Homework #3.

15.1.2. $\hat{H}_{hf} = A\,\vec{S}_1 \cdot \vec{S}_2$.

$[\hat{H}_{hf}, \hat{H}_{Coulomb}]$ since $\hat{H}_{hf}$ and $\hat{H}_{Coulomb}$ act in two different spaces. Thus, they can be diagonalized at the same time. Thus,

$$E = -Ry + E_{hf},$$

where E_{hf} is eigenvalue of $\hat{H}_{hf}$.

$$\hat{H}_{hf} = \frac{A}{2}(\bar{S}_1 + \bar{S}_2)^2 - \frac{A}{2}(\bar{S}_1^2 + \bar{S}_2^2) = \frac{A}{2}\bar{S}^2 - \frac{3}{4}A$$

Singlet corresponds to $\bar{S}^2 = 0$, triplet corresponds to $\bar{S}^2 = 1\cdot(1+1) = 2$.

Thus $E_{triplet} = 2\frac{A}{2} - \frac{3}{4}A = \frac{A}{4}$ //

$E_{singlet} = -\frac{3}{4}A$ //

15.2.1. 1) $\bar{J}^2 |j_1 j_1, j_2 j_2\rangle = (\hat{J}_1^2 + \hat{J}_2 + 2\hat{J}_{12}\hat{J}_{2z} + \hat{J}_{1+}J_{2-} + \hat{J}_{1-}\hat{J}_{2+}) \times$

$\times |j_1 j_1, j_2 j_2\rangle$. $\hat{J}_{1,2+}|j_1 j_1, j_2 j_2\rangle = 0$.

Hence,

$$\bar{J}^2 |j_1 j_1, j_2 j_2\rangle = [j_1(j_1+1) + j_2(j_2+1) + 2j_1 j_2]\,|j_1 j_1, j_2 j_2\rangle =$$

$$= [(j_1+j_2)^2 + (j_1+j_2)]\,|j_1 j_1, j_2 j_2\rangle = (j_1+j_2)(j_1+j_2+1)\,|j_1 j_1, j_2 j_2\rangle$$

15.2.2. 1). $\left(\frac{1}{2}\right) \otimes (1) = \left(\frac{3}{2}\right) \oplus \left(\frac{1}{2}\right)$

$$\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle |1,1\rangle$$

$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \text{const}\, J_- \left[\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle |1,1\rangle\right] = \text{const}\left|1\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle |1,1\rangle +$$

$$+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle |1,0\rangle = \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle |1,1\rangle + \sqrt{\frac{2}{3}}\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle |1,0\rangle \quad \text{(normalization)}$$

Coefficients also must be real.

$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$ – an orthogonal state to $\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle$, and

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \alpha \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle |1,1\rangle + \beta \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle |1,0\rangle, \quad \alpha > 0.$$

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle |1,1\rangle - \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle |1,0\rangle.$$

Thus, $\langle 1,\ 1/2\ 1/2 \mid 3/2\ 3/2\rangle = 1$;

$\langle 1,\ 1/2\ -1/2 \mid 3/2\ 1/2\rangle = \frac{1}{\sqrt{3}}$;

$\langle 1,0,\ 1/2\ 1/2 \mid 3/2\ 1/2\rangle = \sqrt{2/3}$;

$\langle 1,1,\ 1/2\ (-1/2) \mid 1/2\ 1/2\rangle = \sqrt{2/3}$;

$\langle 1,0,\ 1/2\ 1/2 \mid 1/2\ 1/2\rangle = -\sqrt{1/3}$.

