PHY 615 – 2012
PROBLEM SET #1

Problem 1 (14.3.1)

Consider spinor $\begin{pmatrix} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{pmatrix}$

Normalization $\Rightarrow \quad \rho_1^2 + \rho_2^2 = 0$

Parametrize $\quad \rho_1 = \cos\frac{\theta}{2}$

$\rho_2 = \sin\frac{\theta}{2}$

$$\chi = e^{i\frac{\phi_1+\phi_2}{2}} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi_2-\phi_1}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi_2-\phi_1}{2}} \end{pmatrix}$$

$\Rightarrow$ n is along direction

$\theta, \phi \equiv \theta, \frac{\phi_2-\phi_1}{2}$.

Problem 2 (14.3.2)

(1) $\hat{n}\cdot\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$

$$(\hat{n}\cdot\vec{\sigma})|\hat{n}+\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2}\left(\cos\theta\cos\theta/2 + \sin\theta\sin\frac{\theta}{2}\right) \\ e^{i\phi/2}\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix} = |\hat{n}+\rangle$$

$$(\hat{n}\cdot\vec{\sigma})|\hat{n}-\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\phi/2} \\ \cos\theta/2 e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2}\left(-\sin\frac{\theta}{2}\cos\theta + \sin\theta\cos\frac{\theta}{2}\right) \\ e^{i\phi/2}\left(-\sin\theta\sin\theta - \cos\theta\cos\theta/2\right) \end{pmatrix}$$

$$= -|\hat{n},-\rangle$$

(2) $\langle n+|\sigma_1|n+\rangle$

$$= \left(\cos\frac{\theta}{2}e^{+i\phi/2}, \sin\frac{\theta}{2}e^{-i\phi/2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}e^{-i} \\ \sin\frac{\theta}{2}e^{i} \end{pmatrix}$$

$$= \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi}\right)$$

$$= 2\cos\phi\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\cos\phi$$

$$= n_x$$

Thus $\langle n+ | S_1 | n+ \rangle = \frac{\hbar}{2} n_x$

The other relations can be verified similarly

Problem 3 (14.3.3)

From (14.3.32)

$$\mathrm{Tr}\, \sigma_i \cdot \sigma_j = -\mathrm{Tr}(\sigma_j \cdot \sigma_i) = -\mathrm{Tr}(\sigma_i \cdot \sigma_j)$$

(Since $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$)

$$\Rightarrow \mathrm{Tr}\, \sigma_i \cdot \sigma_j = 0 \qquad (i \neq j)$$

From (14.3.33)

$$\sigma_i \cdot \sigma_j = i \epsilon_{ijk} \sigma_k .$$

Thus we have

$$i \epsilon_{ijk}\, \mathrm{Tr}\, \sigma_k = 0$$

Since ϵ_{ijk} is totally antisymmetric this implies

$$\mathrm{Tr}\, \sigma_k = 0 \quad \forall k .$$

eg for $i=1, j=2$

$$i \epsilon_{12k}\, \mathrm{Tr}\, \sigma_k = i\, \mathrm{Tr}\, \sigma_3 = 0$$

Problem 4 (4.3.4)

(1) $(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = A_i B_j \sigma_i \sigma_j$

$\sigma_i \sigma_j = \frac{1}{2}[\sigma_i, \sigma_j] + \frac{1}{2}\{\sigma_i, \sigma_j\}$

$= i\epsilon_{ijk}\sigma_k + \delta_{ij}$

$\Rightarrow$

$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = A_i B_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k)$

$= \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}$

(2) Suppose

$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = m_0 + \sum_{i=1}^{3} m_i \sigma_i$

Then

$m_0 = \frac{1}{2}\mathrm{Tr}[(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma})]$

$= \frac{1}{2} A_i B_j \mathrm{Tr}(\sigma_i \sigma_j)$

$= \frac{1}{2} A_i B_j 2\delta_{ij} = \vec{A}\cdot\vec{B}$

$m_1 = \frac{1}{2} A_i B_j \mathrm{Tr}(\sigma_i \sigma_j \sigma_1)$

The trace is nonzero and equal to $2i$ only if $i=2, j=3$ or $i=3, j=2$

$\Rightarrow m_1 = i(A_2 B_3 - A_3 B_2)$

$= i(\vec{A}\times\vec{B})_1$

Similarly for m_2

Problem 5 (14.3.5)

