Homework Set #5

S15.3.3

i. $$\sum_{q=-1}^{+1} (-1)^q S_1^q T_1^{-q}$$

$$= -S_1^1 T_1^{-1} + S_1^0 T_1^0 - S_1^{-1} T_1^{+1}$$

$$= \frac{1}{2}(S_x + iS_y)(T_x - iT_y) + S_z T_z$$

$$+ \frac{1}{2}(S_x - iS_y)(T_x + iT_y)$$

$$= S_x T_x + S_y T_y + S_z T_z = \vec{S} \cdot \vec{T}$$

ii. First let us prove that

$$D^{(j)}_{-m,-m'} = (-1)^{m-m'} \left(D^{(j)}_{m,m'}\right)^*$$

$$D^{(j)} = e^{-i/\hbar\, \vec{\theta} \cdot \vec{J}^{(j)}}$$

$$D^{(j)*} = e^{+i/\hbar\, \vec{\theta} \cdot \vec{J}^{(j)*}}$$

$$\left(J_z^{(j)}\right)_{m'm} = \delta_{m'm}\, \hbar m$$

$$\left(J_\pm^{(j)}\right)_{m'm} = \left(J_x^{(j)} \pm i J_y^{(j)}\right)_{m'm} = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, \delta_{m',m\pm1}$$

We want to find C such that

$$C(\vec{\theta}\cdot\vec{J}^{(j)*})\,C^{-1} = -\vec{\theta}\cdot\vec{J}^{(j)}$$

Then

$$C\,D^{(j)*}\,C^{-1} = D^{(j)}$$

Under complex conjugation $J_z \to J_z$, $J_x \to J_x$, $J_y \to -J_y$

Thus $J_+ \to J_-$ and $J_- \to J_+$

Note that

$$(J_+^{(j)})_{-m',-m} = \hbar\sqrt{(j+m)(j-m+1)}\,\delta_{-m',-m+1}$$

$$= (J_-^{(j)})_{m'm}$$

Therefore

$$C' J_\pm C' = J_\mp$$

with

$$C' = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \left(C'_{m',m} = \delta_{m',-m}\right)$$

We want

$$C J_z C^{-1} = -J_z$$

$$C J_\pm C^{-1} = -J_\mp$$

The solution is

$$C_{m',m} = (-1)^{j+m} \delta_{m',-m} = (-1)^{j-m'} \delta_{m',-m} \qquad C = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & & \vdots & \vdots \\ (-1)^{2j} & \cdots & 0 & 0 \end{pmatrix}$$

$$C^{-1}_{m'm} = (-1)^{j-m} \delta_{m',-m} = (-1)^{-j+m} \delta_{m',-m} \qquad C^{-1} = \begin{pmatrix} 0 & \cdots & & (-1)^{2j} \\ \vdots & & & \vdots \\ 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

Indeed

$$(C J_z C^{-1})_{m'm} = (-1)^{j+m''} \delta_{m',-m''} \, \hbar m'' \delta_{m'',m'''} (-1)^{-j+m} \delta_{m''',-m}$$

$-m = m'' = m''' = -m'$

$$= -\hbar m \, \delta_{m'm} = -(J_z)_{m'm}$$

$$(C J_+ C^{-1})_{m'm} = (-1)^{j-m'} (C' J_+ C')_{m'm''} (-1)^{-j+m}$$

$$= (-1)^{m-m'} (J_-)_{m'm} \leftarrow \text{vanishes unless } m' = m-1$$

$$= (-1)(J_-)_{m'm}$$

Hence

$$D^{(j)*} = C^{-1} D^{(j)} C$$

$$D^{(j)*}_{m'm} = (-1)^{-j-m''} D^{(j)}_{-m'',-m'} (-1)^{j+m}$$

$$= (-1)^{m-m'} D^{(j)}_{-m',-m}$$

$$D^{(j)}_{-m,-m'} = (-1)^{m-m'} D^{(j)*}_{m\,m'}$$

Then

$$\sum_{q=-k}^{+k} (-1)^q S_k^q T_k^{-q} = S_k^T C T_k (-1)^k$$

$$\rightarrow (D^{(j)T} S_k)^T C (D^{(j)T} T_k) (-1)^k$$

$$= S_k^T D^{(j)} C D^{(j)T} T_k (-1)^k$$

$$= S_k^T C T_k (-1)^k$$

since $D^{(j)*} = D^{(j)T^{-1}} = C^{-1} D^{(j)} C$

implies $D^{(j)} C D^{(j)T} = C$

§15.3.4

$$(j) \otimes (1) = (j+1) \oplus (j) \oplus (j-1)$$

$$|j+1, j+1\rangle = |j, j\rangle |1, 1\rangle$$

$$\frac{1}{\hbar} J_- |j+1, j+1\rangle = \sqrt{2(j+1)}\, |j+1, j\rangle$$

$$= \sqrt{2j}\, |j, j-1\rangle |1, 1\rangle + \sqrt{2}\, |j, j\rangle |1, 0\rangle$$

$$|j+1, j\rangle = \sqrt{\frac{j}{j+1}}\, |j, j-1\rangle |1, 1\rangle + \sqrt{\frac{1}{j+1}}\, |j, j\rangle |1, 0\rangle$$

$$|j, j\rangle = \sqrt{\frac{j}{j+1}}\, |j, j\rangle |1, 0\rangle - \sqrt{\frac{1}{j+1}}\, |j, j-1\rangle |1, 1\rangle$$

