Homework 10

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2.7.4

It is trivial that

$$\{\bar{x},\bar{x}\} = \{\bar{y},\bar{y}\} = \{\bar{p}_x,\bar{p}_x\} = \{\bar{p}_y,\bar{p}_y\} = 0$$

$$\{\bar{x},\bar{y}\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial p_x} - \frac{\partial\bar{x}}{\partial p_x}\frac{\partial\bar{y}}{\partial x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{y}}{\partial p_y} - \frac{\partial\bar{x}}{\partial p_y}\frac{\partial\bar{y}}{\partial y} = 0$$

$$\{\bar{x},\bar{p}_x\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_x}{\partial p_x} - \frac{\partial\bar{x}}{\partial p_x}\frac{\partial\bar{p}_x}{\partial x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_x}{\partial p_y} - \frac{\partial\bar{x}}{\partial p_y}\frac{\partial\bar{p}_x}{\partial y} = \cos^2\theta + \sin^2\theta = 1$$

$$\{\bar{x},\bar{p}_y\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x} - \frac{\partial\bar{x}}{\partial p_x}\frac{\partial\bar{p}_y}{\partial x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} - \frac{\partial\bar{x}}{\partial p_y}\frac{\partial\bar{p}_y}{\partial y} = \cos\theta\sin\theta - \sin\theta\cos\theta = 0$$

$$\{\bar{y},\bar{p}_x\} = \frac{\partial\bar{y}}{\partial x}\frac{\partial\bar{p}_x}{\partial p_x} - \frac{\partial\bar{y}}{\partial p_x}\frac{\partial\bar{p}_x}{\partial x} + \frac{\partial\bar{y}}{\partial y}\frac{\partial\bar{p}_x}{\partial p_y} - \frac{\partial\bar{y}}{\partial p_y}\frac{\partial\bar{p}_y}{\partial y} = \sin\theta\cos\theta - \cos\theta\sin\theta = 0$$

$$\{\bar{y},\bar{p}_y\} = \frac{\partial\bar{y}}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x} - \frac{\partial\bar{y}}{\partial p_x}\frac{\partial\bar{p}_y}{\partial x} + \frac{\partial\bar{y}}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} - \frac{\partial\bar{y}}{\partial p_y}\frac{\partial\bar{p}_y}{\partial y} = \sin^2\theta + \cos^2\theta = 1$$

$$\{\bar{p}_x,\bar{p}_y\} = \frac{\partial\bar{p}_x}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x} - \frac{\partial\bar{p}_x}{\partial p_x}\frac{\partial\bar{p}_y}{\partial x} + \frac{\partial\bar{p}_x}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} - \frac{\partial\bar{p}_x}{\partial p_y}\frac{\partial\bar{p}_y}{\partial y} = 0$$

Therefore, the Poisson Brackets are preserved, and the transformation is a canonical transformation.

2.8.1

$$\bar{q}_1 = q_1 + \epsilon\frac{\partial p}{\partial p_1} = q_1 + \epsilon$$

$$\bar{q}_2 = q_2 + \epsilon\frac{\partial p}{\partial p_2} = q_2 + \epsilon$$

$$\bar{p}_1 = p_1 - \epsilon\frac{\partial p}{\partial q_1} = p_1$$

$$\bar{p}_2 = p_2 - \epsilon\frac{\partial p}{\partial q_2} = p_2$$

Therefore, $p = p_1 + p_2$ is the generator of infinitesimal translations for a two-particle system.

2.8.2

$$\bar{q}_i = q_i + \epsilon\frac{\partial g}{\partial p_i}$$

$$\bar{p}_j = p_j - \epsilon\frac{\partial g}{\partial q_j}$$

$$\{\bar{q}_i,\bar{p}_j\} = \sum_k \frac{\partial\bar{q}_i}{\partial q_k}\frac{\partial\bar{p}_j}{\partial p_k} - \frac{\partial\bar{q}_i}{\partial p_k}\frac{\partial\bar{p}_j}{\partial q_k}$$

$$= \sum_k \left(\delta_{ik} + \epsilon\frac{\partial^2 g}{\partial p_i\partial q_k}\right)\left(\delta_{jk} - \epsilon\frac{\partial^2 g}{\partial q_j\partial p_k}\right) - \epsilon\frac{\partial g}{\partial p_i\partial p_k}\cdot\epsilon\frac{\partial^2 g}{\partial q_i\partial q_k}$$