All other nonzero coefficients are given by (15.2.11)

2). $1 \otimes 1 = 2 \oplus 1 \oplus 0$

$|2,2\rangle = |1,1\rangle|1,1\rangle$

$|2,1\rangle = \alpha J_- |2,2\rangle = \alpha(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle) =$

$= \frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle + \frac{1}{\sqrt{2}}|1,1\rangle|1,0\rangle$

$|2,0\rangle = \alpha J_- |2,1\rangle = \alpha(|1,-1\rangle|1,1\rangle + |1,1\rangle|1,-1\rangle) + 2\alpha|1,0\rangle|1,0\rangle =$

$= \frac{1}{\sqrt{6}}(|1,-1\rangle|1,1\rangle + |1,1\rangle|1,-1\rangle) + \sqrt{\frac{2}{3}}|1,0\rangle|1,0\rangle$

$\langle 2,1|1,1\rangle = 0,\quad |1,1\rangle = \alpha''|1,0\rangle|1,1\rangle + \beta|1,1\rangle|1,0\rangle,\ \beta > 0$

$|1,1\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle)$

$|1,0\rangle = \alpha''' J_- |1,1\rangle = \alpha'''(|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle) =$

$= \frac{1}{\sqrt{2}}(|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle)$

$|1,0\rangle = x|1,1\rangle|1,-1\rangle + y|1,0\rangle|1,0\rangle + z|1,-1\rangle|1,1\rangle$

$\langle 2,0|1,0\rangle = 0$;

$\langle 1,0|1,0\rangle = 0$.

$x - z = 0$;

$x + z + 2y = 0$; $x > 0$.

$|1,0\rangle = \frac{1}{\sqrt{3}}|1,1\rangle|1,-1\rangle - \frac{1}{\sqrt{3}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle|1,1\rangle$

All other coefficients can be found with Eq. (15.2.11)

15.2.3 $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (\frac{1}{2} \otimes \frac{1}{2}) \otimes \frac{1}{2} = (1 \oplus 0) \otimes \frac{1}{2} =$

$= (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) =$

$= (\frac{3}{2} \oplus \frac{1}{2}) \oplus \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$

15.2.7 $|2j_1, 2j_1\rangle = |j_1, j_1\rangle|j_1, j_1\rangle$ – symmetric.

Mathematical induction: let $|2j_1, m\rangle$ is symmetric, i.e. $|2j_1, m\rangle = \sum C_k |j_1, k\rangle|j_1, m-k\rangle$, $C_k = C_{m-k}$.

Show that $|2j_1, m-1\rangle$ is also symmetric.

$|2j_1, m-1\rangle = \text{const}\, J_- |2j_1, m\rangle = \text{const} \sum C_k J_- |j_1 k\rangle|j_1, m-k\rangle) =$

$= \text{const} \sum_k [C_k f(k) |j_1 k-1\rangle|j_1 m-k\rangle + C_k f(m-k) |j_1 k\rangle|j_1, m-k-1\rangle$

$= \text{const} \sum_k |j_1, k\rangle|j_1, m-1-k\rangle (C_k f(m-k) + C_{k+1} f(k+1)) =$

$= \text{const} \sum_k C'_k |j_1, k\rangle|j_1, m-1-k\rangle,$

$C'_k = C_k f(m-k) + C_{k+1} f(k+1)$

$C'_{m-1-k} = C_{m-1-k} f(k+1) + C_{m-k} f(m-k) =$

$= C_{k+1} f(k+1) + C_k f(m-k) = C'_k.$

Thus, $|2j_1, m-1\rangle$ is symmetric.

$|2j_1, 2j_1-1\rangle = \frac{1}{\sqrt{2}} (|j_1, j_1\rangle|j_1, j_1-1\rangle + |j_1, j_1-1\rangle|j_1 j_1\rangle)$

$|2j_1-1, 2j_1-1\rangle = \alpha |j_1, j_1\rangle|j_1 j_1-1\rangle + \beta |j_1, j_1-1\rangle|j_1, j_1\rangle,$

$\langle 2j_1, 2j_1-1 | 2j_1, 2j_1-1\rangle = 0$ Hence, $\alpha = -\beta$.

Now exactly the same induction shows that $|2j_1-1, m\rangle$ is always antisymmetric.

