Problem Set II

S 14.5.1

i The correction due to the proton intrinsic magnetic moment is:

$$H = \ldots - \frac{e}{2Mc} g_{pr} \vec{B} \cdot \vec{S}_{pr} \qquad (e > 0)$$

S_{proton} and $S_{electron}$ are both of order $\hbar$

This compared to $-\left(\frac{-e}{2mc}\right) \vec{B} \cdot (\vec{L}_{el} + 2\vec{S}_{el})$

the correction is of order $\frac{m}{M} \sim \frac{1}{2} \cdot 10^{-3}$

ii $\vec{L}_{proton} = (\vec{r} \times \vec{p})_{proton}$

$\vec{p}_{proton} \sim \vec{p}_{electron}$ (total momentum vanishes in the C.M. frame)

$\vec{r}_{proton} \sim \frac{m}{M} \vec{r}_{electron}$

since the C.M. position $M\vec{r}_{pr} + m\vec{r}_{el} = 0$,

Therefore $\frac{e}{2Mc} \vec{L}_{pr} \cdot \vec{B} \sim \frac{e}{2Mc} \left(\frac{m}{M}\right) \vec{L}_{el} \cdot \vec{B} \sim \left(\frac{m}{M}\right)^2 \frac{e}{mc} \vec{L}_{el} \cdot \vec{B}$

S14.5.3

$$|\vec{S}\cdot\vec{n} = +\frac{\hbar}{2}\rangle = \begin{pmatrix} \cos\theta/2 & e^{-i\varphi/2} \\ \sin\theta/2 & e^{+i\varphi/2} \end{pmatrix}$$

$$|\vec{S}\cdot\vec{n} = -\frac{\hbar}{2}\rangle = \begin{pmatrix} -\sin\theta/2 & e^{-i\varphi/2} \\ \cos\theta/2 & e^{i\varphi/2} \end{pmatrix}$$

in the basis where $S_z = \frac{\hbar}{2}\sigma_z$, and

with $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

In that basis, therefore

$$|S_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \left(\theta = \frac{\pi}{2},\ \varphi = 0\right)$$

$$|S_x = -\frac{\hbar}{2}\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The answer to the first question is

$$\left|\langle S_x = \frac{\hbar}{2} \mid S_z = \frac{\hbar}{2}\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2$$

$$= \frac{1}{2}$$

The answer to the second question is

$$|\langle S_z = \hbar/2 \mid S_x = \hbar/2 \rangle|^2 \, |\langle S_x = \hbar/2 \mid S_z = \hbar/2 \rangle|^2$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} .$$

The answer to the last question depends on precisely how the experiment is performed.

If the $|S_x = +\hbar/2\rangle$ and $|S_x = -\hbar/2\rangle$ beams are made to interfere in phase, the answer is

$$\Big| \langle S_z = -\hbar/2 \mid S_x = \hbar/2 \rangle \langle S_x = \hbar/2 \mid S_z = \hbar/2 \rangle + \langle S_z = -\hbar/2 \mid S_x = -\hbar/2 \rangle \langle S_x = -\hbar/2 \mid S_z = \hbar/2 \rangle \Big|^2$$

$$= |\langle S_z = -\hbar/2 \mid S_z = \hbar/2 \rangle|^2 = 0 .$$

If the $|S_z = \frac{1}{2}\rangle$ and

$|S_z = -\frac{1}{2}\rangle$ beams are kept separate

(so that S_z is measured) or

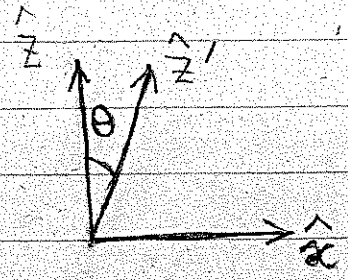
if the two beams are made to

interfere with random phases,

the answer is

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

14.5.4



Spin 1

$$|\vec{S}^{(1)} \cdot \hat{z} = \hbar\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

since $$S_z^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\vec{S}^{(1)} \cdot \hat{z}' = S_z^{(1)} \cos\theta + S_x^{(1)} \sin\theta$$

$$= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cos\theta + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sin\theta$$

$$= \hbar \begin{pmatrix} \cos\theta & \frac{1}{\sqrt{2}}\sin\theta & 0 \\ \frac{1}{\sqrt{2}}\sin\theta & 0 & \frac{1}{\sqrt{2}}\sin\theta \\ 0 & \frac{1}{\sqrt{2}}\sin\theta & -\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta - 1 & \frac{1}{\sqrt{2}}\sin\theta & 0 \\ \frac{1}{\sqrt{2}}\sin\theta & -1 & \frac{1}{\sqrt{2}}\sin\theta \\ 0 & \frac{1}{\sqrt{2}}\sin\theta & -\cos\theta - 1 \end{pmatrix} \left| \vec{S}^{(1)} \cdot \hat{z}' = \hbar \right\rangle = 0$$

$$\left| \vec{S}^{(1)} \cdot \hat{z}' = \hbar \right\rangle = \begin{pmatrix} 1+\cos\theta \\ \sqrt{2}\sin\theta \\ 1-\cos\theta \end{pmatrix} \frac{1}{2}$$