(for $j \geq 1$)

$$\langle j, j | j, j; 1, 0\rangle = \sqrt{\frac{j}{j+1}} \quad \checkmark$$

$\vec{J}$ is a vector operator

$$J_1^0 = J_z$$

$$J_1^{\pm 1} = \mp \frac{1}{\sqrt{2}} (J_x \pm i J_y) = \mp \frac{1}{\sqrt{2}} J_\pm$$

Also, we know

$$J_z |\alpha, j, m\rangle = m\hbar |\alpha, j, m\rangle$$

$$J_\pm |\alpha, j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, |\alpha, j, m \pm 1\rangle$$

Thus

$$\langle \alpha', j', m' | T_1^q | \alpha, j, m\rangle = 0$$

unless $j = j'$, $\alpha = \alpha'$

The Wigner-Eckart theorem tells us that

$$\langle \alpha', j', m' | T_1^q | \alpha, j, m\rangle$$

$$= \langle j, m' | j, m ; 1, q\rangle \cdot \langle \alpha', j' \| T_1 \| \alpha, j\rangle$$

For $q = 0$, $m' = j'$, we have

$$\langle \alpha', j' \| T_1 \| \alpha, j\rangle = \frac{\delta_{\alpha\alpha'}\, \delta_{jj'}\, j\hbar}{\langle j, j | j, j, 1, 0\rangle}$$

$$= \delta_{\alpha\alpha'}\, \delta_{jj'} \sqrt{j(j+1)}\, \hbar$$

11.

$$\langle \alpha' j\, m' | \vec{J} \cdot \vec{A} | \alpha, j, m \rangle$$

$$= \langle \alpha' j\, m' | \left[J_z A_z + \frac{1}{2}(J_+ A_- + J_- A_+) \right] | \alpha, j, m \rangle$$

$$= \langle \alpha' j\, m' | \sum_{q=-1}^{+1} (-1)^q J_1^q A_1^{-q} | \alpha, j, m \rangle$$

$$= \sum_{q=-1}^{+1} (-1)^q \langle j\, m' | J_1^q | j, m'-q \rangle \times$$

$$\langle \alpha', j, m'-q | A_1^{-q} | \alpha, j, m \rangle$$

$$= \sum_{q=-1}^{+1} (-1)^q \langle j\, m' | j\, m'-q, 1, q \rangle \langle j \| J_1 \| j \rangle$$

$$\times \langle j\, m'-q | j, m, 1, -q \rangle \langle \alpha' j \| A_1 \| \alpha, j \rangle$$

$$= \zeta \langle \alpha' j \| A_1 \| \alpha, j \rangle$$

where ζ is independent of α, α' and A.

Let us choose $\vec{A} = \vec{J}$

$$\langle \alpha' j m' | \vec{J} \cdot \vec{J} | \alpha j m \rangle$$

$$= \delta_{\alpha\alpha'} \delta_{mm'} j(j+1) \hbar^2$$

$$= \varepsilon \langle \alpha' j | J_1 | \alpha j \rangle$$

$$= \varepsilon \, \delta_{\alpha\alpha'} \sqrt{j(j+1)}\, \hbar$$

$$\therefore \varepsilon = \delta_{mm'} \sqrt{j(j+1)}\, \hbar$$

iii)

$$\langle \alpha' j m' | A_1^q | \alpha j m \rangle$$

$$= \langle j m' | j m 1 q \rangle \langle \alpha' j \| A_1 \| \alpha j \rangle$$

$$= \langle j m' | j m 1 q \rangle \frac{\langle \alpha' j m | \vec{J} \cdot \vec{A} | \alpha j m \rangle}{\hbar \sqrt{j(j+1)}}$$

$$= \frac{\langle j m' | J_1^q | j m \rangle}{\langle j \| J_1 \| j \rangle} \frac{\langle \alpha' j m | \vec{J} \cdot \vec{A} | \alpha j m \rangle}{\hbar \sqrt{j(j+1)}}$$

$$= \frac{\langle j m' | J_1^q | j m \rangle \langle \alpha' j m | \vec{J} \cdot \vec{A} | \alpha j m \rangle}{\hbar^2 j(j+1)}$$

