PHY 614 - 2011
Problem Set #8

Problem 1 (11.4.1)

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

In coordinate representation

$$\psi(x,t) = e^{-i\hat{H}t/\hbar} \psi(x,0)$$

Suppose

$$\langle x|\Pi|\psi\rangle = \Pi\psi(x,0) = \psi(-x)$$

Then

$$\Pi\psi(x,t) = \Pi e^{-i\hat{H}t/\hbar} \psi(x,0)$$

However since $[\Pi, \hat{H}] = 0$

$$\Pi e^{-i\hat{H}t/\hbar} = e^{-iHt/\hbar} \Pi$$

Thus

$$\Pi\psi(x,t) = e^{-i\hat{H}t/\hbar} \Pi\psi(x,0)$$
$$= e^{-iHt/\hbar} \psi(-x,0)$$
$$= \psi(-x,t)$$

The proof is identical if

$$\Pi\psi(x,0) = -\psi(-x)$$

Problem 2 (12.2.1)

$$U[R]|x,y\rangle = |x - y\varepsilon_z,\ x\varepsilon_z + y\rangle$$

This means

$$\langle x,y|U^\dagger[R] = \langle x - y\varepsilon_z,\ x\varepsilon_z + y|$$

However

$$U^\dagger(R) = 1 + \frac{i\varepsilon_z L_z}{\hbar}$$

Since $L_z = L_z^\dagger$

Thus

$$\langle x,y|U[R] = \langle x + y\varepsilon_z,\ -x\varepsilon_z + y|$$

Thus

$$\langle x,y|U[R]|\psi\rangle = \psi(x + y\varepsilon_z,\ y - x\varepsilon_z)$$

Problem 3 (12.2.3)

$x = r\cos\phi$ $\qquad \tan\phi = \frac{y}{x}$

$y = r\sin\phi$ $\qquad r^2 = x^2 + y^2.$ $\qquad \frac{\partial r}{\partial x} = \frac{x}{r}$

Thus

$$\sec^2\phi\, d\phi = \frac{dy}{x} - \frac{y\,dx}{x^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{x}\frac{x^2}{r^2} = \frac{x}{r^2}$$

$$\frac{\partial\phi}{\partial x} = -\frac{y}{x^2}\frac{x^2}{r^2} = -\frac{y}{r^2}.$$

Thus

$$x\partial_y - y\partial_x = x\left[\frac{x}{r^2}\partial_\phi + \frac{y}{r}\partial_r\right] - y\left[-\frac{y}{r^2}\partial_\phi + \frac{x}{r}\partial_r\right]$$

$$= \frac{x^2+y^2}{r^2}\partial_\phi = \partial_\phi$$

$$\Rightarrow \quad L_z = -i\hbar\partial_\phi$$

By direct method this follows since to lowest order

$$\psi(\rho, \phi - \varepsilon_z) = \left(1 - \varepsilon_z\frac{\partial}{\partial\phi}\right)\psi(\rho,\phi)$$

$$= \left(1 - \frac{i}{\hbar}\left(-i\hbar\frac{\partial}{\partial\phi}\right)\right)\psi(\rho,\phi)$$

Problem 4 (12.2.4)

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\varepsilon_x} \begin{pmatrix} x+\varepsilon_x \\ y \end{pmatrix} \xrightarrow{\varepsilon_z}$$

$$\begin{bmatrix} (x+\varepsilon_x)\cos\varepsilon_z - y\sin\varepsilon_z \\ (x+\varepsilon_x)\sin\varepsilon_z + y\cos\varepsilon_z \end{bmatrix}$$

$$\longrightarrow -\varepsilon_x$$

$$\begin{pmatrix} x\cos\varepsilon_z + \varepsilon_x(\cos\varepsilon_z - 1) - y\sin\varepsilon_z \\ (x+\varepsilon_x)\sin\varepsilon_z + y\cos\varepsilon_z \end{pmatrix}$$

$$\longrightarrow -\varepsilon_z$$

$$\begin{bmatrix} \left(x\cos\varepsilon_z + \varepsilon_x(\cos\varepsilon_z - 1) - y\sin\varepsilon_z\right)\cos\varepsilon_z \\ \quad + \left((x+\varepsilon_x)\sin\varepsilon_z + y\cos\varepsilon_z\right)\sin\varepsilon_z \\ -\left(x\cos\varepsilon_z + \varepsilon_x(\cos\varepsilon_z - 1) - y\sin\varepsilon_z\right)\sin\varepsilon_z \\ \quad + \left((x+\varepsilon_x)\sin\varepsilon_z + y\cos\varepsilon_z\right)\cos\varepsilon_z \end{bmatrix}$$

Keep terms up to $O(\varepsilon_z^2)$

$$\begin{pmatrix} x + \frac{1}{2}\varepsilon_x\varepsilon_z^2 + O(\varepsilon_x\varepsilon_z^4) \\ y + \varepsilon_x\varepsilon_z + O(\varepsilon_x\varepsilon_z^3) \end{pmatrix}$$

Thus the RHS of (12.2.24) is modified to

$$U[R(-\epsilon_z \hat{k})]\, T(-\epsilon_x)\, U[R(\epsilon_z \hat{k})]\, T(\epsilon_x)$$
$$= T\left(\tfrac{1}{2}\epsilon_x \epsilon_z^2 \hat{i} + \epsilon_x \epsilon_z \hat{j}\right)$$

