Homework 2 (1.6.1 ~ 1.8.10)

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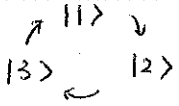
1.6.1

$U|1\rangle = |2\rangle$, $U|2\rangle = |3\rangle$, $U|3\rangle = |1\rangle$



This is a cyclic permutation of the three vectors.
It is also equivalent to rotation of the coordinate axis along (1,1,1) by 120°.

1.6.2

(1) $(\Omega\Lambda)^+ = \Lambda^+\Omega^+ = \Lambda\Omega \neq \Omega\Lambda$ not Hermitian

(2) $(\Omega\Lambda+\Lambda\Omega)^+ = (\Omega\Lambda)^+ + (\Lambda\Omega)^+ = \Lambda^+\Omega^+ + \Omega^+\Lambda^+ = \Lambda\Omega + \Omega\Lambda = \Omega\Lambda + \Lambda\Omega$ Hermitian

(3) $[\Omega,\Lambda]^\dagger = (\Omega\Lambda - \Lambda\Omega)^+ = (\Omega\Lambda)^+ - (\Lambda\Omega)^+ = \Lambda^+\Omega^+ - \Omega^+\Lambda^+ = \Lambda\Omega - \Omega\Lambda$
$= -[\Omega,\Lambda]$ anti-Hermitian

(4) $(i[\Omega,\Lambda])^+ = -i[\Omega,\Lambda]^+ = -i\cdot(-[\Omega,\Lambda]) = i[\Omega,\Lambda]$ Hermitian

1.6.3

Suppose U_1, U_2 are unitary. $U_1^+U_1 = U_2^+U_2 = I$.

Therefore, $(U_1U_2)^+(U_1U_2) = U_2^+U_1^+U_1U_2 = U_2^+(U_1^+U_1)U_2 = U_2^+ I U_2$
$= U_2^+U_2 = I$.

Hence, U_1U_2 is unitary.

1.6.4

$\det(U^+U) = \det I = 1$

$\det(U^+U) = \det(U^+)\det U = \det(U^T)^*\det U = (\det U)^*(\det U)$
$= |\det U|^2$

Therefore, $|\det U| = 1$, which indicates that $\det U$ is a complex number of unit modulus.

1.6.5

According to Example 1.6.1, $R(\frac{1}{2}\pi i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Therefore, $R^+R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$.

Hence, $R(\frac{1}{2}\pi i)$ is unitary.

Unitary

1.6.6

① $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^+ \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$

$\det\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right] = \left(\frac{1}{\sqrt{2}}\right)^2(1\cdot 1 - i\cdot i) = 1 = e^{i\theta}$, when $\theta = 2k\pi$ for integer k.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \neq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ Not Hermitian.

② $\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}^+ \cdot \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = I$ Unitary

$\det\left[\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\right] = \left(\frac{1}{2}\right)^2\left[(1+i)^2 - (1-i)^2\right] = \frac{1}{4}(2i+2i) = i$

$= e^{i\theta}$, when $\theta = 2k\pi + \frac{\pi}{2}$ for integer k.

$\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}^+ = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix} \neq \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$ Not Hermitian.

1.7.1

(1) $\mathrm{Tr}(\Omega\Lambda) = \sum_i (\Omega\Lambda)_{ii} = \sum_i\sum_j \Omega_{ij}\Lambda_{ji} = \sum_j\sum_i \Lambda_{ji}\Omega_{ij} = \sum_j (\Lambda\Omega)_{jj} = \mathrm{Tr}(\Lambda\Omega)$.

(2) $\mathrm{Tr}(\Omega\Lambda\Theta) = \sum_i (\Omega\Lambda\Theta)_{ii} = \sum_i\sum_j\sum_k \Omega_{ij}\Lambda_{jk}\Theta_{ki} = \sum_j\sum_k\sum_i \Lambda_{jk}\Theta_{ki}\Omega_{ij}$

$= \sum_j (\Lambda\Theta\Omega)_{jj} = \mathrm{Tr}(\Lambda\Theta\Omega)$

$= \sum_k\sum_i\sum_j \Theta_{ki}\Omega_{ij}\Lambda_{jk} = \sum_k (\Theta\Omega\Lambda)_{kk} = \mathrm{Tr}(\Theta\Omega\Lambda)$

(3) $\mathrm{Tr}(U^+\Omega U) = \mathrm{Tr}(\Omega UU^+) = \mathrm{Tr}(\Omega I) = \mathrm{Tr}\,\Omega$.

