Homework 5

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4.2.1

(1) The eigenvalues of L_z : $-1, 0, 1$.

(2) The eigenvector of L_z corresponding to $l_z = 1$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\langle L_x \rangle = (1\ 0\ 0)\ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle L_x^2 \rangle = (1\ 0\ 0)\ \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}$$

$$(\Delta L_x)^2 = \langle (L_x - \langle L_x \rangle)^2 \rangle = \langle L_x^2 \rangle - \langle L_x \rangle^2 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\Delta L_x = \frac{1}{\sqrt{2}}.$$

(3)

$$\det(L_x - \lambda I) = \begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda^3 + \lambda = 0$$

$$\lambda = 0, \pm 1.$$

$l_x = -1$: $\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ $\qquad |l_x = -1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

$l_x = 0$: $\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ $\qquad |l_x = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$l_x = 1$: $\begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ $\qquad |l_x = 1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

(4) The eigenvector of L_z corresponding to $l_z = -1$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$|\langle l_x = -1 | l_z = -1 \rangle|^2 = \left| \frac{1}{2} (1\ -\sqrt{2}\ 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

$\Rightarrow$ The probability of the outcome $l_x = -1$ is $\frac{1}{4}$.

$$|\langle l_x = 0 | l_z = -1 \rangle|^2 = \left| \frac{1}{\sqrt{2}} (1\ 0\ -1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$\Rightarrow$ The probability of the outcome $l_x = 0$ is $\frac{1}{2}$.

$$|\langle l_x=1|\, l_z=-1\rangle|^2 = \left|\tfrac{1}{2}(1\ \sqrt{2}\ 1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

$\Rightarrow$ The probability of the outcome $l_x = 1$ is $\frac{1}{4}$.

In fact,

$$|l_z=-1\rangle = |l_x=-1\rangle\langle l_x=-1|\, l_z=-1\rangle + |l_x=0\rangle\langle l_x=0|\, l_z=-1\rangle + |l_x=1\rangle\langle l_x=1|\, l_z=-1\rangle$$
$$= \tfrac{1}{2}|l_x=-1\rangle - \tfrac{1}{\sqrt{2}}|l_x=0\rangle + \tfrac{1}{2}|l_x=1\rangle.$$

(5) $$|\psi\rangle = C_{-1}|l_z=-1\rangle + C_0|l_z=0\rangle + C_1|l_z=1\rangle$$

$$C_{-1} = \langle l_z=-1|\psi\rangle = (0\ 0\ 1)\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$C_0 = \langle l_z=0|\psi\rangle = (0\ 1\ 0)\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{2}$$

$$C_1 = \langle l_z=1|\psi\rangle = (1\ 0\ 0)\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{2}$$

$$\Rightarrow |\psi\rangle = \tfrac{1}{\sqrt{2}}|l_z=-1\rangle + \tfrac{1}{2}|l_z=0\rangle + \tfrac{1}{2}|l_z=1\rangle$$

Since L_z^2 is measured to be $+1$, l_z can be 1 or -1.

The probability of this result is: $|C_{-1}|^2 + |C_1|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

After the measurement, the state is:

$$|\psi'\rangle = C_{-1}'|l_z=-1\rangle + C_1'|l_z=1\rangle$$
$$C_1'/C_{-1}' = C_1/C_{-1} = \frac{1}{\sqrt{2}}$$

Normalization: $C_1'^2 + C_{-1}'^2 = 1 \Rightarrow C_1' = \sqrt{\frac{1}{3}}$, $C_{-1}' = \sqrt{\frac{2}{3}}$

$$|\psi'\rangle = \sqrt{\tfrac{2}{3}}|l_z=-1\rangle + \sqrt{\tfrac{1}{3}}|l_z=1\rangle.$$

If we measure L_z,

outcome $l_z=-1$, probability $|C_{-1}'|^2 = \frac{2}{3}$;

outcome $l_z=1$, probability $|C_1'|^2 = \frac{1}{3}$.

(6) $$|\psi\rangle = C_1|L_z=1\rangle + C_0|L_z=0\rangle + C_{-1}|L_z=-1\rangle$$

$$|C_1|^2 = P(L_z=1) = \tfrac{1}{4} \Rightarrow C_1 = \tfrac{1}{2}e^{i\delta_1}$$
$$|C_0|^2 = P(L_z=0) = \tfrac{1}{2} \Rightarrow C_2 = \tfrac{1}{\sqrt{2}}e^{i\delta_2}$$
$$|C_{-1}|^2 = P(L_z=-1) = \tfrac{1}{4} \Rightarrow C_{-1} = \tfrac{1}{2}e^{i\delta_3}$$

Therefore,

$$|\psi\rangle = \tfrac{1}{2}e^{i\delta_1}|L_z=1\rangle + \tfrac{1}{\sqrt{2}}e^{i\delta_2}|L_z=0\rangle + \tfrac{1}{2}e^{i\delta_3}|L_z=-1\rangle.$$

$\delta_1, \delta_2, \delta_3$ cannot be determined individually.