§15.3.4

1.

$$\vec{\mu} = \gamma_1 \vec{J}_1 + \gamma_2 \vec{J}_2$$

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$$\langle \mu_q^1 \rangle = \langle j m; j_1 j_2 | \mu_q^1 | j, m; j_1, j_2 \rangle$$

$$= \frac{\langle j m; j_1 j_2 | \vec{J} \cdot \vec{\mu} | j, m; j_1 j_2 \rangle \langle j m | J_q^1 | j m \rangle}{\hbar^2 j(j+1)}$$

$$= \delta_0^q \frac{m\hbar}{\hbar^2 j(j+1)} \langle j m; j_1 j_2 | \left[\gamma_1 \vec{J}_1 \cdot \vec{J}_1 + \gamma_2 \vec{J}_2 \cdot \vec{J}_2 + (\gamma_1 + \gamma_2) \vec{J}_1 \cdot \vec{J}_2 \right] | j m; j_1 j_2 \rangle$$

$$= \delta_0^q \frac{m\hbar}{j(j+1)} \left[\gamma_1 j_1(j_1+1) + \gamma_2 j_2(j_2+1) + (\gamma_1 + \gamma_2) \frac{1}{2} \left(j(j+1) - j_1(j_1+1) - j_2(j_2+1) \right) \right]$$

$$= \delta_0^q \, m\hbar \, \frac{1}{2} \left[\gamma_1 + \gamma_2 + \frac{\gamma_1 - \gamma_2}{j(j+1)} \left(j_1(j_1+1) - j_2(j_2+1) \right) \right]$$

ii. The $^2P_{1/2}$ state has $l = 1 \equiv j_1$

$s = 1/2 \equiv j_2$

$j = 1/2$

$\gamma_1 = g_1 \dfrac{e}{2Mc}$ with $g_1 = 1$

$\gamma_2 = g_2 \dfrac{e}{2Mc}$ with $g_2 = 5.6$

$m = \pm 1/2$

$$\langle \mu_z \rangle = \frac{e\hbar}{2Mc} \left(\pm \frac{1}{2}\right) \frac{1}{2} \left[(g_1 + g_2) + (g_1 - g_2) \frac{2 - 3/4}{3/4} \right]$$

$$= \mp (0.26) \frac{e\hbar}{2Mc}$$

iii. Same as before except

$M \to m$, $e \to -e$, $g_2 \to 2$

$$\langle \mu_z \rangle = \pm \frac{1}{3} \left(\frac{-e\hbar}{2mc} \right)$$

S 16.1.2

$$E = \langle \psi | H | \psi \rangle$$

$$= \int_{-a}^{+a} dx\ \psi(x)^* \left(\frac{-\hbar^2}{2m}\right) \frac{d^2}{dx^2} \psi(x)$$

$$= -\frac{\hbar^2}{2m} N^2 \int_{-a}^{+a} dx\ (x-a)(x+a)^2$$

$$= -\frac{\hbar^2}{2m} 4N^2 \int_{0}^{a} dx\ (x-a)(x+a)$$

$$= -\frac{2\hbar^2}{m} N^2 a^3 \int_{0}^{1} dx\ (x^2-1) = \frac{2\hbar^2}{m} N^2 a^3 \frac{2}{3}$$

$$1 = N^2 \int_{-a}^{+a} dx\ (x-a)^2 (x+a)^2$$

$$= N^2 a^5\ 2 \int_{0}^{1} dx\ (x^2-1)^2$$

$$= N^2 a^5\ 2 \left(\frac{1}{5} - 2\frac{1}{3} + 1\right)$$

$$= N^2 a^5\ \frac{16}{15}$$

$$E = \frac{p^2}{m a^2} - \frac{5}{4}$$

$$E_0 = \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2 = \frac{\hbar^2}{ma^2}\,\frac{\pi^2}{8}$$

$$E = \frac{10}{\pi^2} E_0 = 1.013\, E_0$$

[Note:

$$\psi''(x = \pm a) = \psi'(\pm a + \epsilon) - \psi'(\pm a - \epsilon) \Big|_{\epsilon \to 0_+}$$

$$= \mp 2a\, \delta(0) = \mp \infty \,,$$

but $\psi(\pm a) = 0$.

$$E = \int_{-\infty}^{+\infty} dx\; \psi^*(x) \left(-\frac{\hbar^2}{2m}\right) \frac{d^2}{dx^2}\, \psi(x)$$

$$= \int_{-\infty}^{+\infty} dx\; \frac{\hbar^2}{2m} \left|\frac{d\psi}{dx}\right|^2$$

$$= \int_{-a}^{+a} dx\; \frac{\hbar^2}{2m} \left|\frac{d\psi}{dx}\right|^2$$

because $\psi(x) = 0$ for $|x| \geq a$]