pi^2 a_0^6}{\left[1+\left(\frac{a_0|\vec{p}_f-\hbar\vec{k}|}{\hbar}\right)^2\right]^4}\,\delta(E_f^0-E_i^0-\hbar$$

(18.5.23)

This is the average transition rate after a long exposure to coherent light into a precise free particle state $|p_f\rangle$. Realistically, we are more interested in the rate of production of plane wave states pointed in directions lying in a cone of solid angle $d\Omega$, with any allowed energy $E_f = \frac{p_f^2}{2m}$.

$\Rightarrow$ compute $d\Omega\int dp_f\, p_f^2\, R_{i\to f}$. Now $\int\delta(f(x))g(x)dx = \int\delta(y)\, g(f^{-1}(y))\,\frac{df^{-1}(y)}{dy}\,d$

$$= g(f^{-1}(0))\,\frac{df^{-1}(y)}{dy}\bigg|_{y=0}$$

$\Rightarrow$ for $f(x) = \frac{p_f^2}{2m} - E_i^0 - \hbar\omega$, $\frac{df^{-1}(y)}{dy} = \frac{m}{p_f}$ $\Rightarrow$ $g(p_f)\,\frac{m}{p_f}\Big|_{p_f=\sqrt{2m(E_i^0+\hbar\omega}}$

$$\Rightarrow \boxed{R_{i\to d\Omega} = \frac{4e^2a_0^3 p_f\,|\vec{A}_0\cdot(\vec{p}_f-\hbar\vec{k})|^2\,d\Omega}{m\pi\hbar^4c^2\left[1+\left(\frac{a_0|\vec{p}_f-\hbar\vec{k}|}{\hbar}\right)^2\right]^4}\Bigg|_{|\vec{p}_f|=\sqrt{2m(E_i^0+\hbar\omega)}}}$$

but we worked in the Coulomb gauge, where $\vec{A}_0\cdot\vec{k} = 0$

$$\therefore \boxed{R_{i\to d\Omega} = \frac{4e^2a_0^3 p_f\,|\vec{A}_0\cdot\vec{p}_f|^2\,d\Omega}{m\pi\hbar^4c^2\left[1+\frac{a_0^2}{\hbar^2}|\vec{p}_f-\hbar\vec{k}|^2\right]^4}\Bigg|_{|\vec{p}_f|=\sqrt{2m(E_i^0+\hbar\omega)}}}$$

This is maximized when $\vec{p}_f$ and $\vec{k}$ are parallel, reflecting the fact that the electron has an increased component in the high energy plane wave state after the perturbation if the initial e^- momentum and the photon momentum could add constructi

(1) $\sigma = \frac{128 a_0^3 \pi e^2 p_f^3}{3m\hbar^3 \omega c\left[1+\frac{p_f^2 a_0^2}{\hbar^2}\right]^4}$ $\qquad E_f = \frac{p_f^2}{2m} = 10R_y = \frac{10me^4}{2\hbar^2}$

Now $a_0 = \frac{\hbar^2}{me^2}$, so $\frac{p_f^2 a_0^2}{\hbar^2} = \frac{2m \cdot 10me^4\hbar^4}{\hbar^2 \cdot 2\hbar^2 m^2 e^4} = 10$

$$\Rightarrow \sigma = \frac{128\pi e^2}{3m\omega c} \cdot 10^{3/2} \cdot \frac{1}{(1+10)^4}$$

And the e^- started in the Hydrogen ground state, $E_i = -R_y$

$$\Rightarrow 10R_y = -R_y + \hbar\omega$$
$$\Rightarrow \hbar\omega = 11R_y = \frac{11me^4}{2\hbar^2}$$

So $\sigma = \frac{128\pi \cdot 10^{3/2}}{3 \cdot 11^4} \cdot \underbrace{\frac{e^2\hbar}{mc} \cdot \frac{2\hbar^2}{11me^4}}$

$$\frac{2\hbar^3}{11m^2ce^2} = \frac{2\hbar^4 e^2}{11m^2c e^4 \hbar} = \frac{2}{11} a_0^2 \frac{e^2}{\hbar c} = \frac{2}{11}\alpha a_0^2$$

$$\Rightarrow \sigma = \frac{128 \cdot 10^{3/2}}{3 \cdot 11^4} \cdot \frac{2}{11}\alpha \cdot \pi a_0^2 \quad , \quad \alpha = \frac{1}{137} \quad , \quad \sqrt{10} \simeq 3$$

$$\simeq (128\alpha)\left(\frac{10^{1/2}}{3}\right)\left(\frac{10}{11}\right)\cdot\left(\frac{2}{11^4}\right) = \boxed{(\pi a_0^2)\cdot 10^{-4}}$$
$$\simeq 1 \quad \simeq 1 \quad \simeq 1 \quad \simeq 10^{-4}$$

(2) In the limit $\frac{p_f a_0}{\hbar Z} \gg 1$, $\sigma \to \frac{128\pi e^2 \hbar^5}{p_f^5 a_0^5 \cdot 3m\omega c}$. The Z could appear either with e^2 or a_0. But the e^2 came from $(H'_{fi})^2$, the interaction between light & the electron - this doesn't care about the nucleus charge. On the other hand, we know the Bohr radius is tightened to a_0/Z. So $\sigma \sim Z^5$, because $\sigma \sim a_0^{-5}$.