It is true that $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same state, in the sense that θ does not change any physical observable. However, if $|\psi\rangle$ is a linear combination of several eigenstates, as the case shown above, the relative values between the phase factors in front of each eigenstates, $e^{i\delta_1}$, $e^{i\delta_2}$ and $e^{i\delta_3}$ do have physical significance. For example,

$$P(L_x=0) = |\langle L_x=0|\psi\rangle|^2$$
$$= \left|\frac{1}{2}e^{i\delta_1}\langle L_x=0|L_z=1\rangle + \frac{1}{\sqrt{2}}e^{i\delta_2}\langle L_x=0|L_z=2\rangle + \frac{1}{2}e^{i\delta_3}\langle L_x=0|L_z=-1\rangle\right|^2$$

$$\langle L_x=0|L_z=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle L_x=0|L_z=0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$$

$$\langle L_x=0|L_z=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow P(L_x=0) = \left|\frac{1}{2}\cdot\frac{1}{\sqrt{2}}e^{i\delta_1} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}e^{i\delta_3}\right|^2$$
$$= \frac{1}{8}\left(e^{-i\delta_1} - e^{-i\delta_3}\right)\left(e^{i\delta_1} - e^{i\delta_3}\right)$$
$$= \frac{1}{8}\left[2 - \left(e^{-i(\delta_1-\delta_3)} + e^{i(\delta_1-\delta_3)}\right)\right]$$
$$= \frac{1}{8}\left[2 - 2\cdot\frac{e^{i(\delta_1-\delta_3)} + e^{-i(\delta_1-\delta_3)}}{2}\right]$$
$$= \frac{1}{4}\left[1 - \cos(\delta_1-\delta_3)\right]$$

Therefore, the difference between δ_1 and δ_3 can be measured by physical experiments.

4.2.2

Since $\psi(x)$ is real, $\psi^*(x) = \psi(x)$

$$\langle p\rangle = \int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\,dx$$
$$= -i\hbar\int_{-\infty}^{\infty}\psi(x)\frac{\partial}{\partial x}\psi(x)\,dx$$
$$= -\frac{1}{2}i\hbar\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\psi^2(x)\,dx$$
$$= -\frac{1}{2}i\hbar\,\psi^2(x)\Big|_{-\infty}^{\infty} \qquad * \ \psi(x)\to 0 \ \text{ as } \ x\to\pm\infty$$
$$= 0$$

For general case, $\langle p\rangle = \int_{-\infty}^{\infty}[C\psi(x)]^*\left(-i\hbar\frac{\partial}{\partial x}\right)[C\psi(x)] = |C|^2\int_{-\infty}^{\infty}\psi(x)\frac{\partial}{\partial x}\psi(x)dx$

$$= |C|^2\cdot 0 = 0$$

4.2.3

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{\partial}{\partial x} \psi(x)\, dx$$

$$\langle p' \rangle = -i\hbar \int_{-\infty}^{\infty} [\psi^*(x) e^{-ip_0 x/\hbar}] \frac{\partial}{\partial x} [\psi(x) e^{ip_0 x/\hbar}]\, dx$$

$$= -i\hbar \int_{-\infty}^{\infty} [\psi^*(x) e^{-ip_0 x/\hbar}] \cdot [\frac{\partial}{\partial x} \psi(x)] \cdot e^{ip_0 x/\hbar}\, dx$$
$$- i\hbar \int_{-\infty}^{\infty} [\psi^*(x) e^{-ip_0 x/\hbar}] \cdot \psi(x) \cdot i p_0/\hbar \cdot e^{ip_0 x/\hbar}\, dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{\partial}{\partial x} \psi(x)\, dx \quad - i\hbar \cdot i p_0/\hbar \cdot \int_{-\infty}^{\infty} \psi^*(x) \psi(x)\, dx$$

$$= \langle p \rangle + p_0 \int_{-\infty}^{\infty} \psi^*(x) \psi(x)\, dx$$

$$= \langle p \rangle + p_0 .$$

Problem 3.