1.7.2

$$\begin{aligned}\det(U^+\Omega U) &= \det U^+ \det\Omega \det U \\ &= \det\Omega(\det U^+ \det U) \\ &= \det\Omega \det(U^+U) \\ &= \det\Omega \cdot 1 \\ &= \det\Omega .\end{aligned}$$

1.8.1

(1) $$\det(\Omega - I\omega) = \begin{vmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{vmatrix} = (1-\omega)(2-\omega)(4-\omega) = 0$$

$\omega = 1, 2, 4$

$$\omega=1: \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow |\omega=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\omega=2: \begin{pmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow |\omega=2\rangle = \frac{1}{\sqrt{30}}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\omega=4: \begin{pmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow |\omega=4\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

(2) $\Omega^+ \neq \Omega$ NOT Hermitian

$\langle\omega=1|\omega=2\rangle \neq 0$ $\langle\omega=1|\omega=4\rangle \neq 0$ $\langle\omega=2|\omega=4\rangle \neq 0$.

The eigenvectors are not orthogonal.

1.8.2

(1) $\Omega^+ = \Omega$ Hermitian

(2) $$\det(\Omega - I\omega) = \begin{vmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{vmatrix} = -\omega(\omega^2-1) = 0$$

$\omega = 0, 1, -1$.

$\omega = 0$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\omega = 1$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\omega = -1$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(3) $U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad U^+ = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$U^+\Omega U = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1.8.3

(1) $$\det(\Omega - I\omega) = \begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2}-\omega \end{vmatrix} = (1-\omega)\left[\left(\frac{3}{2}-\omega\right)^2 - \frac{1}{4}\right] = 0$$

$\omega = 1$ or $\omega^2 - 3\omega + 2 = 0$

$\Rightarrow \omega_1 = \omega_2 = 1, \quad \omega_3 = 2.$

(2) $\omega = 2$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} x = 0 \\ y + z = 0 \end{matrix}$

Set $y = a$, we have $z = -a$.

Therefore, $|\omega=2\rangle = \frac{1}{\sqrt{2a^2}}\begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$

(3) $\omega = 1$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ x is arbitrary, set it to be b, $y - z = 0$, Set $y = c$, we have $z = c$

Therefore, $|\omega=1\rangle = \frac{1}{\sqrt{b^2+2c^2}}\begin{pmatrix} b \\ c \\ c \end{pmatrix}.$

1.8.4

(1) $\det(\Omega - I\omega) = \begin{vmatrix} 4-\omega & 1 \\ -1 & 2-\omega \end{vmatrix} = \omega^2 - 6\omega + 9 = 0$

$\omega_1 = \omega_2 = 3$.

(2) $\omega = 3$: $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad x + y = 0$

Set $x = a$, we have $y = -a$.

Therefore, $|\omega = 3\rangle = \frac{1}{\sqrt{2a^2}}\begin{pmatrix} a \\ -a \end{pmatrix}$.

This is the only eigenvector we can find.

1.8.5

(1) $\Omega^+\Omega = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

(2) $\det(\Omega - I\omega) = \begin{vmatrix} \cos\theta - \omega & \sin\theta \\ -\sin\theta & \cos\theta - \omega \end{vmatrix} = \omega^2 - 2\omega\cos\theta + 1 = 0$

$\omega = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

(3) $\omega = e^{i\theta}$: $\begin{pmatrix} \cos\theta - e^{i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{i\theta} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$.

$\Rightarrow \quad -ix + y = 0 \quad \Rightarrow \quad |\omega = e^{i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\omega = e^{-i\theta}$: $\begin{pmatrix} \cos\theta - e^{-i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{-i\theta} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$.

$\Rightarrow \quad ix + y = 0 \quad \Rightarrow \quad |\omega = e^{-i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

$\langle \omega = e^{i\theta} | \omega = e^{-i\theta} \rangle = \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1 + i^2) = 0$.

Orthogonal.

(4) $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$

$$U^+\Omega U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

1.8.6

(1) Suppose U is the unitary matrix that transforms Ω in to diagonal matrix.

$\det \Omega = \det(U^+\Omega U) = \det D = \prod_{i=1}^{n} \omega_i$.

Where D is the corresponding diagonal matrix with Ω's eigenvalues ω_i on its diagonal.

(2) $\mathrm{Tr}\,\Omega = \mathrm{Tr}(U^+\Omega U) = \mathrm{Tr}\, D = \sum_{i=1}^{n} \omega_i$.

