Physics 206
Homework #8

18.4.4. 1) $H = \frac{(\bar{p} - \frac{e}{c}\bar{A})^2}{2m} + e\varphi$

2) $\bar{A}' = \bar{A} - \nabla\Lambda, \quad \Phi' = \Phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}$

$$H' = \frac{(\bar{p} - \frac{e}{c}\bar{A} + \frac{e}{c}\nabla\Lambda)^2}{2m} + e\varphi + \frac{e}{c}\frac{\partial\Lambda}{\partial t}.$$

3) $H' e^{-ie\Lambda/\hbar c}\psi = \frac{(\bar{p} - \frac{e}{c}\bar{A} + \frac{e}{c}\nabla\Lambda)(\bar{p} - \frac{e}{c}\bar{A} + \frac{e}{c}\nabla\Lambda)}{2m} e^{-\frac{ie\Lambda}{\hbar c}}\psi +$

$+ \left(e\varphi\psi + \frac{e}{c}\frac{\partial\Lambda}{\partial t}\psi\right) e^{-ie\Lambda/\hbar c} =$

$= \frac{\bar{p} - \frac{e}{c}\bar{A} + \frac{e}{c}\nabla\Lambda}{2m}\left\{\left[(\bar{p} - \frac{e}{c}\bar{A})\psi\right] e^{-\frac{ie\Lambda}{\hbar c}} - \cancel{\frac{e\nabla\Lambda}{c}\psi} + \cancel{\frac{e}{c}\nabla\Lambda\psi}\right\} +$

$+ \left(e\varphi\psi + \frac{e}{c}\frac{\partial\Lambda}{\partial t}\psi\right) e^{-\frac{ie\Lambda}{\hbar c}} =$

$= \left[\frac{(\bar{p} - \frac{e}{c}\bar{A})^2}{2m}\psi + e\varphi\psi + \frac{e}{c}\frac{\partial\Lambda}{\partial t}\psi\right] e^{-\frac{ie\Lambda}{\hbar c}}$

$$i\hbar\frac{\partial}{\partial t}\left[\psi e^{-ie\Lambda/\hbar c}\right] = \left[i\hbar\frac{\partial\psi}{\partial t} + e\frac{\partial\Lambda}{\partial t}\frac{1}{c}\psi\right] e^{-ie\Lambda/\hbar c} =$$

$$= \left[\left(\frac{(\bar{p} - \frac{e}{c}\bar{A})^2}{2m} + e\varphi\right)\psi + \frac{e}{c}\frac{\partial\Lambda}{\partial t}\psi\right] e^{-ie\Lambda/\hbar c} \Rightarrow$$

$$\boxed{i\hbar\frac{\partial}{\partial t}(\psi e^{-ie\Lambda/\hbar c}) = H'\psi e^{-ie\Lambda/\hbar c}}$$

18.5.2. 1). $G = \frac{128\, a_0^3 \pi e^2 p_f^3}{3mh^3\omega c\left[1 + p_f^2 a_0^2/\hbar^2\right]^4}$

$$\frac{m p_f^2}{2} = 10\,Ry$$

$$\hbar\omega = 11\,Ry \approx \cancel{x}\frac{p_f^2}{2m}$$

$$G \approx \frac{128\, a_0^3 \pi e^2 \cdot 2\, p_f}{3\cancel{x}\hbar^2 c\left[1 + \frac{p_f^2 a_0^2}{\hbar^2}\right]^4}$$

$$\frac{p_f^2 a^2}{\hbar^2} = \frac{2mE_f a^2}{\hbar^2} = \frac{20ma^2 Ry}{\hbar^2} = \frac{20ma^2}{2}\frac{me^4}{\hbar^4} = 10\frac{m^2a^2e^4}{\hbar^4} = 10.$$

