QM Honors HW#3 Bixo Huang

12.3.3/ $\psi(\rho,\varphi) = Ae^{-\frac{\rho^2}{2\Delta^2}}\cos^2\varphi \equiv R(\rho)Q(\varphi)$, where $R(\rho)$, $Q(\varphi)$ normalize respectively. Since we only care about the angular momentum part, we can integrate the $R(\rho)$ part out, leaving only $Q(\varphi) = B\cos^2\varphi$.

Now $Q(\varphi) = B\frac{1+\cos 2\varphi}{2} = \tilde{B}\left[1+\frac{e^{i2\varphi}+e^{i(-2\varphi)}}{2}\right]$

$= \sqrt{2\pi}\tilde{B}\left[\Phi_0 + \frac{1}{2}\Phi_2 + \frac{1}{2}\Phi_{-2}\right]$.

(7)

$\Rightarrow P(l_3 = m\hbar) = \frac{\left|\int_0^{2\pi} d\varphi\, \Phi_m^* Q(\varphi)\right|^2}{\int_0^{2\pi} d\varphi\, |Q(\varphi)|^2}$, Use $\int \Phi_{m'}^* \Phi_m d\varphi = \delta_{mm'} \Rightarrow$ denominator $= 1$, just to make calculation simpler

$\Rightarrow P(l_3 = 0) = \frac{|\sqrt{2\pi}\tilde{B}|^2 \, 1^2}{|\sqrt{2\pi}\tilde{B}|^2 \left(1^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)} = \frac{2}{3}$

$P(l_3 = 2\hbar) = P(l_3 = -2\hbar) = \frac{|\sqrt{2\pi}\tilde{B}|^2}{|\sqrt{2\pi}\tilde{B}|^2}\frac{\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{6}$

* Or from the expression of $Q(\varphi)$ we know

$\frac{P(l_3=0)}{P(l_3=2\hbar)} = \frac{P(l_3=0)}{P(l_3=-2\hbar)} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$, and $P(l_3=0)+P(l_3=2\hbar)+P(l_3=-2\hbar) = 1$

$\Rightarrow$ The same result.

12.3.6/ $H = \frac{p^2}{2\mu} = -\frac{\hbar^2}{2\mu}\vec{\nabla}^2 = -\frac{\hbar^2}{2\mu}\left(\vec{e}_\varphi \frac{1}{a}\frac{\partial}{\partial\varphi}\right)^2 = -\frac{\hbar^2}{2\mu a^2}\frac{\partial^2}{\partial\varphi^2}$.

Here we used $\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\varphi \frac{1}{a}\frac{\partial}{\partial\varphi}$ in polar coordinates. But because of the constraint, φ is the only coordinate any wave functions would depend on, so $\vec{\nabla} = \vec{e}_\varphi \frac{1}{a}\frac{\partial}{\partial\varphi}$.

On the other hand $L_z^2 = -\hbar^2\frac{\partial^2}{\partial\varphi^2}$. It is then clear

(5) $$H = \frac{L_z^2}{2\mu a^2}$$

We can use the eigenfunction of L_z:

$$H\Phi_m(\varphi) = \frac{L_z^2}{2\mu a^2}\Phi_m(\varphi) = \frac{m^2\hbar^2}{2\mu a^2}\Phi_m(\varphi).$$

$$\Rightarrow E_m = \frac{\hbar^2}{2\mu a^2}m^2, \quad \text{degeneracy} = 2$$

$\sim \pm m$

12.3.8/ (1) $\vec{A} = \frac{B}{2}(-y\vec{i} + x\vec{j})$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{B}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \frac{B}{2}\cdot[\vec{i}(-\partial_z x) - \vec{j}(\partial_z y) + \vec{k}(\partial_x x + \partial_y y)]$$

(15) $$= \frac{B}{2}\cdot\vec{k}\cdot 2 = B\vec{k}$$

(2) For a classical particle,

$$\mu\frac{v^2}{r} = qvB \Rightarrow v = \frac{qBr}{\mu}$$

$$\omega_0 = \frac{v}{2\pi r} = \frac{qB}{2\pi\mu}$$

If we change the unit $B \to B\frac{2\pi}{c}$, $\omega_0 = \frac{qB}{\mu c}$

(3) $$H = \frac{(P_x + \frac{qB}{2c}Y)^2}{2\mu} + \frac{(P_y - \frac{qB}{2c}X)^2}{2\mu}$$

$$Q = (cP_x + \frac{qB}{2}Y)/qB, \quad P = (P_y - \frac{qB}{2c}X)$$

Clearly $[Q, Q] = [P, P] = 0$ because there is no non-vanishing commutators $[X, P_x] = [Y, P_y] = i\hbar$.

