Problem 1: (1.1.3)

(a) Suppose $f(x), g(x)$ are two fuctions with
$$f(0) = f(L) = 0$$
$$g(0) = g(L) = 0$$

Then

(i) $F(x) \equiv f(x) + g(x)$ also satisfies
$$F(0) = F(L) = 0$$

(ii) $aF(x) = af(x) + ag(x)$

(iii) $(a+b)f(x) = af(x) + bf(x)$

(iv) $a(bf(x)) = ab f(x)$

(v) $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x$

(vi) The null vector is 0, then
$f(x) + 0 = f(x)$ — satisfies b.c trivially

(vii) The inverse is $-f(x)$ which also satisfies the boundary conditions

Thus all properties of a vector space are satisfied

(b) If $f(0) = f(4)$
$g(0) = g(4)$
Then clearly $f(0) + g(0) = f(4) + g(4)$
It may be easily verified that other properties of vector spaces are also satisfied with the null vector being 0 and the inverse of $f(x)$ being $-f(x)$

– So these form a vector space

(c) If $f(0) = 4$ and $g(0) = 4$
clearly $f(0) + g(0) \neq 4$
→ Thus these functions do not form a vector space

Further, the null vector 0 (which is a function which vanishes for all x) does not satisfy the condition that it should be equal to 4 at $x = 0$

Similarly the inverse $-f(x)$ is not a member of the space

⇒ Such functions do not form a vector space

Problem 2 (1.1.4)

Consider $a_1|1\rangle + a_2|2\rangle + a_3|3\rangle = 0$

These equations imply

$$a_2 - 2a_3 = 0$$
$$a_1 + a_2 - a_3 = 0$$
$$a_2 - 2a_3 = 0$$

These equations may be written as

$$\sum_{j=1}^{3} M_{ij} a_j = 0 \qquad \forall\, i$$

where $M = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 1 & -1 \\ 0 & 1 & -2 \end{pmatrix}$

Since $\det M = 0$ these equations have nontrivial solutions. The solutions are of the form

$$a_1 = -\alpha, \quad a_2 = 2\alpha, \quad a_3 = \alpha$$

with arbitrary α

$\Rightarrow$ These vectors are not linearly independent

Problem 3 (1.1.5)

(a) Consider $a_1(1,1,0) + a_2(1,0,1) + a_3(3,2,1) = 0$

Then we must have

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

The determinant of the matrix is

$$-2 - 1 + 3 = 0$$

$\Rightarrow$ 7 nontrivial solutions $(a_1, a_2 a_3)$

(b) For the vectors $(1,1,0), (1,0,1), (0,1,$ the matrix is

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2$$

Thus only solution is $a_1 = a_2 = a_3 = 0$

Problem 4 (1.3.1)

First normalize A

$$a = \frac{\vec{A}}{|A|} = \frac{1}{5}(3\hat{i} + 4\hat{j})$$

$$\langle a|B\rangle = \frac{1}{5}(6 - 24) = -\frac{18}{5}$$

Thus define

$$b' = |B\rangle - \langle a|B\rangle|a\rangle$$

$$= (2\hat{i} - 6\hat{j}) + \frac{18}{25}(3\hat{i} + 4\hat{j})$$

$$= \frac{104}{25}\hat{i} - \frac{78}{25}\hat{j} \qquad |b'| = \frac{26}{5}$$

Normalize b'

$$b = \frac{b'}{|b|} = \frac{1}{5}(4\hat{i} - 3\hat{j})$$

$$\langle a|b\rangle = \langle b|a\rangle = 0$$

$$\langle a|a\rangle = \langle b|b\rangle = 1$$

Prob 5 (1.6.2)

(1) $(\Omega\Lambda)^+ = \Lambda^+\Omega^+ = \Lambda\Omega$

Thus product of hermitian operators which do not commute is not hermitian

(2) $(\Omega\Lambda + \Lambda\Omega)^+ = \Lambda\Omega + \Omega\Lambda$

$\Rightarrow$ Anticommutator is hermitian

(3) $[i[\Omega,\Lambda]]^+ = -i(\Omega\Lambda - \Lambda\Omega)^+$

$= -i(\Lambda\Omega - \Omega\Lambda)$

$= i[\Omega,\Lambda]$

This is also hermitian

Problem 6 (1.6.3)

U, V are unitary.