$$= \sum_k \delta_{ik}\delta_{jk} + \epsilon \frac{\partial^2 g}{\partial p_i \partial q_k}\cdot\delta_{jk} - \delta_{ik}\frac{\partial^2 g}{\partial q_j \partial p_k} + O(\epsilon^2)$$

$$= \delta_{ij} + \epsilon\frac{\partial^2 g}{\partial p_i \partial q_j} - \epsilon\frac{\partial^2 g}{\partial q_j \partial p_i} + O(\epsilon^2)$$

$$= \delta_{ij} + O(\epsilon^2)$$

$$\approx \delta_{ij}$$

— Therefore, the poisson brackets are preserved and g is a canonical transformation.

2.8.3

Rotation of just the coordinates:

$$\begin{cases} \bar{x} = x\cos\theta - y\sin\theta \\ \bar{y} = x\sin\theta + y\cos\theta \end{cases} \qquad \begin{cases} \bar{p}_x = p_x \\ \bar{p}_y = p_y \end{cases}$$

$$\{\bar{x},\bar{y}\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial p_x} - \frac{\partial\bar{x}}{\partial p_x}\frac{\partial\bar{y}}{\partial x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{y}}{\partial p_y} - \frac{\partial\bar{x}}{\partial p_y}\frac{\partial\bar{y}}{\partial y} = 0$$

$$\{\bar{p}_x,\bar{p}_y\} = \{p_x,p_y\} = 0$$

$$\{\bar{x},\bar{p}_y\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x} - \frac{\partial\bar{x}}{\partial p_x}\frac{\partial\bar{p}_y}{\partial x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} - \frac{\partial\bar{x}}{\partial p_y}\frac{\partial\bar{p}_y}{\partial y} = -\sin\theta \neq 0$$

$$\{\bar{x},\bar{p}_x\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_x}{\partial p_x} - \frac{\partial\bar{x}}{\partial p_x}\frac{\partial\bar{p}_x}{\partial x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_x}{\partial p_y} - \frac{\partial\bar{x}}{\partial p_y}\frac{\partial\bar{p}_x}{\partial y} = \cos\theta \neq 1$$

$$\{\bar{y},\bar{p}_x\} = \frac{\partial\bar{y}}{\partial x}\frac{\partial\bar{p}_x}{\partial p_x} - \frac{\partial\bar{y}}{\partial p_x}\frac{\partial\bar{p}_x}{\partial x} + \frac{\partial\bar{y}}{\partial y}\frac{\partial\bar{p}_x}{\partial p_y} - \frac{\partial\bar{y}}{\partial p_y}\frac{\partial\bar{p}_x}{\partial y} = \sin\theta \neq 0$$

$$\{\bar{y},\bar{p}_y\} = \frac{\partial\bar{y}}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x} - \frac{\partial\bar{y}}{\partial p_x}\frac{\partial\bar{p}_y}{\partial x} + \frac{\partial\bar{y}}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} - \frac{\partial\bar{y}}{\partial p_y}\frac{\partial\bar{p}_y}{\partial y} = \cos\theta \neq 1$$

Therefore, some of the poisson brackets are not preserved, and the transformation is not a canonical transformation.

If $\delta H = \epsilon\{H,g\} = \epsilon\left(\frac{\partial H}{\partial x}\frac{\partial g}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial g}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial g}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial g}{\partial y}\right)$,

we have $\delta x = \epsilon\frac{\partial g}{\partial p_x}$, $\delta p_x = -\epsilon\frac{\partial g}{\partial x}$, $\delta y = \epsilon\frac{\partial g}{\partial p_y}$, $\delta p_y = -\epsilon\frac{\partial g}{\partial y}$.

$$\Rightarrow \begin{cases} \bar{x} = x + \epsilon\frac{\partial g}{\partial p_x} \\ \bar{y} = y + \epsilon\frac{\partial g}{\partial p_y} \end{cases} \qquad \begin{cases} \bar{p}_x = p_x - \epsilon\frac{\partial g}{\partial x} \\ \bar{p}_y = p_y - \epsilon\frac{\partial g}{\partial y} \end{cases}$$

According to 2.8.2, this is a canonical transformation. Therefore, there doesn't exist any g, such that $\delta H = \epsilon\{H,g\}$.

