Physics 206 Homework #7

19.3.2 $V(r) = -V_0\,\theta(r_0 - r)$

1). $f(\theta,\varphi) = +\frac{2m}{\hbar^2}\int \frac{\sin qr'}{q} V(r')\,r'\,dr' =$

$= +\frac{2mV_0}{\hbar^2}\int_0^{r_0} \frac{\sin qr}{q}\, r\,dr = -\frac{2mV_0}{\hbar^2 q^2}\int_0^{r_0} r\,d\cos qr =$

$= -\frac{2mV_0}{\hbar^2 q^2}\left(r_0\cos qr_0 - \int_0^{r_0}\cos qr\,dr\right) =$

$= -\frac{2mV_0}{\hbar^2 q^3}(r_0 q\cos qr_0 - \sin qr_0).$

$\frac{d\sigma}{d\Omega} = \frac{4m^2V_0^2}{\hbar^4 q^6}(\sin qr_0 - qr_0\cos qr_0)^2 =$

$= 4r_0^2\left(\frac{mVr_0^2}{\hbar^2}\right)^2 \frac{(\sin qr_0 - qr_0\cos qr_0)^2}{(qr_0)^6}$

2) $kr_0 \to 0 \Rightarrow qr_0 \to 0 \quad (q = 2k\sin\frac{\theta}{2})$

$\frac{d\sigma}{d\Omega} \to 4r_0^2\left(\frac{mVr_0^2}{\hbar^2}\right)^2 \frac{\left[qr_0 - \frac{(qr_0)^3}{6} - qr_0 + \frac{(qr_0)^3}{2}\right]^2}{(qr_0)^6} =$

$= 4r_0^2\left(\frac{mVr_0^2}{\hbar^2}\right)^2 \frac{1}{9}$

$\sigma = 4\pi\frac{d\sigma}{d\Omega} = \frac{16\pi r_0^2}{9}\left(\frac{mVr_0^2}{\hbar^2}\right)^2$

19.3.3. $V(r) = V_0 e^{-r^2/r_0^2}$

$f = -\frac{2m}{\hbar^2}\int \frac{\sin qr}{q}\, r\,dr\, V_0 e^{-r^2/r_0^2} =$

$= -\frac{2mV_0}{q\hbar^2}\int \sin qr\, d\frac{r^2}{2}\, e^{-r^2/r_0^2} =$

$= -\frac{2mV_0}{2qr_0^{-2}\hbar^2}\int \sin qr\, e^{-r^2/r_0^2}\, d\frac{r^2}{r_0^2} =$

$$= \frac{2mV_0}{2q z_0^{-2}\hbar^2}\int \sin qz\, d e^{-z^2/z_0^2} = -\frac{mV_0}{qz_0^2}\int e^{-z^2/z_0^2} d\sin qz =$$

$$= -\frac{mV_0}{\hbar^2 z_0^{-2}}\int e^{-z^2/z_0^2}\sin qz\, dz = -\frac{mV_0}{z_0^2}\,\mathrm{Re}\int_0^\infty e^{-\frac{z^2}{z_0^2}-iqz}\,dz =$$

$$= -\frac{mV_0}{\hbar^2 z_0^{-2}}\left[\frac{1}{2}\int_0^{+\infty} e^{-z^2/z_0^2 - iqz}\,dz + \frac{1}{2}\int_0^{+\infty} e^{-z^2/z_0^2+iqz}\,dz\right] =$$

$$= -\frac{mV_0}{2z_0^{-2}\hbar^2}\int_{-\infty}^{+\infty} dz\, e^{-\frac{z^2}{z_0^2}-iqz} = -\frac{mV_0}{2z_0^{-2}}\int_{-\infty}^{+\infty} z_0\, d\frac{z}{z_0}\, e^{-\left(\frac{z}{z_0}\right)^2 \frac{z_0^2}{2}\frac{2z}{z_0}} =$$

