Homework Set #4

1. It is shown on the next couple of pages that

$$\vec{\Sigma}^{(1)\,T} = \mathcal{U}\ \vec{\Sigma}^{(1)\prime\,T}\ \mathcal{U}^{\dagger}$$

with $$\mathcal{U} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix}$$

(as was stated in class)

Hence

$$\vec{\Sigma}^{(1)} = \mathcal{U}^{*}\ \vec{\Sigma}^{(1)\prime}\ \mathcal{U}^{T}$$

$$\vec{\Sigma}^{(1)\prime} = \mathcal{U}^{T}\ \vec{\Sigma}^{(1)}\ \mathcal{U}^{*}$$

Hence

$$U = \mathcal{U}^{T} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}.$$

Check:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} i \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{1}{2} i \begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} +i & 0 & +i \\ +1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \cdot i \begin{pmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +2i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \checkmark$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix}^T$$

$\checkmark$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ -i & 0 & -i \end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ -i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & -i\sqrt{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

15.2.5

(i) $P_1 = \frac{3}{4}\mathbb{1} + \frac{1}{\hbar^2}(\vec{S}_1 \cdot \vec{S}_2)$

$= \frac{3}{4}\mathbb{1} + \frac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$P_0 = \frac{1}{4}\mathbb{1} - \frac{1}{\hbar^2}(\vec{S}_1 \cdot \vec{S}_2)$

$= \frac{1}{4}\mathbb{1} - \frac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$(P_1)^2 = \frac{9}{16}\mathbb{1} + \frac{3}{8}\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$+ \frac{1}{16}\sigma_{1j}\sigma_{1k}\sigma_{2j}\sigma_{2k}$

$= \frac{9}{16}\mathbb{1} + \frac{3}{8}\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$+ \frac{1}{16}(\delta_{jk}\mathbb{1} + i\epsilon_{jkl}\sigma_{1l}) \cdot (\delta_{jk}\mathbb{1} + i\epsilon_{jkp}\sigma_{2p})$

$= \frac{9}{16}\mathbb{1} + \frac{3}{8}\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$+ \frac{3}{16}\mathbb{1} - \frac{1}{16}2\,\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$= \frac{3}{4}\mathbb{1} + \frac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2 = P_1$ ✓

$$P_0 = 1 - P_1$$

$$(P_0)^2 = 1 - 2P_1 + (P_1)^2$$

$$= 1 - P_1 = P_0 \quad \checkmark$$

$$P_0 P_1 = (1 - P_1) P_1$$

$$= P_1 - P_1 = 0 \quad \checkmark$$

(2)

$$\vec{S} \cdot \vec{S} = (\vec{S}_1 + \vec{S}_2)^2$$

$$= \vec{S}_1 \cdot \vec{S}_1 + \vec{S}_2 \cdot \vec{S}_2 + 2\,\vec{S}_1 \cdot \vec{S}_2$$

$$= \hbar^2 \tfrac{3}{2} + 2\,\vec{S}_1 \cdot \vec{S}_2 = \hbar^2 \Lambda(\Lambda+1)$$

$$\therefore \quad \vec{S}_1 \cdot \vec{S}_2 = \frac{\hbar^2}{2} \Lambda(\Lambda+1) - \hbar^2 \tfrac{3}{4}$$

$$= -\hbar^2 \tfrac{3}{4} \quad \text{for } \Lambda = 0$$

$$= +\hbar^2 \tfrac{1}{4} \quad \text{for } \Lambda = 1 .$$

Since $P_0 |\Lambda = 1\rangle = \left(\frac{1}{4} - \frac{1}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2\right)|\Lambda = 1\rangle$

$= \left(\frac{1}{4} - \frac{1}{4}\right)|\Lambda = 1\rangle = 0$,

P_0 projects onto the $\Lambda = 0$ state

Likewise, since

$P_1 |\Lambda = 0\rangle = \left(\frac{3}{4} + \frac{1}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2\right)|\Lambda = 0\rangle$

$= \left(\frac{3}{4} - \frac{3}{4}\right)|\Lambda = 0\rangle = 0$,

P_1 project onto the $\Lambda = 1$ state.

S 15.2.6

$$\vec{J}\cdot\vec{J} = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S})$$

$$j(j+1) = l(l+1) + s(s+1) + 2\vec{L}\cdot\vec{S}$$

$$\left(l\pm\tfrac{1}{2}\right)\left(l\pm\tfrac{1}{2}+1\right) = l(l+1) + \tfrac{3}{4} + 2\vec{L}\cdot\vec{S}$$

$$= l(l+1) \pm \tfrac{1}{2}(2l+1) + \tfrac{1}{4}$$

$$2\vec{L}\cdot\vec{S} = \pm\left(l+\tfrac{1}{2}\right) - \tfrac{1}{2}$$

$$\left.\begin{aligned} P_{\pm}\,|j = l\pm\tfrac{1}{2}\rangle &= |j = l\pm\tfrac{1}{2}\rangle \\ P_{\pm}\,|j = l\mp\tfrac{1}{2}\rangle &= 0 \end{aligned}\right\} \text{ by definition}$$

$$2\vec{L}\cdot\vec{S}\;|j = l\pm\tfrac{1}{2}\rangle = \left(\pm\left(l+\tfrac{1}{2}\right) - \tfrac{1}{2}\right)|j = l\pm\tfrac{1}{2}\rangle$$

$$P_{\pm} = \frac{4\vec{L}\cdot\vec{S} + 1 \pm (2l+1)}{\pm 2(2l+1)}$$

$$= \frac{1}{2} \pm \frac{1}{l+\frac{1}{2}}\left(\vec{L}\cdot\vec{S} + \frac{1}{4}\right)$$

S15.2.7

$$|2j_1, 2j_1\rangle = \underbrace{|j_1, j_1\rangle_A |j_1, j_1\rangle_B}_{\text{Symmetric}}$$

$|2j_1, m\rangle$ is obtained by applying

$$(J_-)^{2j_1-m} = (J_{A-} + J_{B-})^{2j_1-m}$$

to both sides of the above equation.