Expanding LHS

$$\left(1 + \frac{i}{\hbar}\epsilon_z L_z - \frac{1}{2\hbar^2}\epsilon_z^2 L_z^2\right)\left(1 + \frac{i}{\hbar}\epsilon_x P_x\right)$$
$$\left(1 - \frac{i}{\hbar}\epsilon_z L_z - \frac{1}{2\hbar^2}\epsilon_z^2 L_z^2\right)$$
$$\left(1 - \frac{i}{\hbar}\epsilon_x P_x\right)$$

$$= \Bigg(1 + \frac{i}{\hbar}\epsilon_z L_z - \frac{1}{2\hbar^2}\epsilon_z^2 L_z^2 + \frac{i}{\hbar}\epsilon_x P_x$$
$$- \frac{1}{\hbar^2}\epsilon_x \epsilon_z L_z P_x - \frac{i}{2\hbar^3}\epsilon_x \epsilon_z^2 L_z^2 P_x\Bigg)$$
$$\Bigg(1 - \frac{i}{\hbar}\epsilon_z L_z - \frac{1}{2\hbar^2}\epsilon_z^2 L_z^2 - \frac{i}{\hbar}\epsilon_x P_x$$
$$- \frac{1}{\hbar^2}\epsilon_x \epsilon_z L_z P_x + \frac{i}{2\hbar^3}\epsilon_x \epsilon_z^2 L_z^2 P_x$$

We keep terms which are $O(1)$
$O(\varepsilon_x)$ $O(\varepsilon_z)$ $O(\varepsilon_x \varepsilon_z)$ $O(\varepsilon_x \varepsilon_z^2)$

$O(\varepsilon_x)$: $\frac{i}{\hbar} \varepsilon_x P_x - \frac{i}{\hbar} \varepsilon_x P_x = 0$

$O(\varepsilon_z)$ $\frac{i}{\hbar} \varepsilon_z L_z - \frac{i}{\hbar} \varepsilon_z L_z = 0$

$O(\varepsilon_x \varepsilon_z)$: $\frac{1}{\hbar^2} \varepsilon_z \varepsilon_x L_z P_x + \frac{1}{\hbar^2} \varepsilon_z \varepsilon_x P_x L_z$
$- \frac{1}{\hbar^2} \varepsilon_x \varepsilon_z L_z P_x - \frac{\varepsilon_z \varepsilon_x}{\hbar^2} L_z P_x$

$= \frac{\varepsilon_x \varepsilon_z}{\hbar^2} [P_x, L_z]$

$O(\varepsilon_x \varepsilon_z^2)$ $= \frac{i}{2\hbar^3} \varepsilon_x \varepsilon_z^2 L_z^2 P_x$
$- \frac{i}{\hbar^3} \varepsilon_x \varepsilon_z^2 L_z^2 P_x$
$+ \frac{i}{2\hbar^3} \varepsilon_x \varepsilon_z^2 L_z^2 P_x$
$- \frac{i}{2\hbar^3} \varepsilon_x \varepsilon_z^2 P_x L_z^2$ ✓
$+ \frac{i}{\hbar^3} \varepsilon_x \varepsilon_z^2 L_z P_x L_z$
$- \frac{i}{2\hbar^3} \varepsilon_x \varepsilon_z^2 L_z^2 P_x$ ✓

$= \varepsilon_x \varepsilon_z^2 \left[-\frac{i}{2\hbar^3} L_z^2 P_x - \frac{i}{2\hbar^3} P_x L_z^2 + L_z P_x L_z \right]$

Comparing with RHS must have

$$-2L_z P_x L_z + P_x L_z^2 + L_z^2 P_x = \hbar^2 P_x$$

Since the RHS is

$$1 - \frac{i}{\hbar}\left(\frac{\varepsilon_x \varepsilon_z^2}{2} P_x + \varepsilon_x \varepsilon_z P_y\right)$$

This means we must have

$$[L_z, [L_z, P_x]] = \hbar^2 P_x$$

Since $[L_z, P_x] = i\hbar P_y$

$$[L_z, P_y] = -i\hbar P_x$$

this is satisfied

Note 11/28/2010

Problem 5 (12.3.2)

$$\psi(\rho, \phi+2\pi) = e^{\frac{2\pi i l_z}{\hbar}} \left[A e^{i\phi l_z/\hbar} + B e^{\frac{i\phi l_z'}{\hbar}} e^{\frac{2\pi i (l_z' - l_z)}{\hbar}} \right]$$

If $\psi(\rho, \phi+2\pi)$ represents the same state as $\psi(\rho, \phi)$

$$\psi(\rho, \phi+2\pi) = e^{i\alpha} \psi(\rho, \phi)$$

Thus $$e^{\frac{2\pi i}{\hbar}(l_z' - l_z)} = 1$$

$$\Rightarrow \quad l_z' - l_z = m\hbar$$

$$m = 0, \pm 1, \pm 2, \pm 3 \ldots$$

If $(l_z, -l_z)$ are both allowed values

$$2l_z = m\hbar \Rightarrow l_z = \frac{m}{2}\hbar$$

Problem 6 (12.3-3)

$$\cos^2\phi = \frac{1}{4}\left(e^{i\phi} + e^{-i\phi}\right)^2$$

$$= \frac{1}{4}\left(e^{2i\phi} + e^{-2i\phi} + 2\right)$$

Thus

$$P(l_z=0) : P(l_z=2\hbar) : P(l_z=-2\hbar)$$

$$= 4 : 1 : 1$$

Requiring $\sum P = 1$

We get

$$P(l_z=0) = \frac{4}{6} = \frac{2}{3}$$

$$P(l_z=2\hbar) = P(l_z=-2\hbar) = 1/6$$

Clearly the φ dependence is irrelevant