15.3.1.1) (15.3.10) $\Rightarrow$ $\exp(-i\frac{\hat{\vec{J}}}{\hbar}\vec{n}\theta)\, \hat{T}^q_\kappa \exp(\frac{i\hat{\vec{J}}}{\hbar}\vec{n}\theta) =$

$= \sum_{q'} \langle \kappa q' | \exp(-\frac{i\hat{\vec{J}}\vec{n}\theta}{\hbar}\vec{J}) | \kappa q\rangle \hat{T}^{q'}_\kappa;$

$(\hat{I} - i\frac{\hat{\vec{J}}}{\hbar}\vec{n}\theta)\, \hat{T}^q_\kappa\, (\hat{I} + i\frac{\hat{\vec{J}}\vec{n}\theta}{\hbar}) = \sum_{q'} \langle \kappa q' | \hat{I} - i\frac{\vec{n}\theta\hat{\vec{J}}}{\hbar} | \kappa q\rangle \times$

$\times \hat{T}^{q'}_\kappa$ j

$\Rightarrow \frac{\vec{J}}{\hbar} \hat{T}^q_\kappa + \hat{T}^q_\kappa \frac{\vec{J}}{\hbar} = -\sum_{q'} \langle \kappa q' | \frac{\hat{\vec{J}}}{\hbar} | \kappa q\rangle T^{q'}_\kappa;$

$[\hat{J}_\pm, T^q_\kappa] = \sum_{q'} \langle \kappa q' | \frac{\hat{J}_\pm}{\hbar} | \kappa q\rangle T^{q'}_\kappa \Rightarrow$ (15.3.11)

$[J_z, T^q_\kappa] = \sum_{q'} \langle \kappa q' | \frac{\hat{J}_z}{\hbar} | \kappa q\rangle T^{q'}_\kappa$

HW 3-3

2). $[\hat{V}_i, \hat{J}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{V}_k \quad (15.3.4)$

$[T^0_1, J_z] = 0;$

$[T^0_1, J_x] = i\hbar \varepsilon_{zxy} \overset{1}{V}_y = i\hbar \overset{1}{V}_y$

$[T^0_1, J_y] = -i\hbar V_x;$

$[J_+, T^0_1] = -i\hbar V_y \Leftarrow \hbar V_x = +\hbar T^1_1 \sqrt{(1-0)(1+0+1)}.$

$[J_-, T^0_1] = -i\hbar V_y + \hbar V_x = -\hbar T^{-1}_1 \sqrt{(1+0)(1-0+1)}.$

$[J_z, T^{\pm 1}_1] = \pm \left[\frac{V_x \pm iV_y}{\sqrt{2}}, J_z\right] = \pm \frac{i\hbar}{\sqrt{2}}(-V_y \pm iV_x) =$

$= -\frac{\hbar}{\sqrt{2}}(V_x \pm iV_y) = \hbar(q=\pm 1)\, T^{\pm 1}_1$

$\left[J_\pm, T^{\pm 1}_1\right] = \left[J_x \pm iJ_y, \mp \frac{V_x \pm iV_y}{\sqrt{2}}\right] =$

$= \mp \frac{i}{\sqrt{2}}[J_y, V_x] - \frac{i}{\sqrt{2}}[J_x, V_y] = \pm \frac{i}{\sqrt{2}}[V_x, J_y] + \frac{i}{\sqrt{2}}[V_y, J_x] =$

$= \left(\pm \frac{i\hbar}{\sqrt{2}} V_z - \frac{i\hbar}{\sqrt{2}} V_z\right) i = \frac{\hbar V_z}{\sqrt{2}}(1 \mp 1).$

$[J_+, T^1_1] = 0 //$

$[J_+, T^{-1}_1] = \sqrt{2}\hbar T^0_1 //.$

$[J_-, T^{\pm 1}_1] = \left[J_x - iJ_y, \mp \frac{V_x \pm iV_y}{\sqrt{2}}\right] = \pm \frac{i}{\sqrt{2}}[J_y, V_x] -$

$- \frac{i}{\sqrt{2}}[J_x, V_y] = \mp \frac{i}{\sqrt{2}}[V_x, J_y] + \frac{i}{\sqrt{2}}[V_y, J_x] = \frac{\hbar V_z}{\sqrt{2}}(1 \pm 1).$

$[J_-, T^{-1}_1] = 0 ; //$

$[J_+, T^{-1}_1] = \sqrt{2}\hbar T^0_1 //.$

15.3.4. $\vec{\mu} = \gamma_1 \vec{J}_1 + \gamma_2 \vec{J}_2$

$\langle \mu_x \rangle = \gamma_1 \langle J_{1x} \rangle + \gamma_2 \langle J_{2x} \rangle = \frac{\gamma_1}{2} \langle J^+_1 + J^-_1 \rangle + \frac{\gamma_2}{2} \langle J^+_2 + J^-_2 \rangle.$