Suppose a 2x2 matrix M is expanded as follows

$$M = \sum_{\mu=0}^{3} m_\mu \sigma_\mu \qquad \sigma_0 \equiv \mathbb{I}_{2\times 2}$$

Then $\mathrm{Tr}\, M = 2m_0$

since $\mathrm{Tr}\, \sigma_i = 0 \qquad i = 1, 2, 3 \ldots$

$$\mathrm{Tr}(M\sigma_i) = m_0 \mathrm{Tr}\,\sigma_i + \sum_j m_j \mathrm{Tr}(\sigma_i \sigma_j)$$

Now use

$$\mathrm{Tr}\,\sigma_i \sigma_j = 2\delta_{ij}$$

and $\mathrm{Tr}\,\sigma_i = 0$

$\Rightarrow$

$$\mathrm{Tr}(M\sigma_i) = 2m_i$$

Thus

$$m_0 = \frac{1}{2}\mathrm{Tr}\, M$$

$$m_i = \frac{1}{2}\mathrm{Tr}(M\sigma_i)$$

Direct evaluation

$\Rightarrow$

$$m_0 = \frac{1}{2}(\alpha + \delta)$$

$$m_1 = \frac{1}{2}(\beta + \gamma)$$

$$m_2 = \frac{1}{2} i(\beta - \gamma)$$

$$m_3 = \frac{1}{2}(\alpha - \delta)$$

which may be solved for $\alpha, \beta, \gamma, \delta$

Problem 6 (14.3.7)

(1) $$I + i\sigma_x = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sigma_x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sigma_x \sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\, e^{i\sigma_x \pi/4}$$

Thus

$$(I + i\sigma_x)^{1/2} = 2^{1/4} e^{i\sigma_x \pi/8}$$

$$= 2^{1/4}\left[\cos\frac{\pi}{8} + i\sigma_x \sin\frac{\pi}{8}\right]$$

(2) $$(2I + \sigma_x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(2I + \sigma_x)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}(2I - \sigma_x)$$

(3) $$\sigma_x^{-1} = \sigma_x \quad \text{since } \sigma_x^2 = 1$$

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Problem 5 (14.3.8)

Consider a general matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Then, as in Problem 4

$$M = \sum_{\mu=0}^{3} m_\mu \sigma_\mu$$

(1) $$[M, \sigma_i] = \sum_{\mu=0}^{3} m_\mu [\sigma_\mu, \sigma_i]$$

$$= \sum_{k=1}^{3} m_k [\sigma_k, \sigma_i]$$

$$= 2 \sum_{k=1}^{3} m_k \, i \epsilon_{kij} \sigma_j$$

Now, $$\sum_{i=1}^{3} n_i \sigma_i = 0 \Rightarrow n_i = 0 \quad \forall i$$

Thus

$$\sum_{k} \epsilon_{jki} m_k = 0 \quad \forall (i,j)$$

$$[M, \sigma_1] = 0 \Rightarrow m_2 = m_3 = 0$$

$$[M, \sigma_2] = 0 \Rightarrow m_3 = m_1 = 0$$

Thus only $m_0 \neq 0$ possible

ie M must be a multiple of the unit matrix

(2) For the anticommutator

$$\{M, \sigma_i\} = 2m_0\sigma_i + \sum_j m_j \{\sigma_j, \sigma_i\}$$

However

$$\{\sigma_j, \sigma_i\} = 2\delta_{ji}$$

Thus

$$\{M, \sigma_i\} = 2(m_0\sigma_i + m_i)$$

Thus $\{M, \sigma_i\} = 0 \Rightarrow m_0, m_i = 0$

e.g.

$$0 = \{M, \sigma_3\} = 2(m_0\sigma_3 + m_3)$$

$$= 2\begin{pmatrix} m_0+m_3 & 0 \\ 0 & m_0-m_3 \end{pmatrix}$$

$$\Rightarrow m_0 = m_3 = 0$$

$$\{M, \sigma_1\} = 0 = 2(m_0\sigma_1 + m_1)$$

$$\Rightarrow m_1 = 0$$
$$m_0 = 0$$

$\{M, \sigma_2\} = 0 \Rightarrow m_2 = 0$ as well

Thus such a M must vanish