$$[Q, P] = \left[\frac{cP_x + \frac{qB}{2}Y}{qB}, P_y - \frac{qB}{2c}X\right]$$

$$= \frac{c}{qB}\cdot\frac{qB}{2c}[P_x, -X] + \frac{1}{2}[Y, P_y]$$

$$= i\hbar \Rightarrow Q, P \text{ are canonical.}$$

$$H = \frac{\left(\frac{qB}{c}Q\right)^2}{2\mu} + \frac{P^2}{2\mu} = \frac{q^2B^2}{2\mu c^2}Q^2 + \frac{P^2}{2\mu}$$

Let $\omega_0 = \frac{qB}{\mu c}$, $H = \frac{P^2}{2\mu} + \frac{1}{2}\mu\omega_0^2 Q^2$.

This is the Hamiltonian for 1-D harmonic oscillator

with energy levels $E_n = (n + \frac{1}{2})\hbar\omega_0$

* These levels are called Landau Levels. And it shows that (quantization)

in QM, energy levels get quantized even in some cases where particles are free to travel into infinity, eg. $z \to \infty$ in this case.

(4) $$H = \frac{1}{2\mu}\left(P_x^2 + \left(\frac{qB}{2c}\right)^2 Y^2 + \frac{qB}{c}YP_x + P_y^2 + \left(\frac{qB}{2c}\right)^2 X^2 - \frac{qB}{c}XP_y\right)$$

$$= \frac{P_x^2 + P_y^2}{2\mu} + \frac{\mu\omega_0^2}{8}(X^2 + Y^2) - \frac{\omega_0}{2}L_z$$

$L_z = (XP_y - YP_x)$

$\omega_0 = \frac{qB}{\mu c}$

$$= H\left(\frac{\omega_0}{2}, \mu\right) - \frac{\omega_0}{2}L_z$$

where $H(\frac{\omega_0}{2}, \mu)$ is the Hamiltonian for isotropic 2-D oscillator with frequency $\frac{\omega_0}{2}$

$$[HC(\tfrac{\omega_c}{2}, \mu), L_z] = [HC(\tfrac{\omega_0}{2}, \mu), XP_y - YP_x]$$

$$= \frac{1}{2\mu}\left([P_x^2, XP_y] + [P_y^2, -YP_x] \right)$$

$$+ \frac{\mu\omega_0^2}{8}\left([X^2, -YP_x] + [Y^2, XP_y] \right)$$

$$= \frac{1}{2\mu}\left(2P_xP_y(-i\hbar) + 2(i\hbar)P_yP_x \right)$$

$$+ \frac{\mu\omega_0^2}{8}\left(2YX(-i\hbar) + 2XY(i\hbar) \right)$$

$$= 0 \qquad ([P_x, P_y] = [X, Y] = 0)$$

$\Rightarrow$ For $H(\frac{\omega_0}{2}, \mu)|n\rangle = \hbar(\frac{\omega_0}{2})(n+1)|n\rangle$, where $|n\rangle$ is the basis diagonizing $HC(\frac{\omega_0}{2}, \mu)$,

$$HC(\tfrac{\omega_0}{2}, \mu)(L_z|n\rangle) = L_z HC(\tfrac{\omega_0}{2}, \mu)|n\rangle = \hbar(\tfrac{\omega_0}{2})(n+1)(L_z|n\rangle)$$

$\Rightarrow$ $L_z|n\rangle$ is also an eigenfunction of $HC(\frac{\omega_0}{2}, \mu)$ with quantum number n.