$\langle J^+_1 \rangle = \langle \text{state with } J_z = m \,|\, \text{state with } J_z = m+1 \rangle =$

$= 0.$ Similar argument shows that $\langle J^\pm_{1,2} \rangle = 0 \Rightarrow$

Hw 3-4

$\langle \mu_x \rangle = \langle \mu_y \rangle = 0.$

In accordance with the Wigner-Eckart theorem, $\vec{\mu}$ has the same matrix elements as $\mu_0 \vec{J}$ between the states with the same J. Hence,

$\vec{\mu}\vec{J}$ and $(\mu_0 \vec{J})\vec{J}$

have the same matrix elements.

$\langle (\mu_0 \vec{J})\vec{J} \rangle = \mu_0 \langle \vec{J}^2 \rangle = \mu_0 j(j+1)\hbar^2.$

$$\langle \vec{\mu}\vec{J} \rangle = \left\langle \left(\frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2} \right) \vec{J}_1 \vec{J} \right\rangle + \left\langle \left(\frac{\gamma_1+\gamma_2}{2} - \frac{\gamma_1-\gamma_2}{2} \right) \vec{J}_2 \vec{J} \right\rangle =$$

$$= \frac{\gamma_1+\gamma_2}{2} \langle (\vec{J}_1+\vec{J}_2)\vec{J} \rangle + \frac{\gamma_1-\gamma_2}{2} \langle (\vec{J}_1-\vec{J}_2)(\vec{J}_1+\vec{J}_2) \rangle =$$

$$= \frac{\gamma_1+\gamma_2}{2}\hbar^2 j(j+1) + \frac{\gamma_1-\gamma_2}{2} \langle \vec{J}_1^2 - \vec{J}_2^2 \rangle = \frac{\gamma_1+\gamma_2}{2}\hbar^2 j(j+1) +$$

$$+ \frac{\gamma_1-\gamma_2}{2}\hbar^2 \left(j_1(j_1+1) - j_2(j_2+1) \right)$$

$$\mu_0 = \left[\frac{\gamma_1+\gamma_2}{2} j(j+1) + \frac{\gamma_1-\gamma_2}{2} \left(j_1(j_1+1) - j_2(j_2+1) \right) \right] / j(j+1)$$

$$= \frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2} \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)}.$$

$$\langle \mu_z \rangle = \mu_0 \langle J_z \rangle = m\hbar \left[\frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2} \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)} \right]$$

2). $^2P_{1/2} \Rightarrow j = 1/2,\ s = j_1 = 1/2,\ \ell = j_2 = 1.$

$$\gamma_1 = 5.6 \frac{e}{2Mc}$$

$$\gamma_2 = \frac{e}{2Mc} \quad (14.4.15).$$

$$\langle \mu_z \rangle = \frac{e\hbar}{2Mc} \left(\pm \frac{1}{2} \right) \left[\frac{5.6+1}{2} + \frac{5.6-1}{2} \frac{\frac{3}{4} - 2}{\frac{3}{4}} \right] =$$

$$= \mu_{nuclear} \left(\pm \frac{1}{2} \right) \left[3.3 + 2.3 \left(1 - \frac{8}{3} \right) \right] =$$

$$= \mu_{nuclear} \left(\pm \frac{1}{2} \right) \left[3.3 - 2.3 \cdot \frac{5}{3} \right] = \mu_{nuclear} \left(\pm \frac{1}{2} \right) \left(-\frac{1.6}{3} \right)$$

$$\Rightarrow \pm 0.27\, \mu_{nuclear}.$$

HW 3-5

3). $\gamma_1 = 2\frac{e}{2mc}$, $\gamma_2 = \frac{e}{2mc}$.