$$= -\frac{mV_0}{\hbar^2 2 z_0^{-3}}\int_{-\infty}^{\infty} dy\, e^{-y^2-2y\frac{iqz_0}{2}} = -\frac{mV_0}{2z_0^{-3}}\int_{-\infty}^{+\infty} dy\, e^{-\left(y+\frac{iqz_0}{2}\right)^2 - \frac{q^2z_0^2}{4}} =$$

$$= -\frac{mV_0\sqrt{\pi}}{2\hbar^2 z_0^{-3}}\, e^{-\frac{q^2z_0^2}{4}}$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{\pi m^2 V_0^2}{4 z_0^{-6}\hbar^4} e^{-\frac{q^2z_0^2}{2}} = \frac{\pi z_0^2}{4}\left(\frac{mV_0 z_0^2}{\hbar^2}\right)^2 e^{-\frac{q^2z_0^2}{2}}$$

$$\sigma = \frac{\pi z_0^2}{4}\left(\frac{mV_0 z_0^2}{\hbar^2}\right)^2 \int e^{-\frac{q^2z_0^2}{2}} d\Omega = \frac{\pi^2 z_0^2}{2}\left(\frac{mV_0 z_0^2}{\hbar^2}\right)^2 \times$$

$$\times \int_{q=0}^{q=2k} e^{-\frac{q^2z_0^2}{2}} d\frac{q^2}{2k^2} = \frac{\pi^2 z_0^2}{2}\left(\frac{mV_0 z_0^2}{\hbar^2}\right)^2 \int_0^2 dz\, e^{-\frac{z z_0^2 k^2}{2}} =$$

$$= \frac{\pi z_0^2}{2}\left(\frac{mV_0 z_0^2}{\hbar^2}\right)^2 \frac{1}{z_0^2 k^2}\left(1 - e^{-2k^2 z_0^2}\right) =$$

$$= \frac{\pi}{2k^2}\left(\frac{\mu V_0 z_0^2}{\hbar^2}\right)^2 \left(1 - e^{-2k^2 z_0^2}\right)$$

19.5.1. $m_n c^2 \approx 10^3$ MeV

$$\frac{m_n v^2}{2} = 10^2 \text{ MeV} \Leftrightarrow \frac{p^2}{2m_n} = 10^2 \text{ MeV};$$

$$p^2 = 2\cdot 10^5 (\text{MeV})^2 \frac{1}{c^2}$$

$$(pz_0)^2 = \frac{2\cdot 10^5 (\text{MeV}\cdot F)^2}{c^2} = 2\cdot 10^5 \frac{1}{4\cdot 10^4}\frac{\hbar^2 c^2}{c^2} =$$

$$= 5\hbar^2 \Rightarrow \quad l \leq pz_0 \sim \sqrt{5\hbar^2} \sim 2\hbar$$

19.5.2 $\sigma = \int |f|^2 d\Omega = \int f f^* d\Omega =$

$= \int d\Omega \frac{1}{k^2} \sum_{\ell_1,\ell_2=0}^{\infty} (2\ell_1+1)(2\ell_2+1) e^{i\delta_{\ell_1} - i\delta_{\ell_2}} \sin\delta_{\ell_1} \sin\delta_{\ell_2} P_{\ell_1}(\cos\theta) \times$

$\times P_{\ell_2}(\cos\theta) = \frac{2\pi}{k^2} \sum_{\ell_1,\ell_2=0}^{\infty} (2\ell_1+1)(2\ell_2+1) e^{i(\delta_{\ell_1} - \delta_{\ell_2})} \sin\delta_{\ell_1} \sin\delta_{\ell_2} \times$

$\times \int P_{\ell_1}(x) P_{\ell_2}(x) dx = \frac{2\pi}{k^2} \sum_{\ell_1,\ell_2=0}^{\infty} (2\ell_1+1)(2\ell_2+1) e^{i(\delta_{\ell_1} - \delta_{\ell_2})} \times$