1.8.7

According to 1.8.6, we have

$$\begin{cases} \omega_1 + \omega_2 = \mathrm{Tr}\,\Omega = 1+1 = 2 \\ \omega_1\omega_2 = \det\Omega = 1\times 1 - 2\times 2 = -3 \end{cases}$$

$$\Rightarrow \begin{cases} \omega_1 = -1 \\ \omega_2 = 3 \end{cases} \text{ or } \begin{cases} \omega_1 = 3 \\ \omega_2 = -1 \end{cases}.$$

Verification.

$$\det(\Omega - I\omega) = \begin{vmatrix} 1-\omega & 2 \\ 2 & 1-\omega \end{vmatrix} = 0$$

$$\Rightarrow \omega^2 - 2\omega - 3 = 0 \Rightarrow \omega_1 = 3,\ \omega_2 = -1.$$

1.8.8

(1) $M^iM^j + M^jM^i = 2\delta^{ij} I \Rightarrow M^iM^i = I$

Therefore, $M^iM^i|\omega\rangle = M^i(\omega|\omega\rangle) = \omega^2|\omega\rangle$

and $M^iM^i|\omega\rangle = I|\omega\rangle = |\omega\rangle$.

$\Rightarrow \omega^2 = 1 \Rightarrow \omega = \pm 1$.

(2) $M^i M^j = -M^j M^i$

$\Rightarrow M^j M^i M^j = -M^j M^j M^i = -M^i$

$\Rightarrow \mathrm{Tr}\, M^i = \mathrm{Tr}(-M^j M^i M^j) = -\mathrm{Tr}(M^j M^i M^j)$.

$= -\mathrm{Tr}(M^i M^j M^j) = -\mathrm{Tr}(M^i \underline{I})$

$= -\mathrm{Tr}\, M^i$

$\Rightarrow \mathrm{Tr} M^i = 0$.

(3) According to 1.8.6

$\mathrm{Tr}\, M^i = \sum_{k=1}^{n} \omega_k$, where n is the dimension of the matrix.

Since $\omega_k = \pm 1$, $\mathrm{Tr}\, M^i$ can be zero only if n is even.

* The sum of an odd number of odd numbers is still odd, and cannot be zero.

1.8.9.

$\vec{l} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$

$= \sum_\alpha m_\alpha [\vec{\omega}(\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha(\vec{r}_\alpha \cdot \vec{\omega})]$

$= \sum_\alpha m_\alpha [\vec{\omega} \cdot r_\alpha^2 - \vec{r}_\alpha(\vec{r}_\alpha \cdot \vec{\omega})]$

$\Rightarrow l_i = \sum_\alpha m_\alpha [\omega_i r_\alpha^2 - (\vec{r}_\alpha)_i (\vec{r}_\alpha \cdot \vec{\omega})]$

$= \sum_\alpha m_\alpha [\omega_i r_\alpha^2 - (\vec{r}_\alpha)_i \sum_j (\vec{r}_\alpha)_j \omega_j]$

$= \sum_j \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j] \omega_j$

$\equiv \sum_j M_{ij} \omega_j$,

where $M_{ij} = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j]$.

Or $|l\rangle = M |\omega\rangle$.

(1) No. The angular momentum and angular velocity are not parallel unless $|\omega\rangle$ is an eigenvector of M.

(2) $M_{ji}^* = \left(\sum_\alpha m_\alpha [r_\alpha^2 \delta_{ji} - (\vec{r}_\alpha)_j (\vec{r}_\alpha)_i]\right)^*$

Since m_α, r_α^2, $(\vec{r}_\alpha)_j$, $(\vec{r}_\alpha)_i$ are all real,

$M_{ji}^* = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ji} - (\vec{r}_\alpha)_j (\vec{r}_\alpha)_i] = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j] = M_{ij}$.

Hermitian.

(3) Since M is Hermitian, we can always find three eigenvectors which are orthogonal to each other by solving the eigen-problem $M|\omega\rangle = \omega|\omega\rangle$. And these three eigenvectors denote the three directions for $\vec{\omega}$ we want to find in the 3-D Euclidean space.

(4) The complete symmetry of sphere means all directions are equivalent eigendirections. Therefore, the eigenvalues are degenerate.

1.8.10

$$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} = 0$$

Since the two Hermitian matrices commute, they can be diagonalized simultaneously.

Starting from Λ,

$$\det(\Lambda - I\omega) = \begin{vmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{vmatrix} = (2-\omega)(\omega+1)(\omega-3) = 0$$

$\omega = -1, 2, 3$.

$\omega = -1$: $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=-1\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\omega = 2$: $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\omega = 3$: $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |\omega=3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U^{+}\Lambda U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$U^{+}\Omega U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$