Homework #5
Physics 206

17.2.1. 1) $H_0 = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$; $x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+)$

$$\lambda x^4 = \frac{\hbar^2\lambda}{4m^2\omega^2}(\hat{a} + \hat{a}^+)^4 = \frac{\hbar^2\lambda}{4m^2\omega^2}(\hat{a}^+ + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a})^2 =$$

$$= \frac{\lambda\hbar^2}{4m^2\omega^2}(\hat{a}^2 + \hat{a}^{+2} + 2\hat{a}^+\hat{a} + 1)^2 = \frac{\lambda\hbar^2}{4m^2\omega^2}(\hat{a}^4 + \hat{a}^{+4} + \hat{a}^2\hat{a}^{+2} + \hat{a}^{+2}\hat{a}^2 +$$

$$+ (\hat{a}^2 + \hat{a}^{+2})2\hat{a}^+\hat{a} + 2\hat{a}^+\hat{a}(\hat{a}^2 + \hat{a}^{+2}) + 4\hat{a}^+\hat{a}\,\hat{a}^+\hat{a} + 2\hat{a}^2 + 2\hat{a}^{+2} +$$

$$+ 4\hat{a}^+\hat{a} + 1)$$

$$\Delta E = \langle H^1 \rangle$$

$$\langle \hat{a}^4 \rangle = \langle \hat{a}^{+4} \rangle = \langle (\hat{a}^2 + \hat{a}^{+2})2\hat{a}^+\hat{a} \rangle = \langle 2\hat{a}^+\hat{a}(\hat{a}^2 + \hat{a}^{+2}) \rangle$$

$$= \langle 2\hat{a}^2 \rangle = \langle 2\hat{a}^{+2} \rangle = 0$$

$$\langle H^1 \rangle = \frac{\lambda\hbar^2}{4m^2\omega^2}\left[\sqrt{n+1}\sqrt{n+2}\sqrt{n+2}\sqrt{n+1} + \sqrt{n}\sqrt{n-1}\sqrt{n-1}\sqrt{n} + \right.$$

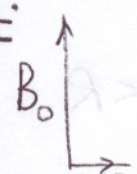
$$\left. + 4\sqrt{n}\sqrt{n}\sqrt{n}\sqrt{n} + 4\sqrt{n}\sqrt{n} + 1\right] = \frac{\lambda\hbar^2}{4m^2\omega^2}[(n+1)(n+2) + n(n-1) + 4n^2 +$$

$$+ 4n + 1] = \frac{\lambda\hbar^2}{4m^2\omega^2}[6n^2 + 6n + 3] = \frac{3\lambda\hbar^2}{4m^2\omega^2}[2n^2 + 2n + 1].$$

2). For any λ $\lambda n^2 >> \hbar\omega n$ at large n.

So the perturbation theory makes no sense.

17.2.2.



$$\psi_0 = |1/2\rangle$$

$$E_1 = -\frac{\gamma}{2}\langle 1/2 | B_6 \times | 1/2 \rangle = 0$$

$$E_2 = +\frac{\gamma^2}{4}\frac{|\langle 1/2 | B_6 \times | -1/2 \rangle|^2}{+E_{1/2} - E_{-1/2}} = \frac{B^2\gamma^2/4}{-2\gamma\,1/2\,B_0} = \frac{-B^2\gamma^2/4}{\gamma B_0} =$$

$$= -\frac{B^2\gamma}{2B_0}$$

$$\psi = |1/2\rangle + |\psi_1\rangle = |1/2\rangle + |-1/2\rangle \frac{\langle 1/2 | B_6 \times | -1/2 \rangle (-\frac{\gamma}{2})}{-2\gamma\frac{1}{2}B_0}$$

$$= |1/2\rangle + |-1/2\rangle \frac{B}{\gamma B_0}(-\frac{\gamma}{2}) = |1/2\rangle + |-1/2\rangle \frac{B}{2B_0}$$

Exact energy $-\frac{\gamma}{2}\sqrt{B_0^2 + B^2} = -\frac{\gamma}{2}B_0 - \frac{\gamma}{4}\frac{B^2}{B_0}$.