$\Rightarrow$ $L_z|n\rangle = \tilde{k}|n\rangle$. # This is just the eigenfunction for L_z, so we know $\tilde{k} \equiv l_z = m\hbar$, where m is another quantum number specifying the degeneracy in $|n\rangle$ (Recall $n = n_x + n_y$)

See exercise 10.2.2

$$\Rightarrow \quad H|n\rangle = \left(HC\frac{\omega_0}{2}, \mu\right) - \frac{\omega_0}{2}L_3\Big)|n\rangle$$

$$= \left(\hbar\frac{\omega_0}{2}(n+1) - \frac{\omega_0}{2}m\hbar\right)|n\rangle$$

So $|n\rangle$ also diagonizes H. The energy levels are

$$E_{n,m} = \frac{\hbar\omega_0}{2}(n+1) - \frac{\hbar\omega_0}{2}m$$

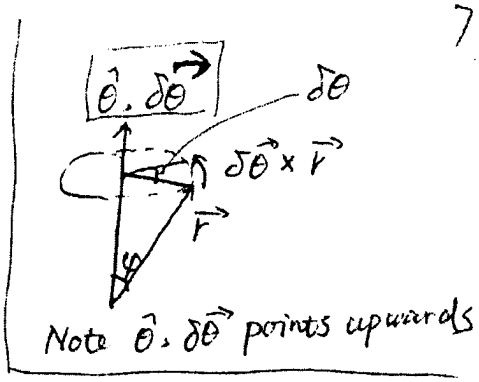
Take $n \equiv 2k + |m|$, $\quad E = \hbar\omega_0\left(k + \frac{|m|}{2} - \frac{m}{2} + \frac{1}{2}\right)$

For $m \geq 0$, $\quad E = \hbar\omega_0\left(k + \frac{1}{2}\right)$

$m < 0$, $\quad E = \hbar\omega_0\left(k - m + \frac{1}{2}\right)$

Thus, levels are the same as $E = \hbar\omega_0\left(n + \frac{1}{2}\right)$

12.4.3/(1) According to the picture, when $\vec{r}$ is rotated by an angle $\delta\theta$ along $\hat{\theta}$ (denote $\delta\vec{\theta} = \delta\theta\,\hat{\theta}$), it moves (the axis) the distance of $r\sin\varphi\,\delta\theta$ along the direction $\hat{\theta}\times\vec{r}$. Thus, we can summarized it as



$$\vec{r} \to \vec{r} + \delta\vec{\theta}\times\vec{r}$$

(2) $$U[R(\delta\vec{\theta})]\,\psi(\vec{r}) = \left(I - \frac{i}{\hbar}\delta\theta\, L_{\hat{\theta}}\right)\psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{\hbar}\delta\theta\, L_{\hat{\theta}}\,\psi(\vec{r}) \qquad (1)$$

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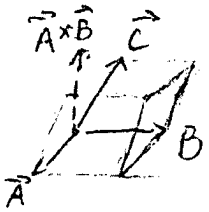
Besides, $$U[R(\delta\vec{\theta})]\,\psi(\vec{r}) = \psi(\vec{r} - \delta\vec{\theta}\times\vec{r})$$

$$= \psi(\vec{r}) - (\delta\vec{\theta}\times\vec{r})\cdot\vec{\nabla}\psi(\vec{r}).$$

Use the identity: $(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{B}\times\vec{C})$ ≠



(This represents the volume of a "leaning" cube with $\vec{A}$, $\vec{B}$, $\vec{C}$ as three edges)

$$\Rightarrow U[R(\delta\vec{\theta})]\psi(\vec{r}) = \psi(\vec{r}) - \delta\vec{\theta}\cdot(\vec{r}\times\vec{\nabla})\psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{i\hbar}\delta\theta\,(\hat{\theta}\cdot(\vec{r}\times\vec{p}))\,\psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{\hbar}\delta\theta\,(\hat{\theta}\cdot\vec{L})\,\psi(\vec{r}) \qquad (2)$$

(1)(2) $\Rightarrow$ $L_{\hat{\theta}} = \hat{\theta}\cdot\vec{L}$

12.5.3/ $J_\pm = J_x \pm i J_y$

$$\Rightarrow \begin{cases} J_x = \dfrac{J_+ + J_-}{2} & J_x^2 = \dfrac{J_+^2 + J_-^2 + J_+J_- + J_-J_+}{4} \\ J_y = \dfrac{J_+ - J_-}{2i} & J_y^2 = \dfrac{-J_+^2 - J_-^2 + J_+J_- + J_-J_+}{4} \end{cases}$$

(1) Since $J_\pm |j, m\rangle \rightarrow |j, m\pm 1\rangle$, we have

$\langle J_\pm \rangle \equiv \langle jm | J_\pm | jm \rangle = 0$, and therefore