Then
$$(UV)^\dagger UV = V^\dagger U^\dagger U V$$
$$= V^\dagger V$$
$$= 1$$

Since $U^\dagger U = 1$, $V^\dagger V = 1$

Thus (UV) is unitary

Problem 7 (1.6.4)

$$\det(U^\dagger U) = \det(1) = 1$$
$$\Rightarrow (\det U^\dagger)(\det U) = 1$$
$$\Rightarrow (\det U^*)(\det U) = 1$$
$$\Rightarrow |\det U|^2 = 1$$
$$\Rightarrow |\det U| = 1$$

Problem 8 (1.6.5)

$$A \equiv R\left(\frac{\pi i}{2}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Then

$$A^* = A$$

and

$$A^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$A^\dagger A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 9 (1.7.2)

$$\begin{aligned} \det(U^\dagger \Omega U) &= (\det U^\dagger)(\det \Omega)(\det U) \\ &= (\det U^\dagger)(\det U)(\det \Omega) \\ &= \det(U^\dagger U)\det(\Omega) \\ &= \det \Omega \end{aligned}$$

Problem 10 (1.8.1)

The characteristic eqn is

$$\det \begin{pmatrix} 1-\lambda & 3 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 4-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(4-\lambda) = 0$$

Thus eigenvalues are $\lambda = 1, 2, 4$

For $\lambda = 1$

$$\begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} 3u_2 + u_3 &= 0 \\ u_2 &= 0 \\ u_2 + 3u_3 &= 0 \end{aligned}\right] \Rightarrow u_2 = u_3 = 0$$

Thus normalized eigenvector

$$|w=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 2$

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$-u_1 + 3u_2 + u_3 = 0$$

$$u_2 + 2u_3 = 0$$

$$\Rightarrow u_1 = -5u_3 \qquad u_2 = -2u_3$$

Thus normalized eigenvector is

$$\frac{1}{\sqrt{30}}\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda = 4$

$$\begin{pmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$\Rightarrow$

$$-3u_1 + 3u_2 + u_3 = 0$$

$$u_2 = 0$$

Thus $u_3 = 3u_1$

$\Rightarrow$

$$|\omega = 4\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

The matrix is clearly not hermitian

$$\Omega^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 4 \end{pmatrix} \neq \Omega$$

and by inspection the eigenvectors are <u>not</u> orthogonal

Problem 4 (1.7.2)

$$\det(U^+ \Omega U) = (\det U^+)(\det \Omega)(\det U)$$
$$= (\det U^+)(\det U)(\det \Omega)$$
$$= \det(U^+ U) \det \Omega$$
$$= \det \Omega$$

Problem 11 (1.8.3)

$$\det(\Omega - \omega I) = \det \begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -1/2 \\ 0 & -1/2 & \frac{3}{2}-\omega \end{vmatrix}$$

$$= (1-\omega)\left(\left(\frac{3}{2}-\omega\right)^2 - \frac{1}{4} \right) = 0$$

Thus solns are $\omega = 1, 1, 2$

For $\omega = 2$ $\quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & -1/2 \\ 0 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

The soln. to this eqn is

$$x_1 = 0$$
$$x_2 = -x_3 = a$$

Thus the eigenvector is

$$|2\rangle = N \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$$

Normalizing $\langle 2|2\rangle = 1$ gives

$$N = \frac{1}{(2a^2)^{1/2}}$$

For $\omega = 1$ we get the eqn

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Thus x_1 is not determined
and $x_2 = x_3$