Exact ground state: $H \sim B_0 \sigma_z + B\sigma_x = \begin{pmatrix} B_0 & B \\ B & -B_0 \end{pmatrix}$

$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\quad \begin{pmatrix} B_0 & B \\ B & -B_0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \sqrt{B_0^2+B^2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\alpha\left[\sqrt{B_0^2+B^2} - B_0\right] = B\beta; \quad \alpha \approx 1;$

$$\beta = \frac{\alpha B^2}{\sqrt{B_0^2+B^2}+B_0}\frac{1}{B} \approx \frac{B}{2B_0} //$$

$$\alpha = \sqrt{1-\beta^2} = 1 + O(B^2) //.$$

17.2.3. 1) Gauss theorem:

$R < r \Rightarrow \quad 4\pi r^2 E = 4\pi e \Rightarrow E = \frac{e}{r^2} \Rightarrow \varphi(r) = -\frac{e^2}{r}$

$R > r \Rightarrow \quad 4\pi r^2 E = 4\pi \frac{4\pi}{3} r^3 \frac{e}{\frac{4\pi}{3}R^3} = 4\pi e\left(\frac{r}{R}\right)^3;$

$$E = \frac{er}{R^3} \Rightarrow \varphi = \text{const} + \frac{e^2 r^2}{2R^3}.$$

$$\varphi(R) = -\frac{e^2}{R} \Rightarrow \varphi = -\frac{3e^2}{2R} + \frac{e^2 r^2}{2R^3}$$

2) $\psi = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \approx \frac{1}{\sqrt{\pi a_0^3}}$ for $r < R$

$$H_1 = \frac{e^2}{r} - \frac{3e^2}{2R} + \frac{e^2 r^2}{2R^3}$$

$$\langle H_1 \rangle \propto \frac{4\pi e^2}{\pi a_0^3}\int_0^R r^2 dr\left[\frac{1}{r} - \frac{3}{2R} + \frac{r^2}{2R^3}\right] =$$

$$= \frac{4e^2}{a_0^3}\int_0^R \left[r\,dr - \frac{3r^2 dr}{2R} + \frac{r^4 dr}{2R^3}\right] =$$

$$= \frac{2e^2}{a_0^3}\int_0^R \left[2r\,dr - \frac{3r^2 dr}{R} + \frac{r^4 dr}{R^3}\right] = \frac{2e^2}{a_0^3}\left[R^2 - \frac{3R^2}{3} + \frac{R^2}{5}\right]$$

$$= \frac{2e^2 R^2}{5a_0^3} //$$

17.2.4. 1) $\sum_{n'} (E_{n'} - E_n)|\langle n'|x|n\rangle|^2 = \sum_{n'} \langle n|x|n'\rangle\langle n'|[H,x]|n\rangle =$

$$= \sum_{n'} \langle n|x|n'\rangle\langle n'|\left[\frac{p^2}{2M}, x\right]|n\rangle$$

$$\left[\frac{p^2}{2M}, x\right] = -\frac{\hbar^2}{2M}\left[\hat\Delta\hat x - \hat x\hat\Delta\right] = -\frac{\hbar^2}{M}\frac{\partial}{\partial x} = -\frac{\hbar}{m}\hat p$$

For 1D motion all $|n'\rangle$ can be chosen to be real. Hence, $\langle n|x|n'\rangle = \left(\langle n'|x|n\rangle\right)^* = \langle n'|x|n\rangle$.

$\langle n'|\left[\frac{\hat p^2}{2m}, x\right]|n\rangle = \langle n'|-\frac{i\hbar}{m}\hat p|n\rangle = \left(\langle n|\left(-\frac{i\hbar}{m}\hat p\right)^+|n'\rangle\right)^* =$

$= \left(\langle n|\frac{i\hbar}{m}\hat p|n'\rangle\right)^* = \langle n|\frac{i\hbar}{m}\hat p|n'\rangle$ (since $\frac{i\hbar}{m}\hat p = \frac{\hbar}{m}\frac{\partial}{\partial x}$ is a real operator).