10.2.2

(1) We start with the following Hamiltonian:

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2$$

Introduce creation and annihilation operators as follows:

$$\begin{aligned}
a_x &\equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_x}{\hbar}}x + i\sqrt{\frac{1}{m\hbar\omega_x}}p_x\right)\\
a_x^\dagger &\equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_x}{\hbar}}x - i\sqrt{\frac{1}{m\hbar\omega_x}}p_x\right)\\
a_y &\equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_y}{\hbar}}y + i\sqrt{\frac{1}{m\hbar\omega_y}}p_y\right)\\
a_y^\dagger &\equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_y}{\hbar}}y - i\sqrt{\frac{1}{m\hbar\omega_y}}p_y\right)
\end{aligned}$$

Then we have that:

$$\begin{aligned}
[a_x, a_y] = [a_x, a_y^\dagger] = [a_x^\dagger, a_y] = [a_x^\dagger, a_y^\dagger] &= 0\\
[a_x, a_x^\dagger] &= \frac{1}{2}\frac{i}{\hbar}\left([p,x] - [x,p]\right)\\
&= \frac{1}{2}\frac{i}{\hbar}\cdot -2i\hbar\\
&= 1\\
[a_y, a_y^\dagger] &= 1, \text{ analogously}
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\hbar\omega_x\left(a_x^\dagger a_x + \frac{1}{2}\right) &= \frac{1}{2}\hbar\omega_x\left(\frac{m\omega_x}{\hbar}x^2 + \frac{1}{m\hbar\omega}p_x^2 - \frac{i}{\hbar}[p_x, x] + 1\right)\\
&= \frac{1}{2}m\omega_x^2x^2 + \frac{p_x^2}{2m} + \hbar\omega\left(-\frac{i}{\hbar}\cdot -i\hbar + 1\right)\\
&= \frac{p_x^2}{2m} + \frac{1}{2}m\omega_x^2x^2
\end{aligned}$$

Thus

$$H = \hbar\omega_x\left(a_x^\dagger a_x + \frac{1}{2}\right) + \hbar\omega_y\left(a_y^\dagger a_y + \frac{1}{2}\right)$$

We know the spectrum of a single harmonic oscillator. (If you don't remember, it comes from the fact that $a^\dagger a$ is a "number operator", that is, it has eigenvalues of all non-negative integers.) Using this and the fact that the x operators (that is, $a_x, a_x^\dagger$) and y operators commute, we have:

$$E = \hbar\omega_x\left(n_x + \frac{1}{2}\right) + \hbar\omega_y\left(n_y + \frac{1}{2}\right)$$

for non-negative integers n_x, n_y.

(2) The combined harmonic oscillator will have a ground state wavefunction that is the product of ground state wavefunctions for an oscillator in the x direction (with frequency ω_x) and in the y direction (with frequency ω_y). This can be seen in an easy way: we know that there is exactly one ground state of the combined harmonic oscillator, with energy $E = \frac{1}{2}\hbar(\omega_x + \omega_y)$ so we attempt a solution of the time independent Schrödinger equation using a seperable solution $\psi(x,y) = \psi_x(x)\psi_y(y)$. The time independent Schrödinger equation is exactly the

equation we derived previously without the modification due to oscillator coupling. We show how to solve this:

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{2}m\omega_x^2x^2+\frac{1}{2}m\omega_y^2y^2\right)\psi_x(x)\psi_y(y)=\frac{1}{2}\hbar(\omega_x+\omega_y)\psi_x(x)\psi_y(y)$$

$$\underbrace{\frac{1}{\psi_x}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi_x}{\partial x^2}+\frac{1}{2}m\omega_x^2x^2\psi_x\right)}_{=f(x)}+\underbrace{\frac{1}{\psi_y}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi_y}{\partial y^2}+\frac{1}{2}m\omega_y^2y^2\psi_y\right)}_{=g(y)}=\frac{1}{2}\hbar(\omega_x+\omega_y)$$

thus the highlighted parts are each constants since we have equated a function of x only, a function of y only, and a constant. We are free to try the solution $f(x)=\frac{1}{2}\hbar\omega_x$, $g(x)=\frac{1}{2}\hbar\omega_y$, which produces the following equations:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_x}{dx^2}+\frac{1}{2}m\omega_x^2x^2\psi_x=\frac{1}{2}\hbar\omega_x\psi_x$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_y}{dy^2}+\frac{1}{2}m\omega_y^2y^2\psi_y=\frac{1}{2}\hbar\omega_y\psi_y,$$

