# Homework 4 (1.10.4 ~ 2.7.1)

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## 1.10.4

We start from Eq.(1.10.55)

$$|\psi(t)\rangle = \sum_{m=1}^{\infty} |m\rangle\langle m|\psi(0)\rangle \cos\omega_m t, \quad \omega_m = \frac{m\pi}{L}$$

$$\psi(x,t) = \langle x|\psi(t)\rangle = \sum_{m=1}^{\infty} \langle x|m\rangle\langle m|\psi(0)\rangle \cos\omega_m t$$

$$\langle x|m\rangle = \psi_m(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{m\pi x}{L} \qquad (\text{Eq. (1.10.50)})$$

$$\begin{aligned}\langle m|\psi(0)\rangle &= \int_0^L \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{m\pi x}{L}\cdot\psi(x,0)\,dx \\ &= \left(\frac{2}{L}\right)^{\frac{1}{2}}\left[\int_0^{\frac{L}{2}} \frac{2xh}{L}\sin\frac{m\pi x}{L}dx + \int_{\frac{L}{2}}^{L}\frac{2h}{L}(L-x)\sin\frac{m\pi x}{L}dx\right] \\ &\equiv \left(\frac{2}{L}\right)^{\frac{1}{2}}[\,\mathrm{I}+\mathrm{II}\,]\end{aligned}$$

$$\begin{aligned}\mathrm{I} &= \int_0^{\frac{L}{2}} \frac{2xh}{L}\sin\frac{m\pi x}{L}dx \\ &= -\frac{2h}{L}\cdot\frac{L}{m\pi}\int_0^{\frac{L}{2}} x\,d\cos\frac{m\pi x}{L} \\ &= -\frac{2h}{L}\cdot\frac{L}{m\pi}\,x\cos\frac{m\pi x}{L}\Big|_0^{\frac{L}{2}} + \frac{2h}{L}\cdot\frac{L}{m\pi}\int_0^{\frac{L}{2}}\cos\frac{m\pi x}{L}dx \\ &= -\frac{2h}{m\pi}\cdot\frac{L}{2}\cos\frac{m\pi}{2} + \frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi x}{L}\Big|_0^{\frac{L}{2}} \\ &= -\frac{hL}{m\pi}\cos\frac{m\pi}{2} + \frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}\end{aligned}$$

$$\begin{aligned}\mathrm{II} &= \int_{\frac{L}{2}}^{L}\frac{2h}{L}\cdot L\sin\frac{m\pi x}{L}dx - \int_{\frac{L}{2}}^{L}\frac{2hx}{L}\sin\frac{m\pi x}{L}dx \\ &= -2h\cdot\frac{L}{m\pi}\cos\frac{m\pi x}{L}\Big|_{\frac{L}{2}}^{L} + \frac{2h}{L}\cdot\frac{L}{m\pi}\int_{\frac{L}{2}}^{L} x\,d\cos\frac{m\pi x}{L} \\ &= \frac{2hL}{m\pi}\cos\frac{m\pi}{2} - \frac{2hL}{m\pi}\cos m\pi + \frac{2h}{L}\cdot\frac{L}{m\pi}\,x\cos\frac{m\pi x}{L}\Big|_{\frac{L}{2}}^{L} - \frac{2h}{m\pi}\int_{\frac{L}{2}}^{L}\cos\frac{m\pi x}{L}dx \\ &= \frac{2hL}{m\pi}\cos\frac{m\pi}{2} - \frac{2hL}{m\pi}\cos m\pi + \frac{2hL}{m\pi}\cos m\pi - \frac{hL}{m\pi}\cos\frac{m\pi}{2} - \frac{2h}{m\pi}\cdot\frac{L}{m\pi}\sin\frac{m\pi x}{L}\Big|_{\frac{L}{2}}^{L} \\ &= \frac{hL}{m\pi}\cos\frac{m\pi}{2} - \underbrace{\frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin m\pi}_{0} + \frac{2h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}\end{aligned}$$

$$\mathrm{I}+\mathrm{II} = \frac{4h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}$$

Therefore,

$$\begin{aligned}\psi(x,t) &= \sum_{m=1}^{\infty}\left(\frac{2}{L}\right)^{\frac{1}{2}}\sin\frac{m\pi x}{L}\cdot\left(\frac{2}{L}\right)^{\frac{1}{2}}\cdot\frac{4h}{L}\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi}{2}\cos\omega_m t \\ &= \sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t\cdot\left(\frac{8h}{\pi^2 m^2}\right)\sin\left(\frac{\pi m}{2}\right).\end{aligned}$$

## 2.1.1

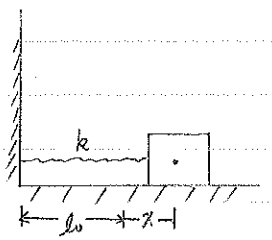
$$T = \frac{1}{2}m\dot{x}^2$$

$$V = \frac{1}{2}kx^2$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0.$$

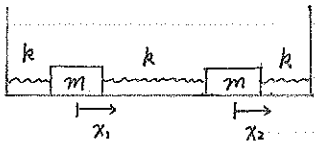


2.1.2

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2-x_1)^2 + \frac{1}{2}kx_2^2$$

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2) - k(x_1^2 - x_1x_2 + x_2^2)$$




$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \qquad \frac{\partial L}{\partial x_1} = -2kx_1 + kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \Rightarrow m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \Rightarrow \ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \quad ①$$

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \qquad \frac{\partial L}{\partial x_2} = kx_1 - 2kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \Rightarrow \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2 \quad ②$$

① and ② are the same as Eqs. (1.8.24) and (1.8.25).