$$\langle \mu_z \rangle = \frac{e\hbar}{2mc}\left(\pm\frac{1}{2}\right)\left[\frac{2+1}{2} + \frac{2-1}{2}\;\frac{\frac{3}{4}-2}{3/4}\right] =$$

$$= \mu_{Bohr}\left(\pm\frac{1}{2}\right)\left[\frac{3}{2} + \frac{1}{2}\left(-\frac{5}{3}\right)\right] = \mu_{Bohr}\left[\frac{1}{2}\;\frac{4}{3}\right]\left(\pm\frac{1}{2}\right) =$$

$$= \pm\frac{1}{3}\mu_{Bohr} \;//$$

16.1.2. $\psi = x^2 - a^2$.

$$\int_{-a}^{a}|\psi|^2 dx \qquad \langle E\rangle = \int_{-a}^{a}\left(-\frac{\hbar^2}{2m}\psi''\right)\psi\,dx = +\int_{-a}^{a}\frac{\hbar^2}{2m}(\psi')^2\,dx -$$

$$-\frac{\hbar^2}{2m}\psi'\psi\Big|_{-a}^{a} = \int_{-a}^{a}\frac{\hbar^2}{2m}(\psi')^2 dx = \int_{-a}^{a}\frac{\hbar^2}{2m}4x^2 dx =$$

$$= \frac{\hbar^2}{2m}\frac{8}{3}a^3 = \frac{4}{3}\frac{\hbar^2 a^3}{m}$$

$$\int_{-a}^{a}|\psi|^2 dx = \int_{-a}^{a}\left(x^4 - 2x^2a^2 + a^4\right)dx = \frac{2}{5}a^5 - \frac{4}{3}a^5 + 2a^5 =$$

$$= a^5\left(\frac{6}{15} - \frac{20}{15} + \frac{30}{15}\right) = a^5\frac{16}{15}.$$

$$\langle E\rangle = \frac{4}{3}\frac{\hbar^2 a^3}{m}\frac{15}{16a^5} = \frac{20}{16}\frac{\hbar^2}{ma^2} = \frac{5}{4}\frac{\hbar^2}{ma^2}$$

$$E_0 = \frac{\hbar^2\pi^2}{8ma^2} \Rightarrow \langle E\rangle = \frac{10}{\pi^2}E_0. \;//$$

16.1.3. $\psi = e^{-\alpha x^2/2}$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{+\infty}e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}.$$

$$\langle\psi|H|\psi\rangle = -aV_0 + \int\frac{\hbar^2}{2m}(\psi')^2 dx =$$

$$= \int\frac{\hbar^2}{2m}(\alpha x)^2 e^{-\alpha x^2}dx - aV_0 = -aV_0 + \frac{\hbar^2\alpha^2}{2m}\left(-\frac{d}{d\alpha}\right)\int e^{-\alpha x^2}dx =$$

$$= -aV_0 + \frac{\hbar^2\alpha^2}{2m}\left(-\frac{d}{d\alpha}\right)\sqrt{\frac{\pi}{\alpha}} = -aV_0 + \frac{\hbar^2\sqrt{\pi}\sqrt{\alpha}}{4m}.$$

$$E = -aV_0\sqrt{\frac{\alpha}{\pi}} + \frac{\hbar^2\alpha}{4m}$$

$$\frac{dE}{d\sqrt{\alpha}} = -\frac{aV_0}{\sqrt{\pi}} + \frac{\hbar^2\sqrt{\alpha}}{2m} \Rightarrow \sqrt{\alpha} = \frac{2maV_0}{\sqrt{\pi}\hbar^2}.$$

$$E = -\frac{2maV_0^2}{\pi\hbar^2} + \frac{4m^2a^2V_0^2\hbar^2}{4m\pi\hbar^4} = -\frac{ma^2V_0^2}{\pi\hbar^2}. \;//$$

Hw 3–6

16.1.5. $\psi = Y_1^m(\theta,\varphi) R(r) = Y_1^m(\theta,\varphi) \frac{U(r)}{r}$.

$U(r)$ does not depend on m.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{e^2}{r} + \frac{l(l+1)\hbar^2}{2mr^2}\right) U = E U;$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{e^2}{r} + \frac{\hbar^2}{mr^2}\right) U = EU \Rightarrow H_{eff} = -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{r} + \frac{\hbar^2}{mr^2}.$$

$U_{r\to 0} \sim r^{l+1} = r^2$. $U_{r\to\infty} \sim \exp\left(-\frac{\alpha r}{2}\right)$.