Thus, $\sum_{n'} \langle n|x|n'\rangle\langle n'|\left(-\frac{i\hbar}{m}\hat p\right)|n\rangle =$

$= \frac{1}{2}\sum_{n'}\left(\langle n|x|n'\rangle\langle n'|-\frac{i\hbar}{m}\hat p\right)|n\rangle + \langle n|\frac{i\hbar}{m}\hat p|n'\rangle\langle n'|x|n\rangle\Big)$

$= \frac{1}{2}\langle n|\frac{i\hbar}{m}\hat p x - \frac{i\hbar x\hat p}{m}|n\rangle = \frac{1}{2}\langle n|\frac{\hbar^2}{m}|n\rangle = \frac{\hbar^2}{2m}$

2). $\sum_{n'}(E_{n'}-E_n)|\langle n'|x|n\rangle|^2 = \sum_{n'}(E_{n'}-E_n)\frac{\hbar}{2m\omega}|\langle n'|\hat a+\hat a^+|n\rangle|^2$

$= \sum_{n'}\frac{\hbar^2}{2m}(n'-n)\left[\delta_{n,n'+1}\left(\sqrt{n}\right)^2 + \delta_{n,n'-1}\left(\sqrt{n+1}\right)^2\right] =$

$= \frac{\hbar^2}{2m}\left[-n+n+1\right] = \frac{\hbar^2}{2m}$ //

5.1. $\langle 0|bx|0\rangle = 0$

$\Delta E = \sum_{n>0}\frac{|\langle n|bx|0\rangle|^2}{\frac{\hbar\omega}{2}-(n+\frac12)\hbar\omega} = \frac{|\langle 1|bx|0\rangle|^2}{-\hbar\omega} = -\frac{b^2}{\hbar\omega}\frac{\hbar}{2m\omega} = -\frac{b^2}{2m\omega^2}$

Exact solution

$H = \frac{\hat p^2}{2m} + \frac{m\omega^2x^2}{2} + bx = \frac{\hat p^2}{2m} + \frac{m\omega^2}{2}\left(x+\frac{b}{m\omega^2}\right)^2 -$

$-\frac{b^2}{m^2\omega^4}\frac{m\omega^2}{2} = \frac{\hat p^2}{2m} + \frac{m\omega^2}{2}\tilde x^2 - \frac{b^2}{2m\omega^2}$.

$E_0 = \frac{\hbar\omega}{2} - \frac{b^2}{2m\omega^2}$ //

5.3. $\psi_0 = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}$, $E = 2\frac{\hbar^2\pi^2}{2mL^2} = \frac{\hbar^2\pi^2}{mL^2}$.

$\psi_{1,a} = \frac{2}{L}\sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}$, $E = \frac{5}{2}\frac{\hbar^2\pi^2}{mL^2}$

$\psi_{1,b} = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L}$, $E = \frac{5}{2}\frac{\hbar^2\pi^2}{mL^2}$

$V_1 = \lambda xy$.

1) ψ_0 does not change in the 0th order.

$$\delta E_0 = \lambda\langle xy\rangle = \frac{4\lambda}{L^2}\int dx\,dy\,\sin^2\frac{2\pi x}{L}\sin^2\frac{\pi y}{L}\,xy =$$

$$= \frac{4\lambda}{L^2}\left(\int dx\sin^2\frac{\pi x}{L}\,x\right)^2 = \frac{4\lambda}{L^2}\left(\int_0^{L/2}\left[\sin^2\frac{2\pi x}{L}\right)x + \right.$$

$$\left. + \sin^2\frac{\pi}{L}(L-x)(L-x)\right]dx\Big)^2 = \frac{4\lambda}{L^2}\left(\int_0^{L/2} dx\sin^2\frac{\pi x}{L}\,L\right)^2 =$$

$$= 4\lambda\left(\int_0^{L/2} dx\sin^2\frac{\pi x}{L}\right)^2 = 4\lambda\frac{L^2}{\pi^2}\left(\int_0^{L/2} d\frac{\pi x}{L}\sin^2\frac{\pi x}{L}\right)^2 =$$

$$= \frac{4\lambda L^2}{\pi^2}\left(\int_0^{\pi/2} dy\sin^2 y\right)^2 = \frac{4\lambda L^2}{\pi^2}\left(\frac{\pi}{4}\right)^2 = \frac{\lambda L^2}{4}$$

