6 3 2 H solutions

HW #1

Ans 9.4.1 $\langle n=1 | \frac{1}{x^2} | n=1 \rangle$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar\, 2^{2(1)}}\right)^{1/4} \exp\left(\frac{-m\omega x^2}{2\hbar}\right) 2\left(\frac{m\omega}{\hbar}\right)^{1/2} x$$

$$\left\langle \frac{1}{x^2} \right\rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar\,4}\right)^{1/2} 4 \exp\left(\frac{-m\omega x^2}{\hbar}\right) \left(\frac{m\omega}{\hbar}\right) \frac{1}{x^2}\, x^2\, dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-m\omega x^2}{\hbar}\right) dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) 2\int_{0}^{\infty} \exp\left(\frac{-m\omega x^2}{\hbar}\right) dx = 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \left(\frac{\pi\hbar}{m\omega}\right)^{1/2}$$

$$= 2\left(\frac{m\omega}{\hbar}\right)$$

(7)

$$\langle n=1 | x^2 | n=1 \rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar\,4}\right)^{1/2} 4 \exp\left(\frac{-m\omega x^2}{\hbar}\right) \left(\frac{m\omega}{\hbar}\right) x^4\, dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \int_{-\infty}^{\infty} x^4 \exp\left(\frac{-m\omega x^2}{\hbar}\right) dx$$

$$= 4\left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \sqrt{\frac{\hbar\pi}{m\omega}}\, \frac{(2(2)-1)!!}{\left(2\frac{m\omega}{\hbar}\right)^2} = \frac{4}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right) \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} \frac{3}{4\left(\frac{m\omega}{\hbar}\right)^2}$$

$$= \frac{3\hbar}{2(m\omega)} \quad \therefore \quad \frac{1}{\langle n=1 | x^2 | n=1 \rangle} = \frac{2m\omega}{3\hbar}$$

Thus,

$$\left\langle \frac{1}{x^2} \right\rangle \simeq \frac{1}{\langle x^2 \rangle} \simeq \frac{m\omega}{\hbar}$$

dm 9.4.2. $\psi = \frac{1}{(\pi a_0^3)^{1/2}} e^{-r/a_0}$

$\langle \psi | \psi \rangle = 4\pi \int_0^\infty \frac{1}{(\pi a_0^3)} e^{-2r/a_0} r^2 dr = \frac{4}{a_0^3} \int_0^\infty e^{-2r/a_0} r^2 dr$

$= \frac{4}{a_0^3} \frac{2}{(2/a_0)^3} = \frac{8}{8} \frac{a_0^3}{a_0^3} = \boxed{1}$ $\quad \alpha = \left(\frac{2}{a_0}\right)$, $n = 2$

(2) $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$

We have spherical wavefunctions. So $\langle x \rangle = \langle y \rangle = \langle z \rangle = 0$

$\therefore (\Delta x)^2 = \langle x^2 \rangle$ $\quad \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = \langle r^2 \rangle$

↳ ∵ spherical, symmetric w.r.t x, y & z

(8) $\therefore \langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle$

$= \frac{1}{3} 4\pi \int \frac{1}{\pi a_0^3} e^{-2r/a_0} r^4 dr$

$= \frac{4}{3 a_0^3} \int e^{-2r/a_0} r^4 dr = \frac{4}{3a_0^3} \frac{4!}{(2/a_0)^5} = \frac{4}{3} a_0^2 \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2}$

$\therefore (\Delta x)^2 = a_0^2 \Rightarrow \boxed{(\Delta x) = a_0 = \frac{\hbar^2}{me^2}}$

(3) $\left\langle \frac{1}{r} \right\rangle \xrightarrow{\text{in coordinate space}} 4\pi \int_0^\infty \frac{1}{(\pi a_0^3)} e^{-2r/a_0} r \, dr$

$= \frac{4}{a_0^3} \int_0^\infty r e^{-2r/a_0} dr = \frac{4}{a_0^3} \frac{1}{\left(\frac{2}{a_0}\right)^2} = \frac{1}{a_0} = \frac{me^2}{\hbar^2}$

