Homework 3 (1.8.11 ~ 1.10.3)

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1.8.11

(1) Equation of motion $\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Set $\begin{vmatrix} -\frac{2k}{m}+\omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m}+\omega^2 \end{vmatrix} = 0$, we have $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{k}{m}}$.

The corresponding eigenvectors are $|\omega_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $|\omega_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Set $\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. $\Lambda^+\begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}\Lambda = \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix}$.

In eigenbasis,

$$\begin{pmatrix} \ddot{X}_I \\ \ddot{X}_{II} \end{pmatrix} = \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix}\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}$$

$$\Rightarrow \begin{cases} X_I(t) = X_I(0)\cos\omega_1 t \\ X_{II}(t) = X_{II}(0)\cos\omega_2 t. \end{cases} \quad (1)$$

In this problem, $|1\rangle = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We first transform it into eigenbasis.

$$\begin{pmatrix} X_I(0) \\ X_I(0) \end{pmatrix} = \Lambda^+\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the eigenbasis, the state evolves according to (1),

$$\begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\omega_1 t \\ \frac{1}{\sqrt{2}}\cos\omega_2 t \end{pmatrix}.$$

Then, we tranform it back to the original basis.

$$|1(t)\rangle = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \Lambda\begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}\cos\omega_1 t \\ \frac{1}{\sqrt{2}}\cos\omega_2 t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos\omega_1 t + \frac{1}{2}\cos\omega_2 t \\ -\frac{1}{2}\cos\omega_1 t + \frac{1}{2}\cos\omega_2 t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\cos\sqrt{\frac{3k}{m}}t + \frac{1}{2}\cos\sqrt{\frac{k}{m}}t \\ -\frac{1}{2}\cos\sqrt{\frac{3k}{m}}t + \frac{1}{2}\cos\sqrt{\frac{k}{m}}t \end{pmatrix}.$$

(2) By substituting $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Eq.(1.8.31), we can get the same solution.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos\sqrt{\frac{k}{m}}t + \frac{1}{2}\cos\sqrt{\frac{3k}{m}}t \\ \frac{1}{2}\cos\sqrt{\frac{k}{m}}t - \frac{1}{2}\cos\sqrt{\frac{3k}{m}}t \end{pmatrix}.$$

1.8.12

(1) $|\ddot{x}(t)\rangle = \Omega\,|x(t)\rangle$

$|x(t)\rangle = U(t)\,|x(0)\rangle$

$\Rightarrow \frac{d^2}{dt}U(t)\,|x(0)\rangle = \Omega\,U(t)\,|x(0)\rangle$

$\Rightarrow \left(\frac{d^2}{dt} - \Omega\right)U(t)\,|x(0)\rangle = 0$.

Since $|x(0)\rangle$ is arbitrary, we have

$\frac{d^2}{dt^2}U(t) - \Omega\,U(t) = 0$.

(2) From 1.8.11, we know the $\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ can diagonalize Ω, and therefore, it can also diagonalize U.

In this common basis, we have

$$\begin{pmatrix} \ddot{U}_{11}(t) & 0 \\ 0 & \ddot{U}_{22}(t) \end{pmatrix} - \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix}\begin{pmatrix} U_{11}(t) & 0 \\ 0 & U_{22}(t) \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} \ddot{U}_{11}(t) + \omega_1^2 U_{11}(t) = 0 \\ \ddot{U}_{22}(t) + \omega_2^2 U_{22}(t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} U_{11} = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t \\ U_{22} = A_2 \cos\omega_2 t + B_2 \sin\omega_2 t \end{cases}$$

$$|\dot{x}(0)\rangle = \frac{d}{dt}\left[U(t)\,|x(0)\rangle\right]\Big|_{t=0} = \left(\frac{d}{dt}U(t)\right)\Big|_{t=0}|x(0)\rangle = 0$$

$\Rightarrow \frac{d}{dt}U(t)\Big|_{t=0} = 0 \Rightarrow \dot{U}_{11}(0) = \dot{U}_{22}(0) = 0 \Rightarrow B_1 = B_2 = 0$.

To satisfy that U is unitary, $A_1 = A_2 = 1$.