The fraction that leaves the first SG apparatus and passes the second

$$\left| \left\langle \vec{S}^{(1)} \cdot \hat{z}' = \hbar \,\middle|\, \vec{S}^{(1)} \cdot \hat{z} = \hbar \right\rangle \right|^2$$

$$= \left(\frac{1}{2}(1+\cos\theta) \right)^2 = \frac{1}{4}(1+\cos\theta)^2$$

S15.1.2 $H = H_c + A \vec{S}_p \cdot \vec{S}_e$

a. There are four states to consider

$$|\uparrow,\uparrow\rangle = |S_{pz} = \tfrac{\hbar}{2}\rangle |S_{ez} = \tfrac{\hbar}{2}\rangle$$

$$|\downarrow,\uparrow\rangle = |S_{pz} = -\tfrac{\hbar}{2}\rangle |S_{ez} = \tfrac{\hbar}{2}\rangle$$

$$|\uparrow,\downarrow\rangle = |S_{pz} = \tfrac{\hbar}{2}\rangle |S_{ez} = -\tfrac{\hbar}{2}\rangle$$

$$|\downarrow,\downarrow\rangle = |S_{pz} = -\tfrac{\hbar}{2}\rangle |S_{ez} = -\tfrac{\hbar}{2}\rangle$$

For all four states $H_c = -R_y$,

whereas

$$\vec{S}_p \cdot \vec{S}_e = S_{pz} S_{ez} + S_{px} S_{ex} + S_{py} S_{ey}$$

$$= \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix} + \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \right.$$

$$\left. + i \begin{pmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix} i \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \right]$$

$$\vec{S}_p \cdot \vec{S}_e = \frac{\hbar^2}{4}\left[\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}$$

$$\det\begin{pmatrix}-1-\lambda & 2\\ 2 & -1-\lambda\end{pmatrix} = (1+\lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$
$$= (\lambda+3)(\lambda-1)$$

$\lambda = 1$: $\quad \begin{pmatrix}-2&2\\2&-2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = 0$

$\lambda = -3$: $\quad \begin{pmatrix}2&2\\2&2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = 0$

There are three states with energy

$$E_+ = -R_y + \frac{\hbar^2}{4}A \quad , \text{ namely :}$$

$|\uparrow,\uparrow\rangle$

$\frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle\right)$

$|\downarrow,\downarrow\rangle$

The remaining state $\frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\right)$

has energy $E_- = -Ry - \frac{3\hbar^2}{4}A$.

Note that the triplet are total angular momentum ($\vec{J} = \vec{S}_p + \vec{S}_e$) $j=1$ whereas the singlet is $j=0$.

It would have been more efficient to note first that $[\vec{J}, H] = 0$.

ii)

$$\mathcal{H}_{hf} \sim \frac{\vec{\mu}_e \cdot \vec{\mu}_p}{a_0^3}$$

$$\sim \frac{e}{2mc} 2\left(\frac{\hbar}{2}\right) \frac{e}{2Mc} (5.6)\left(\frac{\hbar}{2}\right) \frac{1}{a_0^3}$$

$$\therefore A \sim \frac{2e}{2mc} (5.6) \frac{e}{2Mc} \frac{1}{a_0^3}$$

$$\Delta E = E_+ - E_- = A\hbar^2\left(\frac{1}{4} - \left(-\frac{3}{4}\right)\right) = A\hbar^2$$

$$\sim \left(\frac{5.6}{2}\right) \frac{e^2}{Mmc^2} \left(\frac{mc^2}{\hbar^2}\right)^3 \hbar^2$$

$$\Delta E \sim 2.8 \left(\frac{e^2}{\hbar c}\right)^4 \frac{m^2}{M} c^2$$

$$\sim 2.8\, \alpha^4 \left(\frac{0.511}{940}\right) 0.511 \text{ MeV}$$

$$\sim 2.2\ 10^{-6} \text{ eV}$$

$$\Delta E = h\nu \qquad \lambda = \frac{c}{\nu}$$

$$\hbar c = 1.97\ 10^{-5} \text{ eV cm} \qquad \hbar = \frac{h}{2\pi}$$

$$\lambda = \frac{hc}{\Delta E} \sim 2\pi \frac{1.97\ 10^{-5} \text{ eV cm}}{2.2\ 10^{-6} \text{ eV}}$$

$$\sim 56 \text{ cm}$$

iii. The actual value of

$$\Delta E \simeq 2.2\ 10^{-6} \text{ eV} \frac{56 \text{ cm}}{21 \text{ cm}}$$

$$\simeq 5.9\ 10^{-6} \text{ eV}$$

$$= k_B T_{hf}$$

$k_B = 8.6\ 10^{-5}$ eV K

$T_{hf} \simeq 0.07$ K $\ll$ room temperature

$$\frac{P(\text{triplet})}{P(\text{singlet})} \simeq 3$$