$$\langle x \rangle = \langle \psi | x | \psi \rangle$$

$$= \iint dp\, dp' \ \langle \psi | p \rangle \langle p | x | p' \rangle \langle p' | \psi \rangle$$

$$= \iint dp\, dp' \ \psi^*(p) \langle p | x | p' \rangle \psi(p')$$

$$= \iint dp\, dp' \ \psi^*(p)\, i\hbar \frac{\partial}{\partial p} \delta(p - p')\, \psi(p')$$

$$= \int \psi^*(p) \left(i\hbar \frac{\partial}{\partial p} \right) \psi(p)\, dp$$

Where we use the relation $\langle p' | x | p'' \rangle = i\hbar \frac{\partial}{\partial p'} \delta(p' - p'')$

Prove: $\langle p' | x | p'' \rangle$

$$= \iint dx'\, dx'' \ \langle p' | x' \rangle \langle x' | x | x'' \rangle \langle x'' | p'' \rangle$$

$$= \iint dx'\, dx'' \ \frac{1}{\sqrt{2\pi\hbar}} e^{-ip'x'/\hbar} \cdot x'' \delta(x' - x'') \cdot \frac{1}{\sqrt{2\pi\hbar}} e^{ip''x''/\hbar}$$

$$= \int dx' \ \frac{1}{2\pi\hbar} x' e^{-i(p' - p'')x'/\hbar}$$

$$= \frac{1}{2\pi\hbar} \int dx' \left(-\frac{\hbar}{i} \right) \frac{\partial}{\partial p'} e^{-i(p' - p'')x'/\hbar}$$

$$= \frac{1}{2\pi\hbar} (i\hbar) \frac{\partial}{\partial p'} \int dx' \ e^{-i(p' - p'')x'/\hbar}$$

$$= \frac{1}{2\pi\hbar} (i\hbar) \frac{\partial}{\partial p'} [2\pi\hbar\, \delta(p' - p'')]$$

$$= i\hbar \frac{\partial}{\partial p'} \delta(p' - p'')$$

Problem 4

In the energy eigenbasis, $\hat{H} = \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & \ddots \end{pmatrix}$.

Therefore, $e^{-i\hat{H}t/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!} (-i\hat{H}t/\hbar)^n$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[-i \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & \ddots \end{pmatrix} t/\hbar \right]^n$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} (-iE_1 t/\hbar)^n & & \\ & \sum_{n=0}^{\infty} \frac{1}{n!} (-iE_2 t/\hbar)^n & \\ & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} e^{-iE_1 t/\hbar} & & \\ & e^{-iE_2 t/\hbar} & \\ & & \ddots \end{pmatrix}$$

$$= \sum_E |E\rangle\langle E| e^{-iEt/\hbar}$$

Thus, $U(t) = \sum_E |E\rangle\langle E| e^{-iEt/\hbar} = e^{-i\hat{H}t/\hbar}$.

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} \left(e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \right)$$

$$= i\hbar (-i\hat{H}/\hbar) e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

$$= \hat{H} \left(e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \right)$$

$$= \hat{H} |\psi(t)\rangle.$$

Therefore, $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$ satisfies the Schrödinger equation.

Problem 5.

$$|\psi\rangle = a|\psi_1(t)\rangle + b|\psi_2(t)\rangle$$

$$= a|\psi_1\rangle e^{-iE_1 t/\hbar} + b|\psi_2\rangle e^{-iE_2 t/\hbar}$$

$$P(x) = |\langle x|\psi\rangle|^2 = \left(a^* e^{iE_1 t/\hbar} \langle \psi_1|x\rangle + b^* e^{iE_2 t/\hbar} \langle \psi_2|x\rangle \right) \cdot \left(a e^{-iE_1 t/\hbar} \langle x|\psi_1\rangle + b e^{-iE_2 t/\hbar} \langle x|\psi_2\rangle \right)$$

$$= |a\psi_1(x)|^2 + |b\psi_2(x)|^2 + a^* b \psi_1^*(x)\psi_2(x) e^{i(E_1 - E_2)t/\hbar} + ab^* \psi_1(x)\psi_2^*(x) e^{i(E_2 - E_1)t}$$

Therefore, the probability of finding the particle at some position changes with time.

Problem 2 (Done by Sean Finch)

Make Plots of the probability density in space and momentum space for a Gaussian wavepacket for several values of Delta to illustrate the X and P uncertianty relation.