$\langle r \rangle = 4\pi \int_0^\infty \frac{1}{\pi a_0^3} e^{-2r/a_0} r^3 dr = \left(\frac{4\pi}{\pi a_0^3}\right)\left(\frac{3 a_0^4}{8}\right) = \frac{3}{2} a_0 = \frac{3}{2} \frac{\hbar^2}{me^2}$

$\therefore \frac{1}{\langle r \rangle} = \frac{2}{3} \frac{me^2}{\hbar^2}$ $\quad \therefore \left\langle \frac{1}{r} \right\rangle \simeq \frac{1}{\langle r \rangle} \simeq \frac{me^2}{\hbar^2}$

Ans 3.4.3. $H = \frac{p^2}{2m} - \frac{e^2}{(R^2)^{1/2}}$; $\langle H \rangle = \frac{\langle p^2 \rangle}{2m} - \frac{e^2}{\langle R^2 \rangle^{1/2}}$

$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$

$(\Delta R)^2 = \langle R^2 \rangle - \langle R \rangle^2$

We are interested in ground state $\Rightarrow \langle H \rangle$ minimum. So, work with states in which $\langle p \rangle = \langle R \rangle = 0$

(7) $\therefore \langle H \rangle \approx \frac{(\Delta p)^2}{2m} - \frac{e^2}{[(\Delta R)^2]^{1/2}}$ $\quad (\Delta p)(\Delta R) \geq \frac{\hbar}{2}$

$\therefore \langle H \rangle \geq \frac{\hbar^2}{8(\Delta R)^2 m} - \frac{e^2}{(\Delta R)}$ $\quad \therefore (\Delta p) \geq \frac{\hbar}{2(\Delta R)}$

Differentiating, $\frac{\hbar^2(-2)}{8m(\Delta R)^3} + \frac{e^2}{(\Delta R)^2} = 0$

$\therefore \frac{e^2}{(\Delta R)^2} = \frac{2\hbar^2}{8m(\Delta R)^3}$ $\quad \therefore e^2 = \frac{\hbar^2}{4m(\Delta R)}$ $\quad \therefore (\Delta R) = \frac{\hbar^2}{4me^2}$

$\therefore \langle H \rangle \geq \frac{\hbar^2}{8m\left(\frac{\hbar^2}{4me^2}\right)^2} - \frac{e^2(4me^2)}{\hbar^2}$

$\geq \frac{2\hbar^2 4m^2e^4}{4m\hbar^4} - \frac{4e^4m}{\hbar^2}$

$\geq \frac{2e^4m}{\hbar^2} - \frac{4e^4m}{\hbar^2}$ $\quad \therefore \boxed{\langle H \rangle \geq -\frac{2me^4}{\hbar^2}}$

$\therefore$ The ground state energy is of the order of $-\frac{2me^4}{\hbar^2}$

Ans 9.4.4

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\langle\psi|[\hat{\omega},\hat{\lambda}]_+|\psi\rangle^2 + \frac{1}{4}\langle\psi|\hat{\Gamma}|\psi\rangle^2$$

where, $\hat{\omega} = \hat{\Omega} - \bar{\Omega}$, $\hat{\lambda} = \hat{\Lambda} - \bar{\Lambda}$; $[\hat{\omega},\hat{\lambda}] = [\hat{\Omega},\hat{\Lambda}] = i\hat{\Gamma}$

$$[T,x] = \frac{1}{2m}[p^2,x] = p[p,x] + [p,x]p = p(-i\hbar) - i\hbar p$$

$$= \frac{-2i\hbar p}{2m}$$

$\therefore$ Here $\hat{\Gamma} = -2\hbar\hat{p}$.

$[\hat{\omega},\hat{\lambda}]_+$ is Hermitean $\therefore \langle[\hat{\omega},\hat{\lambda}]_+\rangle^2 \geq 0$

(a) $\therefore (\Delta T)^2(\Delta x)^2 \geq \frac{1}{4}\frac{\langle\hat{p}\rangle^2}{4m^2}4\hbar^2$ $\quad (\Delta T)(\Delta x) \geq \frac{|k\theta|}{2m}$

This relation is not proper because it involves the particle momentum $\langle p\rangle$ & therefore is not universal.

Also for bound states $\langle p\rangle = 0$ $\therefore$ This relation just gives $(\Delta T)(\Delta x) \geq 0$, which is trivial information.