2). $\langle\psi_{1,a}|\lambda xy|\psi_{1a}\rangle = \frac{4\lambda}{L^2}\int dx\,\sin^2\frac{2\pi x}{L}\,x\int dy\,\sin^2\frac{\pi y}{L}\,y =$

$$= \frac{4\lambda}{L^2}\frac{L^2}{4}\int dx\,\sin^2\frac{2\pi x}{L}\,x = \lambda\left[\int_0^{L/2} dx\left(\sin^2\frac{2\pi x}{L}\,x + \sin^2\frac{2\pi}{L}(L-x)(L-x)\right)\right]$$

$$= \lambda L\int_0^{L/2} dx\sin^2\frac{2\pi x}{L} = \lambda L\frac{L}{2\pi}\int_0^{L/2} d\frac{2\pi x}{L}\sin^2\frac{2\pi x}{L} = \frac{\lambda L^2}{2\pi}\int_0^{\pi} dy\sin^2 y =$$

$$= \frac{\lambda L^2}{2\pi}\frac{\pi}{4} = \frac{\lambda L^2}{4}.$$

$$\langle\psi_{1,b}|\lambda xy|\psi_{1,b}\rangle = \langle\psi_{1,a}|\lambda xy|\psi_{1,a}\rangle = \frac{\lambda L^2}{4}$$

$$\langle\psi_{1,b}|\lambda xy|\psi_{1,a}\rangle = \frac{4\lambda}{L^2}\int_0^L dx\,dy\,xy\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{2\pi y}{L} =$$

$$= \frac{4\lambda}{L^2}\left(\int_0^L x\,dx\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\right)^2 = \frac{4\lambda}{L^2}\left(\int_0^L\frac{x\,dx}{2}\left[\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right]\right)^2 =$$

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$$= \frac{\lambda}{L^2}\left(\int_0^L x\,dx\left[\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right]\right)^2 = \frac{\lambda}{L^2}\left[\int_0^L x\,d\left(\frac{L}{\pi}\sin\frac{\pi x}{L} - \frac{L}{3\pi}\sin\frac{3\pi x}{L}\right)\right]^2 =$$

$$= \frac{\lambda}{\pi^2}\left[\int_0^L x\,d\left(\sin\frac{\pi x}{L} - \frac{1}{3}\sin\frac{3\pi x}{L}\right)\right]^2 = \frac{\lambda}{\pi^2}\left[\int_0^L \left(\sin\frac{\pi x}{L} - \frac{1}{3}\sin\frac{3\pi x}{L}\right)dx\right]$$

$$= \frac{\lambda}{\pi^2}\left[\frac{L}{\pi}(\cos 0 - \cos\pi) - \frac{L}{9\pi}(\cos 0 - \cos 3\pi)\right]^2 =$$

$$= \frac{\lambda}{\pi^2}\left[\frac{2L}{\pi} - \frac{2L}{9\pi}\right]^2 = \frac{\lambda}{\pi^2}\left[\frac{16L}{9\pi}\right]^2 = \frac{256}{81\pi^4}\lambda L^2.$$

$$H^{(1)} = \lambda L^4 \begin{pmatrix} 1/4 & \frac{256}{81\pi^4} \\ \frac{256}{81\pi^4} & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} - \frac{E^{(1)}}{\lambda L^2} & \frac{256}{81\pi^4} \\ \frac{256}{81\pi^4} & \frac{1}{4} - \frac{E^{(1)}}{\lambda L^2} \end{pmatrix} = 0$$

$$\frac{E^{(1)}}{\lambda L^4} = \frac{1}{4} \pm \frac{256}{81\pi^4};$$

$$E^{(1)} = \frac{\lambda L^4}{4} \pm \frac{256}{81\pi^4}\lambda L^4.$$

$$\begin{pmatrix} \mp\frac{256}{81\pi^4} & \frac{256}{81\pi^4} \\ \frac{256}{81\pi^4} & \mp\frac{256}{81\pi^4} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \text{ gives}$$

the zeroth order eigenfunctions $\alpha\psi_{1a} + \beta\psi_{1b}$.

These are

$(\psi_{1a} \pm \psi_{1b})/\sqrt{2}$.