No zeroes $\Rightarrow$ $U \sim r^2 \exp\left(-\frac{\alpha r}{2}\right)$.

$$\int_0^\infty U^2 dr = \int_0^\infty r^4 e^{-\alpha r} dr = \frac{1}{\alpha^5}\int_0^\infty x^4 e^{-x} dx = \frac{\Gamma(5)}{\alpha^5} = \frac{24}{\alpha^5}.$$

$$\langle U|H_{eff}|U\rangle = -e^2\int \exp(-\alpha r)\, r^3 dr + \frac{\hbar^2}{m}\int_0^\infty \exp(-\alpha r) r^2 dr +$$

$$+ \frac{\hbar^2}{2m}\int_0^\infty (U'(r))^2 dr = -\frac{e^2}{\alpha^4}\Gamma(4) + \frac{\hbar^2}{m\alpha^3}\Gamma(3) + \frac{\hbar^2}{2m}\int_0^\infty (U'(r))^2 dr$$

$$\frac{\hbar^2}{2m}\int_0^\infty (U'(r))^2 dr = \frac{\hbar^2}{2m}\int_0^\infty \left(2r - \frac{\alpha r^2}{2}\right)^2 e^{-\alpha r} dr =$$

$$= \frac{\hbar^2}{2m}\int_0^\infty \left(4r^2 - 2\alpha r^3 + \frac{\alpha^2 r^4}{4}\right) e^{-\alpha r} dr = \frac{\hbar^2}{2m}\left(\frac{4\Gamma(3)}{\alpha^3} - \frac{2\alpha}{\alpha^4}\Gamma(4) + \right.$$

$$\left. + \frac{\alpha^2}{4}\frac{\Gamma(5)}{\alpha^5}\right) = \frac{\hbar^2}{2m\alpha^3}\left(8 - 12 + \frac{24}{4}\right) = \frac{\hbar^2}{m\alpha^3}.$$

$$\langle E\rangle = \left(-\frac{6e^2}{\alpha^4} + \frac{2\hbar^2}{m\alpha^3} + \frac{\hbar^2}{m\alpha^3}\right) \Big/ \frac{24}{\alpha^5} = \frac{3\hbar^2\alpha^2}{24m} - \frac{6e^2\alpha}{24} =$$

$$= \frac{\hbar^2\alpha^2}{8m} - \frac{e^2\alpha}{4}.$$

$$\frac{d\langle E\rangle}{d\alpha} = 0 \Rightarrow \frac{\hbar^2\alpha}{4m} = \frac{e^2}{4};\ \alpha = \frac{me^2}{\hbar^2}.$$

$$\langle E\rangle = \frac{me^4}{\hbar^2}\left(\frac{1}{8} - \frac{1}{4}\right) = -\frac{me^4}{8\hbar^2}$$ – exact result

Hw 3-7

Last problem. Just use expressions for $Y_2^m(\theta,\varphi)$

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi} = \sqrt{\frac{15}{32\pi}}\left(\sin^2\theta\cos 2\varphi \pm i\sin^2\theta\sin 2\varphi\right) =$$

$$= \sqrt{\frac{15}{32\pi}}\left(\sin^2\theta\cos^2\varphi - \sin^2\theta\sin^2\varphi \pm 2i\sin^2\theta\sin\varphi\cos\varphi\right) \sim$$

$$\sim \sqrt{\frac{15}{32\pi}}\left(x^2 - y^2 \pm 2ixy\right) = \sqrt{\frac{15}{32\pi}}\left(x \pm iy\right)^2 //.$$

$$Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi} \sim \mp\sqrt{\frac{15}{8\pi}}\, z\left(x \pm iy\right).$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) = \sqrt{\frac{5}{16\pi}}\left(2\cos^2\theta - \sin^2\theta\right) \sim$$

$$\sim \sqrt{\frac{5}{16\pi}}\left(2z^2 - x^2 - y^2\right).$$

Hw 3-8