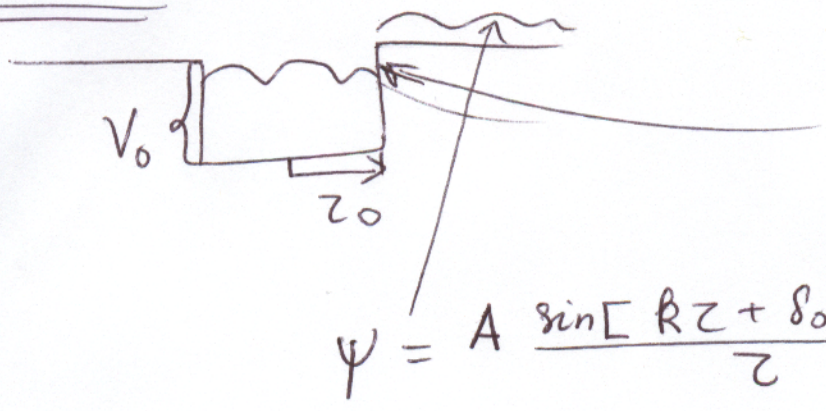
$\times \sin\delta_{\ell_1} \sin\delta_{\ell_2} \frac{2}{2\ell_1+1} \delta_{\ell_1,\ell_2} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell .$

$\mathrm{Im}\, f(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin\delta_\ell P_\ell(\cos 0) \,\mathrm{Im}\, e^{i\delta_\ell} =$

$= \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin\delta_\ell \cdot 1 \cdot \sin\delta_\ell = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell \Rightarrow$

$\Rightarrow \sigma = \frac{4\pi}{k} \mathrm{Im}\, f(0)$

19.5.4.



$\psi = \frac{\sin k' r}{r}$ ← general solution for a free particle near $r = 0$

$\psi = A \frac{\sin[kr + \delta_0(k)]}{r}$ ← general solution for large r.

At $r = r_0$

$\frac{(r\psi)'}{(r\psi)} = \frac{k' \cos k' r_0}{\sin k' r_0} = \frac{k \cos[kr_0 + \delta_0(k)]}{\sin[kr_0 + \delta_0(k)]}$

$\frac{k}{k'} \mathrm{tg}[k' r_0] = \mathrm{tg}[kr_0 + \delta_0(k)] \Rightarrow$

$\Rightarrow \delta_0(k) = -kr_0 + \tan^{-1}\left[\frac{k}{k'} \tan k' r_0\right]$

HW 7-3

Let $k' = (2n+1)\frac{\pi}{2r_0} + \Delta k' \approx (2n+1)\frac{\pi}{2r_0}$, $k = k_n + \Delta k \approx k_n$

$$\delta_0 = -(k_n + \Delta k) r_0 + \tan^{-1}\left(\frac{k_n + \Delta k}{k_n' + \Delta k'} \tan\left(\frac{2n+1}{2}\pi + \Delta k' r_0 \right) \right) \approx$$

$$\approx -k_n r_0 + \tan^{-1}\left(\frac{k_n}{k_n'} \frac{-1}{\tan(\Delta k' r_0)} \right) \approx$$

$$\approx -k_n r_0 + \tan^{-1}\left[-\frac{k_n}{k_n' \Delta k' r_0} \right]$$

$$E = \frac{\hbar^2 k'^2}{2m} + V_0$$

$$E_0 - E = -\Delta E = -\frac{\hbar^2 k' \Delta k'}{m} \quad ; (\Delta k') = \frac{m(E_0 - E)}{\hbar^2 k'}$$

$$\delta_0 = -k_n r_0 + \tan^{-1}\left[+\frac{k_n}{k_n' \frac{m r_0}{\hbar^2 k_n'} (E_0 - E)} \right] =$$

$$= \delta_B + \tan^{-1}\left[\frac{\hbar^2 k_n}{m r_0 (E_0 - E)} \right] = \delta_B + \tan^{-1}\left[\frac{\Gamma/2}{E_0 - E} \right].$$

$k_0' = \frac{\pi}{2r_0}$;

$$E = 0 = V_0 + \frac{\hbar^2 k_0'^2}{2m} = V_0 + \frac{\pi^2 \hbar^2}{8 m r_0^2} \Rightarrow V_0 = -\frac{\pi^2 \hbar^2}{8 m r_0^2}.$$