but we already know a solution to these! The solution is the harmonic oscillator ground state wavefunctions for x and y. Thus we have found a solution for ψ, and it satisfies the appropriate boundary conditions (we can set the normalization with a constant, and we have that $\psi\to 0$ as we approach infinity in any direction. Thus it is the solution for which we have been looking. We know furthermore that the state corresponding to n_x, n_y is $(a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}|0\rangle$, and since the $a_x^\dagger$ operators will only affect the x part of the wavefunction, and similarly for the $a_y^\dagger$ operators for the y part of the wavefunction, we have that $\psi_{n_x,n_y}=\psi_{n_x}(x)\psi_{n_y}(y)$, where ψ_{n_x} is the wavefunction for the n_xth energy eigenstate for a harmonic oscillator with frequency ω_x, and similarly for ψ_{n_y} (with frequency ω_y).

Under a parity transformation $x\to -x, y\to -y$, the wavefunction picks up a factor of $(-1)^{n_x}$ from changing $x\to -x$ in $\psi_{n_x}(x)$, and a similar factor of $(-1)^{n_y}$ from $\psi_{n_y}(y)$, so thus the overall factor is $(-1)^{n_x+n_y}$. Thus the eigenstates have definite parity, and that parity is $(-1)^{n_x+n_y}$ (where 1 indicates even under parity, -1 indicates odd under parity; this is standard notation, and if you read about the parity operator you'll see where it comes from).

(3) For $\omega_x=\omega_y=\omega$, we know the general eigenstate has wavefunction:

$$\psi_{n_x,n_y}(x,y)=$$
$$\left(\frac{m\omega}{\pi\hbar 2^{n_x+n_y}n_x!n_y!}\right)^{1/2}\exp\left(-\frac{m\omega(x^2+y^2)}{2\hbar}\right)H_{n_x}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}x\right]H_{n_y}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}y\right]$$

We can easily write this down for the lowest three energy eigenstates, which correspond to $n_x=n_y=0$ (lowest), and the two with $n_x=1,n_y=0$, $n_x=0,n_y=1$ (degenerate, next lowest):

$$\psi_{0,0}(x,y)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m\omega(x^2+y^2)}{2\hbar}\right)$$
$$\psi_{1,0}(x,y)=\sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)x\exp\left(-\frac{m\omega(x^2+y^2)}{2\hbar}\right)$$
$$\psi_{0,1}(x,y)=\sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)y\exp\left(-\frac{m\omega(x^2+y^2)}{2\hbar}\right)$$

These are also easily re-expressable in terms of polar coordinates:

$$\psi_{0,0}(\rho,\phi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega\rho^2}{2\hbar}\right)$$
$$\psi_{1,0}(\rho,\phi) = \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)\rho\cos(\phi)\exp\left(-\frac{m\omega\rho^2}{2\hbar}\right)$$
$$\psi_{0,1}(\rho,\phi) = \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)\rho\sin(\phi)\exp\left(-\frac{m\omega\rho^2}{2\hbar}\right)$$

We also have that the formula for the energy given n_x, n_y is

$$E = \hbar\omega(n_x + n_y + 1)$$

Thus the states with $n_x + n_y = n$ are all degenerate. So the state with $E = (n+1)\hbar\omega$ is $n+1$ degenerate: there are $n+1$ ways to choose $n_x + n_y = n$ if n_x, n_y are non-negative integers. (n_x can take any value 0 to n, hence we are not "off by 1").

10.3.3

(1) The particles are to be considered distinguishable. Any number of particles can be in any one state.

So each of three particles A, B, C can be placed in any of one of three states. Hence there exist a total of ${}_3\Pi_3 = 3^3 = 27$ possible states. $\Rightarrow$ The particles are said to obey "Maxwell-Boltzmann Statistics"

(2) The particles are to be considered indistinguishable. This implies that A = B = C, so that the states in the case (1) which differed only in the interchange of particles are now no longer to be considered distinct. Hence there exist a total of 10 possible states. (Refer to the table below.)

(3) The particles are to be considered indistinguishable. No more than one particle can be in any state. Hence there exists only one possible state.

Case (2)			Case (3)		
a	b	c	a	b	c
AAA			A	A	A
	AAA				
		AAA			
A	AA				
A		AA			
AA	A				
	A	AA			
	AA	A			
AA		A			
A	A	A			