HW8-1

$$\sigma \approx \frac{256\, a_0^3 \pi e^2 p_f}{3\hbar^2 c\, 11^4} = \frac{256}{3\cdot 11^4}\pi a_0^2 \frac{e^2 a_0 p_f}{\hbar^2 c} =$$

$$= \frac{256}{3\cdot 11^4}\pi a_0^2 \frac{p_f}{mc} \approx \frac{10^2}{10^4}\pi a_0^2 \frac{\sqrt{2m\cdot 10\, Ry}}{mc} \approx$$

$$\approx 10^{-2}\pi a_0^2 \frac{\sqrt{20\cdot 10^{-27} g\cdot 10^6 K}}{10^{-27} g\cdot 3\cdot 10^{10}\frac{cm}{c}} \approx 10^{-2}\pi a_0^2 \sqrt{\frac{20\cdot 10^{-27}\cdot 10^{-10}}{3\cdot 10^{-17}}} =$$

$$= 10^{-2}\pi a_0^2 \sqrt{\frac{2\cdot 10^{-36}}{3\cdot 10^{-17}}} \approx 10^{-2}\pi a_0^2\, 10^{-1} \sim \underline{\underline{10^{-3}\pi a_0^2}}$$

2).

$$\left.\frac{d\sigma}{d\Omega}\right|_{H,Z=1} = \frac{32 e^2 \hbar^5 \cos^2\theta}{m c \omega p_f^5 a_0^5}$$ ← goes from $\frac{e}{c}\bar{A}\bar{p}$

$$\frac{p^2}{2m} \sim \frac{Ze^2}{r} \Rightarrow \frac{\hbar^2}{ma^2} \sim \frac{Ze^2}{a} \Rightarrow a \sim \frac{\hbar^2}{me^2}\frac{1}{Z},\ p \sim \frac{Zme^2}{\hbar},$$

$$E \sim \frac{Ze^2}{a} \sim \frac{Z^2 e^4 m}{\hbar^2}$$

← does not renormalize

$$\left.\frac{d\sigma}{d\Omega}\right|_Z \sim \frac{32 e^2 \hbar^5 \cos^2\theta}{m c \omega p_f^5 \left(\frac{a_0}{Z}\right)^5} \sim Z^5 //$$

← renormalized atom size

20.1.1.

$$P = \psi^+\psi,\ \bar{J} = c\psi^+\bar{\alpha}\psi.$$

$$P = \psi_a^*\psi_a;$$

$$\vec{J} = c\psi_a^*\bar{\alpha}_{ab}\psi_b$$

$$\bar{\nabla}\bar{J} = c\frac{\partial}{\partial x_k}\left(\psi_a^*\alpha_{ab}^k\psi_b\right)$$

$$\frac{\partial}{\partial t}\psi_a^*\psi_a = \psi_a\frac{\partial}{\partial t}\psi_a^* + \psi_a^*\frac{\partial}{\partial t}\psi_a =$$

$$= \psi_a\left[\frac{1}{-i\hbar}(c\alpha^*\vec{p}^* + \beta mc^2)\psi_b^*\right] + \psi_a^*\left[\frac{1}{i\hbar}(c\bar{\alpha}\bar{p} + \beta mc^2)\psi_b\right] =$$

$$= \psi_a\frac{i}{\hbar}c\alpha_k^* i\hbar\frac{\partial}{\partial x_k}\psi_b^* + \psi_a^*\frac{(-i)}{\hbar}c\bar{\alpha}_k(-i)\hbar\frac{\partial}{\partial x_k}\psi_b =$$

$$= -\psi_a c\alpha_{ab}^{*k}\frac{\partial}{\partial x_k}\psi_b^* - c\psi_a^*\alpha_{ab}^k\frac{\partial}{\partial x_k}\psi_b =$$

$$= -c\frac{\partial}{\partial x_k}\psi_b^*\alpha_{ba}^k\psi_a - c\psi_a^*\alpha_{ab}^k\frac{\partial}{\partial x_k}\psi_b = -c\frac{\partial}{\partial x_k}\left(\psi_a^*\alpha_{ab}^k\psi_b\right) \Rightarrow$$

$$\Rightarrow \underline{\frac{\partial P}{\partial t} + \bar{\nabla}\bar{J} = 0}$$

1) $\vec{E} = -4\pi c \vec{\Pi} = -4\pi c (-i)\sqrt{\frac{\hbar}{64\pi^4 c^2}} \sum_\lambda \int d^3k \sqrt{\omega} \left[\hat{a}\, \vec{\varepsilon}^\lambda e^{i\vec{k}\vec{z}} - \hat{a}^+ \vec{\varepsilon}^\lambda e^{-i\vec{k}\vec{z}}\right] =$