Therefore, $U = \begin{pmatrix} \cos\omega_1 t & 0 \\ 0 & \cos\omega_2 t \end{pmatrix}$,

which is the same as Eq. (1.8.43).

1.9.1

In the eigenbasis, $\Omega = \begin{pmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_m \end{pmatrix}$, where ω_i are eigenvalues.

$$f(\Omega) = \sum_{n=0}^{\infty}\Omega^n = \begin{pmatrix} \sum_{n=0}^{\infty}\omega_1^n & & & \\ & \ddots & & \\ & & & \sum_{n=0}^{\infty}\omega_m^n \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\omega_1} & & & \\ & \ddots & & \\ & & & \frac{1}{1-\omega_m} \end{pmatrix} = \frac{1}{1-\Omega}.$$

This equality holds iff $|\omega_i| < 1$ for $i = 1, \ldots, m$.

Therefore, $f(\Omega)$ can be defined as $\frac{1}{1-\Omega}$ if and only if the absolute value of each of Ω's eigenvalue is less than 1.

1.9.2

$$U^\dagger = (e^{iH})^\dagger = e^{-iH^\dagger} = e^{-iH}$$

↑ H is Hermitian

$$U^\dagger U = e^{-iH} e^{iH} = e^{-iH+iH} = 1 \leftarrow \text{unitary}$$

↑ ☆ This equality only holds for commuting operators.

1.9.3

In the eigenbasis of H, $U = e^{iH} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(i\epsilon_1)^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{(i\epsilon_m)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{i\epsilon_1} & & \\ & \ddots & \\ & & e^{i\epsilon_m} \end{pmatrix}$.

where $\epsilon_1, \cdots, \epsilon_m$ are eigenvalues of H. i.e $H = \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_m \end{pmatrix}$.

Therefore,

$$\det U = \prod_{i=1}^{m} e^{i\epsilon_i} = e^{i\sum_{i=1}^{m}\epsilon_i} = e^{i\,\mathrm{Tr}H}.$$

1.10.1

Since $\delta(x) = \delta(-x)$, we have $\delta(ax) = \delta(|a|x)$.

Therefore,

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \int_{-\infty}^{\infty} \delta(|a|x)\,dx = \int_{-\infty}^{\infty} \delta(|a|x)\cdot \frac{1}{|a|}\,d(|a|x)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(|a|x)\,d(|a|x) \qquad (\text{change } |a|x \text{ to } x)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(x)\,dx.$$

Thus, $\delta(ax) = \delta(x)/|a|$.

1.10.2

Expand $f(x)$ around x_i, where $f(x_i) = 0$:

$$f(x) = f(x_i) + f'(x_i)(x-x_i) + \frac{1}{2}f''(x_i)(x-x_i)^2 + \cdots$$
$$= 0 + f'(x_i)(x-x_i) + O[(x-x_i)^2]$$
$$\approx f'(x_i)(x-x_i)$$

Introduce a test function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\,\delta(f(x))\,dx = \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\delta(f(x))$$
$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\delta\left(\left.\frac{df}{dx}\right|_{x=x_i}(x-x_i)\right)dx$$
$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\delta\left[\left|\left.\frac{df}{dx}\right|_{x=x_i}\right|(x-x_i)\right]dx$$

See 1.10.1 →

$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\,\frac{\delta(x-x_i)}{\left|\left.df/dx\right|_{x=x_i}\right|}\,dx.$$

Therefore, $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{\left|\left.df/dx\right|_{x=x_i}\right|}$.

1.10.3

Introduce a test function $g(x)$. * $g(\infty)$ and $g(-\infty)$ are finite.

$$\int_{-\infty}^{\infty} g(x)\,\frac{d}{dx}\theta(x-x')\,dx$$
$$= \int_{-\infty}^{\infty} g(x)\,d\,\theta(x-x')$$
$$= \theta(x-x')\,g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\theta(x-x')\,g'(x)\,dx$$
$$= 1\cdot g(\infty) - 0\cdot g(-\infty) - \int_0^{\infty} g'(x)\,dx$$
$$= g(\infty) - [g(\infty) - g(0)]$$
$$= g(0)$$
$$= \int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx.$$

Therefore, $\frac{d}{dx}\theta(x-x') = \delta(x)$.