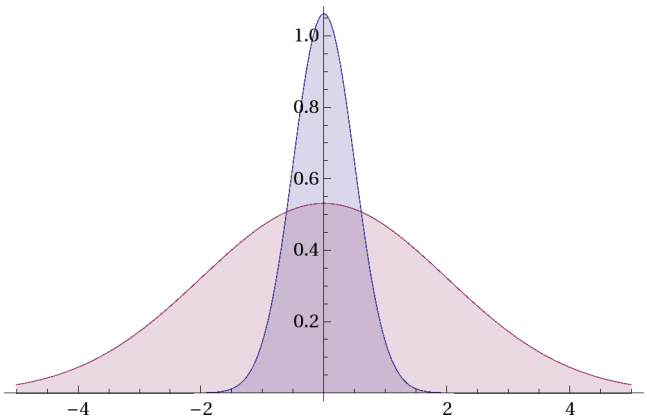
Note: here we set a = 0 (the wavepacket centers at x=0) and hbar = 1 (makes the scaling of the graphs easier to read).

Definitions of the wave packet functions:

```
X[x_, d_] := 1 / (Pi * d^2)^.25 * Exp[-x^2 / (2 d^2)]
P[p_, d_] := (d^2 / Pi)^.25 * Exp[-(p * d)^2 / (2)]
```

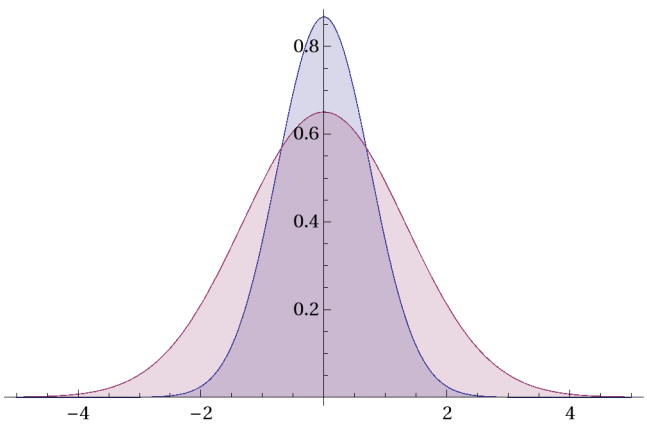
Here is a plot with Delta = .5: (this means the position is more localized)

```
Plot[{X[x, .5], P[x, .5]}, {x, -5, 5}, Filling → Axis]
```



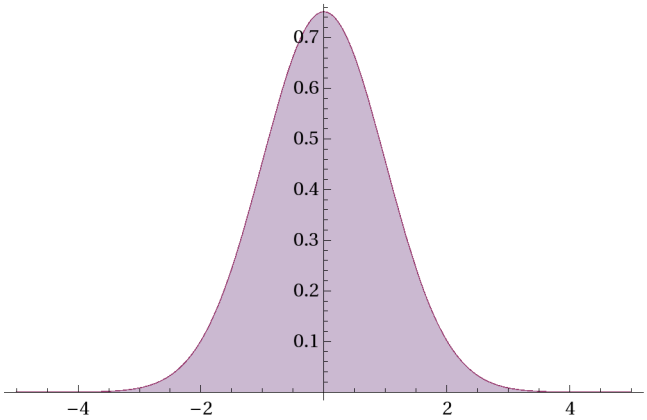
Delta = .75: (position is more localized)

```
Plot[{X[x, .75], P[x, .75]}, {x, -5, 5}, Filling → Axis]
```



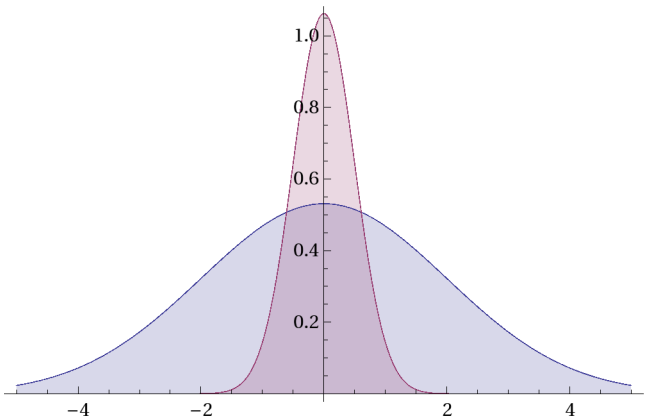
Delta = 1 (X and P know to the same amount of percision)

```
Plot[{X[x, 1], P[x, 1]}, {x, -5, 5}, Filling → Axis]
```



Delta = 2: (momentum is more localized)

```
Plot[{X[x, 2], P[x, 2]}, {x, -5, 5}, Filling → Axis]
```



Delta = 2: (momentum in very small range)

```
Plot[{X[x, 5], P[x, 5]}, {x, -10, 10},
 PlotRange → Full, Filling → Axis]
```