$\langle J_x \rangle = \langle J_y \rangle = 0$

(2) $\langle J_x^2 \rangle = \frac{1}{4} \langle jm | (J_+J_- + J_-J_+) | jm \rangle$

Here we have $J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 + i(J_yJ_x - J_xJ_y)$

$= J^2 - J_z^2 + \hbar J_z$

$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J^2 - J_z^2 - \hbar J_z$

$$\Rightarrow \langle J_x^2 \rangle = \frac{1}{4} \langle jm | 2J^2 - 2J_z^2 | jm \rangle = \frac{1}{2}[j(j+1)\hbar^2 - m^2\hbar^2]$$

$$= \frac{\hbar^2}{2}(j(j+1) - m^2)$$

$$\langle J_y^2 \rangle = \frac{1}{4} \langle jm | (J_+J_- + J_-J_+) | jm \rangle = \langle J_x^2 \rangle .$$

(3) $\Delta J_x = \Delta J_y = \sqrt{\langle J_x^2 \rangle} = \hbar\sqrt{\frac{1}{2}[j(j+1) - m^2]}$

$$\Delta J_x \cdot \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - m^2]$$

From (9.2.9), $(\Delta J_x)^2(\Delta J_y)^2 \geq |\langle jm | J_x J_y | jm \rangle|^2$

$$= \left|\langle jm| \frac{J_+J_+ - J_+J_+}{4i} |jm\rangle\right|^2$$

$$\Rightarrow \Delta J_x \cdot \Delta J_y \geq \left|\langle jm| \frac{2\hbar J_z}{4} |jm\rangle\right| = \frac{\hbar}{2}|m|\hbar$$

$$= \frac{|m|}{2}\hbar^2$$

We can see that $[j(j+1) - |m|^2] - |m| = j(j+1) - |m|(|m|+1)$

Since $|m| \leq j$, $j(j+1) - |m|(|m|+1) \geq 0$

$\Rightarrow \Delta J_x \cdot \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - m^2] \geq \frac{|m|}{2}\hbar^2$ is satisfied

(4) When $m = \pm j \Rightarrow$

$\Delta J_x \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - j^2] = \frac{\hbar^2}{2}j = \frac{\hbar^2}{2}|m|$, the inequality is saturated.

12.5.13/(1) $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$

$= \mp\sqrt{\frac{3}{8\pi}}\frac{1}{r}[r\sin\theta\cos\varphi \pm i r\sin\theta\sin\varphi]$

$= \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}$

* $\begin{cases} Y_1^1 = \sqrt{\frac{3}{8\pi}}\frac{-x-iy}{r} \\ Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\frac{x-iy}{r} \end{cases}$

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$

(2) $\psi = N(x+y+2z)e^{-ar}$

Here $\begin{cases} x = (Y_1^{-1} - Y_1^1)\sqrt{\frac{2\pi}{3}}r \\ y = i(Y_1^{-1} + Y_1^1)\sqrt{\frac{2\pi}{3}}r \end{cases}$ $\quad z = Y_0^1\sqrt{\frac{4\pi}{3}}r$

$\Rightarrow \psi = Ne^{-ar}\left[\sqrt{\frac{2\pi}{3}}r\left(Y_1^{-1}(1+i) - Y_1^1(1-i)\right) + 2\sqrt{\frac{4\pi}{3}}r\,Y_1^0\right]$

$\Rightarrow P(l_z=0) = \frac{\left|\int d\Omega\, Y_1^{0*}\psi\right|^2}{\int|\psi|^2 d\Omega} = \frac{(2\sqrt{2})^2}{(2\sqrt{2})^2 + |1+i|^2 + |1-i|^2} = \frac{2}{3}$

* Here we drop the integration over r because this variable can be separated in ψ and this part is integrated into unity in both numerator & denominator.

Similarly $P(l_z=\pm\hbar) = \frac{|d\Omega\, Y_1^{\pm 1*}\psi|^2}{\int|\psi|^2 d\Omega} = \frac{|1\pm i|^2}{(2\sqrt{2})^2 + |1+i|^2 + |1-i|^2}$

$= \frac{1}{6}$

→ * $\psi = Ne^{-ar}\sqrt{\frac{2\pi}{3}}r\left[Y_1^{-1}(1+i) - Y_1^1(1-i) + 2\sqrt{2}\,Y_1^0\right]$