$\Rightarrow$ Normalized eigenket is

$$|1\rangle = \frac{1}{(b^2+2c^2)^{1/2}} \begin{pmatrix} b \\ c \\ c \end{pmatrix} \quad \text{for any } b, c$$

Alternatively, requiring that $\langle 2|1\rangle = 0$ we get

$$(0, a, -a) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow a(x_2 - x_3) = 0 \quad \text{or } x_2 = x_3$$

Problem 12 (1.8.4)

To determine eigenvalues, solve

$$\begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda-2)(\lambda-4)+1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$$

An eigenvector

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \Rightarrow |3,1\rangle = \frac{1}{\sqrt{2a^2}}\begin{pmatrix} a \\ a \end{pmatrix}$$

This is the most general form of the eigenvector. Clearly we cannot pick a to find a pair of orthogonal eigenvectors

Problem 13 (1.8.6)

(1) $$\Omega^\dagger = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Omega^\dagger \Omega = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2) $$\begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$$(\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

$$(\lambda - \cos\theta + i\sin\theta)(\lambda - \cos\theta - i\sin\theta) = 0$$

Thus $\lambda = e^{-i\theta}, e^{i\theta}$

(3) For $\lambda = e^{-i\theta} = \cos\theta - i\sin\theta$

$$\begin{pmatrix} \cos\theta - (\cos\theta - i\sin\theta) & \sin\theta \\ -\sin\theta & \cos\theta - (\cos\theta - i\sin\theta) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} +i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad i u_1 + u_2 = 0 \quad \Rightarrow \quad u_1 = i u_2$$

$$\Rightarrow$$

$$|\omega = e^{-i\theta}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{normalized}$$

Similarly for $\lambda = e^{i\theta}$

$$|\omega = e^{i\theta}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Clearly

$$\langle e^{-i\theta} | e^{i\theta} \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} (-i, 1) \begin{pmatrix} -i \\ 1 \end{pmatrix} = 0$$

(4) The matrix U is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$U^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$U^+ U = \frac{1}{2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 1$$

$$U^\dagger \Omega U = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} ie^{i\theta} & -ie^{-i\theta} \\ e^{i\theta} & e^{-i\theta} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2e^{i\theta} & 0 \\ 0 & 2e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Problem 14 (1.8.8)

(1) Suppose $M^i|a\rangle = \lambda|a\rangle$

Then

$$(M^i)^2|a\rangle = \lambda M^i|a\rangle = \lambda^2|a\rangle$$

However $(M^i)^2 = I \Rightarrow \lambda^2 = 1$

$$\Rightarrow \lambda = \pm 1$$

(2) Consider any of the matrices, e.g. M^1. Then consider

$$\mathrm{Tr}\,(M^1(M^i)^2) = \mathrm{Tr}(M^1 I) = \mathrm{Tr}\,M^1$$

Now take $i \neq i$

$$\mathrm{Tr}\,(M^1(M^i)^2) = \mathrm{Tr}(M^1 M^i M^i)$$

$$= -\mathrm{Tr}(M^i M^1 M^i) \quad (M^1M^j = -M^jM^i,\ i\neq j)$$

$$= -\mathrm{Tr}\,((M^i)^2 M^1) \quad \text{(cyclicity)}$$

$$= -\mathrm{Tr}(M^1).$$

Thus $\mathrm{Tr}\,M^1 = -\mathrm{Tr}\,M^1 \Rightarrow \mathrm{Tr}\,M^1 = 0$

Similary for any of the others

(3) Consider the basis in which say M^1 is diagonal

$M^1 = \text{diag}(1, 1, -1, -1, 1, \ldots)$

etc. (since eigenvalues are ± 1)

Since $\text{Tr} M^1 = 0$

half of the diagonal elements have to be $+1$, other half have to be -1

This is impossible if M^1 is odd-dimensional