2.1.3

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$

$$V = V(r)$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{\partial V(r)}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \underline{\underline{m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\,\dot{\phi}^2 + \frac{\partial V(r)}{\partial r} = 0}}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \underline{\underline{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0}}$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}, \quad \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}(mr^2\sin^2\theta\,\dot{\phi}) = 0$$

$$\Rightarrow mr^2\sin^2\theta\,\dot{\phi} = l \qquad (l \text{ is a constant})$$

$$\Rightarrow \underline{\underline{\dot{\phi} = \frac{l}{mr^2\sin^2\theta}}}.$$

Note: $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\ \dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \\ \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{cases}$$

$$T = \frac{1}{2}m\big[\dot{r}^2\sin^2\theta\cos^2\phi + r^2\dot{\theta}^2\cos^2\theta\cos^2\phi + r^2\dot{\phi}^2\sin^2\theta\sin^2\phi + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi$$
$$- 2r\dot{r}\sin^2\theta\cos\phi\sin\phi - 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi + \dot{r}^2\sin^2\theta\sin^2\phi +$$
$$r^2\dot{\theta}^2\cos^2\theta\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta\cos^2\phi + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\sin^2\phi + 2\dot{r}r\dot{\phi}\sin^2\theta\sin\phi\cos\phi$$
$$+ 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi + \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\big]$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$$

2.5.1

$$p_s = \frac{\partial T}{\partial \dot{q}_s} = \sum_i\sum_j T_{ij}(q)\,\dot{q}_i\,\delta_{js} + \sum_i\sum_j T_{ij}(q)\,\delta_{is}\,\dot{q}_j$$
$$= \sum_i T_{is}(q)\,\dot{q}_i + \sum_j T_{sj}(q)\,\dot{q}_j$$
$$\sum_s p_s\dot{q}_s = \sum_s\sum_i T_{is}(q)\,\dot{q}_i\dot{q}_s + \sum_s\sum_j T_{sj}(q)\,\dot{q}_s\dot{q}_j$$
$$= T + T$$
$$= 2T.$$

2.5.2

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$
$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$
$$H = \dot{x}p - L = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Since $L$ is not an explicit function of $t$, $H$ is a conservative.

Set $H = E$, where $E$ is a constant, we have

$$\frac{1}{2}kx^2 + \frac{p^2}{2m} = E$$
$$\Rightarrow \frac{x^2}{\frac{2E}{k}} + \frac{p^2}{2mE} = 1.$$

Therefore, if we set $a^2 = 2E/k$, $b^2 = 2mE$, we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{p}{b}\right)^2 = 1.$$

2.5.3

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + x_2^2 - x_1x_2)$$
$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \quad\Rightarrow\quad \dot{x}_1 = \frac{p_1}{m}$$
$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \quad\Rightarrow\quad \dot{x}_2 = \frac{p_2}{m}$$

$$H = \dot{x}_1 p_1 + \dot{x}_2 p_2 - L$$
$$= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + k(x_1^2 + x_2^2 - x_1 x_2)$$

$$\begin{cases} \dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m} & ① \\ \dot{p}_1 = -\frac{\partial H}{\partial x_1} = -2kx_1 + kx_2 & ② \end{cases}$$

From ①, we know $p_1 = m\dot{x}_1$. Therefore, $\dot{p}_1 = m\ddot{x}_1$. Substitute it into ②, we have $m\ddot{x}_1 = -2kx_1 + kx_2$.

$$\Rightarrow \ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2. \quad ③$$

$$\begin{cases} \dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m} & ④ \\ \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -2kx_2 + kx_1 & ⑤ \end{cases}$$

From ④, we know $p_2 = m\dot{x}_2$. Therefore, $\dot{p}_2 = m\ddot{x}_2$. Substitute it into ⑤, we have $m\ddot{x}_2 = -2kx_2 + kx_1$.

$$\Rightarrow \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2. \quad ⑥$$

2.7.1

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) = -\sum_i \left(\frac{\partial \lambda}{\partial q_i}\frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i}\frac{\partial \omega}{\partial q_i}\right) = -\{\lambda, \omega\}.$$

$$\{\omega, \lambda+\sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\cdot\frac{\partial(\lambda+\sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\cdot\frac{\partial(\lambda+\sigma)}{\partial q_i}\right)$$
$$= \sum_i \left[\frac{\partial \omega}{\partial q_i}\cdot\left(\frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i}\right) - \frac{\partial \omega}{\partial p_i}\cdot\left(\frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i}\right)\right]$$
$$= \sum_i \left[\left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) + \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right)\right]$$
$$= \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) + \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right)$$
$$= \{\omega, \lambda\} + \{\omega, \sigma\}.$$

$$\{\omega, \lambda\sigma\} = \sum_i \left[\frac{\partial \omega}{\partial q_i}\frac{\partial(\lambda\sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial(\lambda\sigma)}{\partial q_i}\right]$$
$$= \sum_i \left[\lambda\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} + \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i}\sigma - \lambda\frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\sigma\right]$$
$$= \lambda\sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\right) + \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right)\cdot\sigma$$
$$= \lambda\{\omega, \sigma\} + \{\omega, \lambda\}\sigma.$$