5.4. $H_0 = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)$

a) $E_0 = 2\frac{\hbar\omega}{2} = \hbar\omega \qquad \psi = |0\rangle|0\rangle$

$E_1 = \frac{\hbar\omega}{2} + \frac{3}{2}\hbar\omega = 2\hbar\omega$, degeneracy $= 2 \qquad \psi = |0\rangle|1\rangle$ & $|1\rangle|0\rangle$

$E_2 = \frac{\hbar\omega}{2} + \frac{5}{2}\hbar\omega = \frac{3}{2}\hbar\omega + \frac{3}{2}\hbar\omega = 3\hbar\omega$, degeneracy $= 3$

$\psi = |0\rangle|2\rangle,\ |2\rangle|0\rangle,\ |1\rangle|1\rangle$

B). $V = \delta m\omega^2 xy$

$\psi_0 = |0\rangle$

$\delta E = \langle 0|V|0\rangle = \langle 0|x|0\rangle\langle 0|y|0\rangle = 0.$

$\psi_{1,1} = |0\rangle|1\rangle$, $\psi_{1,2} = |1\rangle|0\rangle$

$\langle\psi_{1,1}|V|\psi_{1,1}\rangle \sim \langle 0|x|0\rangle = 0$

$\langle\psi_{1,2}|V|\psi_{1,2}\rangle = 0$

$\langle\psi_{1,1}|V|\psi_{1,2}\rangle = \langle 0|\langle 1|\delta m\omega^2 xy|1\rangle|0\rangle =$

$= |\langle 0|x|1\rangle|^2 \delta m\omega^2 = \frac{\hbar\delta m\omega^2}{2m\omega} = \frac{\delta\hbar\omega}{2}.$

$H_{(1)} = \begin{vmatrix} 0 & \frac{\delta\hbar\omega}{2} \\ \delta\frac{\hbar\omega}{2} & 0 \end{vmatrix} \Rightarrow$ functions of the 0th approximation are $\frac{\psi_{1,1} \pm \psi_{1,2}}{\sqrt{2}}$

$\det(H_{(1)} - E_{(1)}) = 0 \Rightarrow E_{(1)} = \pm\frac{\delta\hbar\omega}{2}.$

$\psi_{1,1} = |0\rangle|2\rangle$, $\psi_{1,2} = |1\rangle|1\rangle$, $\psi_{1,3} = |2\rangle|0\rangle$

$\langle\psi_{1,1}|V|\psi_{1,1}\rangle = 0$; $\langle\psi_{1,2}|V|\psi_{1,2}\rangle = 0$, $\langle\psi_{1,3}|V|\psi_{1,3}\rangle = 0$

$\langle\psi_{1,1}|V|\psi_{1,3}\rangle = \langle 0|x|2\rangle\langle 2|y|0\rangle\delta m\omega^2 = 0$

$\langle\psi_{1,1}|V|\psi_{1,2}\rangle = \langle 0|x|1\rangle\langle 2|y|1\rangle\delta m\omega^2 =$

$= \frac{\hbar}{2m\omega}\delta m\omega^2\sqrt{2} = \frac{\delta\hbar\omega}{\sqrt{2}}$; $\langle\psi_{1,2}|V|\psi_{1,3}\rangle = \frac{\delta\hbar\omega}{\sqrt{2}}.$

$$H_{(1)} = \frac{\delta\hbar\omega}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det\left(\frac{\delta\hbar\omega}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - E_{(1)}\right) = 0$$

$$\begin{vmatrix} -\frac{E_{(1)}\sqrt{2}}{\delta\hbar\omega} & 1 & 0 \\ 1 & -\frac{E_{(1)}\sqrt{2}}{\delta\hbar\omega} & 1 \\ 0 & 1 & -\frac{E_{(1)}\sqrt{2}}{\delta\hbar\omega} \end{vmatrix} = 0;$$

$$-\frac{E_{(1)}\sqrt{2}}{\delta\hbar\omega}\left[\frac{E_{(1)}^2 2}{\delta^2\hbar^2\omega^2} - 1\right] - 1\left[-\frac{E_{(1)}\sqrt{2}}{\delta\hbar\omega}\right] = 0;$$

$$+\frac{E_{(1)}\sqrt{2}}{\delta\hbar\omega}\left[\frac{2E_{(1)}^2}{\delta^2\hbar^2\omega^2} - 2\right] = 0;$$

$$E_{(1)} = 0 \quad \text{or} \quad E_{(1)} = \pm\delta\hbar\omega$$

Zeroth order wave functions

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$\beta = 0$

$\alpha + \gamma = 0 \Rightarrow \dfrac{|\psi_{1,1}\rangle - |\psi_{1,3}\rangle}{\sqrt{2}}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \pm\sqrt{2}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$\beta = \pm\sqrt{2}\alpha;$