$= \frac{4\pi c i \sqrt{\hbar}}{8\pi^2 c} \sum_\lambda \int [\dots] = \frac{i\sqrt{\hbar}}{2\pi} \sum_\lambda \int d^3k \sqrt{\omega} \left[\hat{a}\, \vec{\varepsilon}^\lambda e^{i\vec{k}\vec{z}} - \hat{a}^+ \vec{\varepsilon}^\lambda e^{-i\vec{k}\vec{z}}\right]$

$\vec{H} = \vec{\nabla} \times \vec{A} = \sqrt{\frac{\hbar c^2}{4\pi^2}}\, \varepsilon^{\alpha\beta\gamma} \frac{\partial}{\partial z_\beta} \sum_\lambda \int \frac{d^3k}{\sqrt{\omega}} \left[\hat{a}\, \varepsilon^\lambda_\gamma e^{i\vec{k}\vec{z}} + \hat{a}^+ \varepsilon^\lambda_\gamma e^{-i\vec{k}\vec{z}}\right] =$

$= \frac{i\sqrt{\hbar}\, c}{2\pi} \varepsilon^{\alpha\beta\gamma} \sum_\lambda \int \frac{d^3k}{\sqrt{\omega}}\, k_\beta\, \varepsilon^\lambda_\gamma \left[\hat{a}\, e^{i\vec{k}\vec{z}} - \hat{a}^+ e^{-i\vec{k}\vec{z}}\right]$

$[H_\alpha, E_\delta] = -\frac{\hbar c}{4\pi^2} \varepsilon^{\alpha\beta\gamma} \sum_{\lambda_H \lambda_E} \int \frac{d^3k_H}{\sqrt{\omega_H}}\, d^3k_E \sqrt{\omega_E}\, k^H_\beta\, \varepsilon^{\lambda_H}_\gamma \varepsilon^{\lambda_E}_\delta \times$

$\times \left[\left([\hat{a}_H, \hat{a}^+_E]\left(-e^{i\vec{k}_H\vec{z} - i\vec{k}_E\vec{z}'}\right) + [a^+_H, \hat{a}_E]\left(-e^{-i\vec{k}_H\vec{z} + i\vec{k}_E\vec{z}'}\right)\right)\right]$

$= -\frac{\hbar c}{4\pi^2} \varepsilon^{\alpha\beta\gamma} \sum_\lambda \int \frac{d^3k}{\sqrt{\omega}} \sqrt{\omega}\, k_\beta\, \varepsilon^\lambda_\gamma \varepsilon^\lambda_\delta \left[-e^{i\vec{k}\Delta\vec{z}} + e^{-i\vec{k}\Delta\vec{z}}\right] =$

$= -\frac{\hbar c}{4\pi^2} \varepsilon^{\alpha\beta\gamma} \int d^3k\, k_\beta \left(\delta_{\gamma\delta} - \frac{k_\gamma k_\delta}{k^2}\right)\left(-e^{i\vec{k}\Delta\vec{z}} + e^{-i\vec{k}\Delta\vec{z}}\right) =$

$\sum_\lambda \varepsilon^\lambda_\gamma \varepsilon^\lambda_\delta = \delta_{\gamma\delta} - \frac{k_\gamma k_\delta}{k^2}$ since $\vec{\varepsilon}^1, \vec{\varepsilon}^2, \frac{\vec{k}}{k}$ form a basis

$= +\frac{\hbar c}{4\pi^2} \varepsilon^{\alpha\beta\delta} \int d^3k\, k_\beta \left(e^{i\vec{k}\Delta\vec{z}} - e^{-i\vec{k}\Delta\vec{z}}\right) =$

$= \frac{\hbar c}{4\pi^2 i} \varepsilon^{\alpha\beta\gamma} \frac{\partial}{\partial \Delta z_\beta} \int d^3k \left(e^{i\vec{k}\Delta\vec{z}} + e^{-i\vec{k}\Delta\vec{z}}\right) =$

$= -\frac{\hbar c i}{4\pi^2} 8\pi^3\, \varepsilon^{\alpha\beta\gamma} \frac{\partial}{\partial \Delta z_\beta}\, 2\, \delta(\Delta\vec{z}) =$

$= -4\pi\hbar c i\, \varepsilon^{\alpha\beta\gamma}\, \delta'_{(\beta)}(\vec{z} - \vec{z}')\,.$