The RHS is obviously symmetric under the interchange $A \leftrightarrow B$

$$|2j_1, 2j_1-1\rangle = \frac{1}{\sqrt{2}} |j_1, j_1\rangle_A |j_1, j_1-1\rangle_B + \frac{1}{\sqrt{2}} |j_1, j_1-1\rangle_A |j_1, j_1\rangle_B$$

$$|2j_1-1, 2j_1-1\rangle = \frac{1}{\sqrt{2}} |j_1, j_1\rangle_A |j_1, j_1-1\rangle_B - \frac{1}{\sqrt{2}} |j_1, j_1-1\rangle_A |j_1, j_1\rangle_B$$

is anti-symmetric under $A \to B$

$$|2j_1-1, m\rangle = C_m \underbrace{\left(J_{A-} + J_{B-}\right)^{2j_1-1-m}}_{\text{symm}} \underbrace{|2j_1-1, 2j_1-1\rangle}_{\text{anti-symm.}}$$

is anti-symmetric under the interchange $A \leftrightarrow B$ for all m

F.

15.3.1

$$e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}\, T_k^q\, e^{+\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}} = \sum_{q'=-k}^{+k} D^{(k)}_{q'q}\, T_k^{q'}$$

with $$D^{(k)}_{qq'} = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}^{(k)}} \right)_{qq'}$$

and $$\left(J_z^{(k)}\right)_{q'q} = q\hbar\,\delta_{qq'}$$

$$\left(J_\pm^{(k)}\right)_{q'q} = \hbar\,\delta_{q',q\pm1}\sqrt{(q\mp k)(q\pm k+1)}$$

a

For small $\vec{\theta}$

$$e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}\, T_k^q\, e^{\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}$$

$$= T_k^q - \frac{i}{\hbar}\theta_m \left[J_m, T_k^q\right] \qquad m = 1,2,3$$

$$= \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}^{(k)}} \right)_{q'q} T_k^{q'}$$

$$= T_k^q - \frac{i}{\hbar}\theta_m \left(J_m^{(k)}\right)_{q'q} T_k^{q'}$$

$$[J_m, T^q_k] = J^{(k)}_{m\,q'q}\, T^{q'}_k$$

$$[J_z, T^q_k] = \hbar q\, T^q_k$$

$$[J_\pm, T^q_k] = \pm\hbar\sqrt{(k\mp q)(k\pm q+1)}\; T^{q\pm1}_k$$

b

$$[J_i, V_j] = i\,\epsilon_{ijk}\, V_k$$

$$[J_z, V^0_1] = [J_z, V_z] = 0 \quad \checkmark$$

$$[J_z, V^{\pm1}_1] = \mp\left[J_z, \frac{1}{\sqrt{2}}(V_x \pm i V_y)\right]$$

$$= \mp\frac{\hbar}{\sqrt{2}}(iV_y \pm V_x) = -\hbar(V_x \pm iV_y)\frac{1}{\sqrt{2}}$$

$$= \pm\hbar V^{\pm1}_1 \quad \checkmark$$

$$[J_\pm, V^0_1] = [J_x \pm iJ_y, V_z]$$

$$= (-iV_y \mp V_x)\hbar = \mp(V_x \pm iV_y)\hbar$$

$$= \hbar\sqrt{2}\, V^{\pm1}_1 \quad \checkmark$$

$$\left[J_\pm, V_1^{+1}\right] = \left[J_x \pm i J_y, \frac{-1}{\sqrt{2}}(V_x + i V_y)\right]$$

$$= \left(\mp V_z + V_z\right)\frac{1}{\sqrt{2}} = \begin{pmatrix} 0 & \checkmark \\ \sqrt{2}\, V_1^0 & \checkmark \end{pmatrix}$$

$$\left[J_\pm, V_1^{-1}\right] = \left[J_x \pm i J_y, \frac{1}{\sqrt{2}}(V_x - i V_y)\right]$$

$$= \left(\pm V_z + V_z\right)\frac{1}{\sqrt{2}} = \begin{pmatrix} \sqrt{2}\, V_1^0 & \checkmark \\ 0 & \checkmark \end{pmatrix}$$

S15.35

$$\langle j\,m \,|\, T^q_k \,|\, j\,m \rangle$$

$$= \underbrace{\langle j\,m \,|\, k\,q,\, j\,m \rangle}_{\text{vanishes unless}} \langle j \,\|\, T_k \,\|\, j \rangle$$

$$j + k \geq j \geq |j - k|$$

If $k > 2j$, then $|j-k| = k - j > j$.