$\alpha + \gamma = \pm\sqrt{2}\beta;$

$\beta = \pm\sqrt{2}\gamma \Rightarrow \alpha = \gamma = \pm\frac{\beta}{\sqrt{2}}.$

$$|\psi\rangle = \frac{1}{2}(|\psi_{1,1}\rangle + |\psi_{1,3}\rangle) \pm \frac{1}{\sqrt{2}}|\psi_{1,2}\rangle$$

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c). $H_0 = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(x^2+2\delta xy+y^2) =$

$= \frac{\hat{p}_{\frac{x+y}{\sqrt2}}^2}{2m} + \frac{\hat{p}_{\frac{x-y}{\sqrt2}}^2}{2m} + \frac{m\omega^2}{2}\left[(1+\delta)\left(\frac{x+y}{\sqrt2}\right)^2 + (1-\delta)\left(\frac{x-y}{\sqrt2}\right)^2\right] =$

$= \frac{\hat{p}_{x'}^2}{2m} + \frac{\hat{p}_{y'}^2}{2m} + \frac{m\omega^2}{2}\left[(1+\delta)x'^2 + (1-\delta)y'^2\right].$

$E_0 = \frac{\hbar\omega}{2}\left[\sqrt{1+\delta} + \sqrt{1-\delta}\right] = \hbar\omega(1+O(\delta^2))$ //

$E_1 \Rightarrow E_0 + \hbar\omega\sqrt{1\pm\delta} = \hbar\omega \pm \frac{\hbar\omega\delta}{2} + O(\delta^2)$ //

$E_2 \Rightarrow \begin{cases} 3E_0 = 3\hbar\omega + O(\delta^2) \\ E_0 + 2\hbar\omega\sqrt{1\pm\delta} = 3\hbar\omega \pm \delta\hbar\omega \end{cases}$ //

Thus, exact energies concide with our perturbative results at small δ.

5.9a). $|n, \ell=1, m=\pm1, 0\rangle$

$V = \lambda(x^2-y^2)$

$|m=\pm1\rangle \sim \mp\frac{x\pm iy}{\sqrt2}$

$|m=0\rangle \sim z.$

Consider $|m=0\rangle \sim z$ and

$\frac{|m=1\rangle \pm |m=-1\rangle}{\sqrt2} \sim x, y.$

$\langle z_\alpha | x^2-y^2 | z_\beta \rangle = 0$ for $\alpha \neq \beta$

from parity considerations

i.e. $\int x(x^2-y^2)y\, d^3x = 0$ since

for any x $\quad x(x^2-y^2)y + x(x^2-(-y)^2)(-y) = 0$

etc.

Thus, the result is $|m=0\rangle$, $\frac{|m=1\rangle \pm |m=-1\rangle}{\sqrt2}$

5.11 a). $\begin{pmatrix} E_1^\circ & \lambda\Delta \\ \lambda\Delta & E_2^\circ \end{pmatrix}$

$$(E-E_1^\circ)(E-E_2^\circ) = \lambda^2\Delta^2;$$

$$E^2 - E(E_1^\circ + E_2^\circ) + E_1^\circ E_2^\circ - \lambda^2\Delta^2 = 0;$$

$$E = \frac{E_1^\circ + E_2^\circ}{2} \pm \frac{1}{2}\sqrt{(E_1^\circ + E_2^\circ)^2 - 4E_1^\circ E_2^\circ + 4\lambda^2\Delta^2} =$$

$$= \frac{E_1^\circ + E_2^\circ}{2} \pm \sqrt{\frac{(E_1^\circ - E_2^\circ)^2}{4} + \lambda^2\Delta^2}.$$

$$(E_1^\circ - E)\alpha + \lambda\Delta\beta = 0 \Rightarrow$$

$$\alpha = \frac{\lambda\Delta\beta}{E_1^\circ - E} = \frac{\lambda\Delta\,\beta}{\frac{E_1^\circ - E_2^\circ}{2} \mp \sqrt{\frac{(E_1^\circ - E_2^\circ)^2}{4} + \lambda^2\Delta^2}}.$$

$$\alpha^2 + \beta^2 = 1 \Rightarrow$$

$$\beta^2\left[1 + \frac{\Delta^2\lambda^2}{\frac{(E_1^\circ - E_2^\circ)^2}{2} + \lambda^2\Delta^2 \mp (E_1^\circ - E_2^\circ)\sqrt{\quad}}\right] = 1$$

$$\beta = \sqrt{\frac{\frac{(E_1^\circ - E_2)^2}{2} + \lambda^2\Delta^2 \mp (E_1^\circ - E_2^\circ)\sqrt{\quad}}{\frac{(E_1^\circ - E_2^\circ)^2}{2} + 2\lambda^2\Delta^2 \mp (E_1^\circ - E_2^\circ)\sqrt{\quad}}}.$$

в). First, expand the exact results in powers of $\lambda\Delta$.

$$E = \frac{E_1^\circ + E_2^\circ}{2} \pm \frac{E_1^\circ - E_2^\circ}{2}\sqrt{1 + \frac{4\lambda^2\Delta^2}{(E_1^\circ - E_2^\circ)^2}} = \frac{E_1^\circ + E_2^\circ}{2} \pm$$

$$\pm \frac{E_1 - E_2^\circ}{2}\left(1 + \frac{2\lambda^2\Delta^2}{(E_1^\circ - E_2^\circ)^2}\right) = \begin{cases} E_1^\circ + \dfrac{\lambda^2\Delta^2}{E_1^\circ - E_2^\circ} \\ E_2^\circ + \dfrac{\lambda^2\Delta^2}{E_2^\circ - E_1^\circ} \end{cases}$$

But this is exactly the second order perturbation theory result!

HW5-9

What about α and β?

1) $\beta \approx 1$ (+ sign in our exact expression, assume that $E_1^\circ > E_2^\circ$)

Then $\alpha = \dfrac{\lambda\Delta}{E_1^\circ - E} \quad \beta = \dfrac{\lambda\Delta}{\frac{E_1^\circ - E_2^\circ}{2} + \sqrt{\frac{(E_1^\circ - E_2^\circ)^2}{4} + \lambda^2\Delta^2}} \quad \beta \approx$

$\approx \dfrac{\lambda\Delta}{E_1^\circ - E_2^\circ}$ //

Again, this is the perturbation theory result.

2) The "$-$" sign

$$\beta = \sqrt{\frac{\lambda^2\Delta^2 - \frac{(E_1^\circ - E_2^\circ)^2}{2}\left[-1 + \sqrt{1 + \frac{4\lambda^2\Delta^2}{(E_1^\circ - E_2^\circ)^2}}\right]}{2\lambda^2\Delta^2 - \frac{(E_1^\circ - E_2^\circ)^2}{2}\left[-1 + \sqrt{1 + \frac{4\lambda^2\Delta^2}{(E_1^\circ - E_2^\circ)^2}}\right]}} \approx$$

$$\approx \sqrt{\frac{\lambda^2\Delta^2 - \frac{(E_1^\circ - E_2^\circ)^2}{2}\left[\frac{2\lambda^2\Delta^2}{(E_1^\circ - E_2^\circ)^2} - \frac{1}{8}\frac{16\lambda^4\Delta^4}{(E_1^\circ - E_2^\circ)^4}\right]}{\lambda^2\Delta^2}} =$$

$$= \sqrt{\frac{\lambda^4\Delta^4}{\lambda^2\Delta^2 (E_1^\circ - E_2^\circ)^2}} = \frac{\lambda\Delta}{E_1^\circ - E_2^\circ}$$ //

$$\alpha = \frac{\lambda\Delta\beta}{\frac{E_1^\circ - E_2^\circ}{2} - \frac{E_1^\circ - E_2^\circ}{2}\sqrt{1 + \frac{\lambda^2\Delta^2 4}{(E_1 - E_2^\circ)^2}}} \approx -1.$$ //

This again agrees with the perturbation theory.

c). The degenerate theory for this case was developed in class: $\psi = \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2)$ //

$E = E^\circ \pm \lambda|\Delta|$ //

For $E_1^\circ \approx E_2^\circ$

$E = \dfrac{E_1^\circ + E_2^\circ}{2} \pm \lambda\Delta$ //

$\beta \approx \sqrt{\dfrac{\lambda^2\Delta^2}{2\lambda^2\Delta^2}} = \dfrac{1}{\sqrt{2}}$ //.

HW 5-10