PHY 615-2012
Problem Set # 5

Problem 1: Possible values of j are

$$j = j_2 + 1,\ j_2,\ j_2 - 1$$

For a given j, and eigenvalue m of J_z, the values of m_1, m_2 must be such that

$$m_1 + m_2 = m$$

Since possible values of m_1 are $1, 0, -1$ possible values of m_2 are $m-1,\ m,\ m+1$

Thus write

$$|j,m\rangle = c_1 |1,1;j_2,m-1\rangle + c_2 |1,0;j_2,m\rangle + c_3 |1,-1;j_2,m+1\rangle$$

Consider

$$J^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2\,\vec{J}_1\cdot\vec{J}_2$$
$$= \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2\,J_{1z}J_{2z} + J_{1-}J_{2+} + J_{2-}J_{1+}$$

This gives

$$J^2|j,m\rangle$$
$$= c_1\left(2 + j_2(j_2+1) + 2(m-1) + J_{1-}J_{2+}\right)|1,1;j_2,m-1\rangle$$
$$+ c_2\left(2 + j_2(j_2+1) + J_{1-}J_{2+} + J_{2-}J_{1+}\right)|1,0;j_2,m\rangle$$
$$+ c_3\left(2 + j_2(j_2+1) - 2(m+1) + J_{1+}J_{2-}\right)|1,-1;j_2,m+1\rangle$$

where we have used

$$J_{1+}|1,1;j_2,m-1\rangle = J_{1-}|1,-1;j_2,m+1\rangle = 0$$

$$J_{1-}\,|\,j_1, m_1\rangle = \sqrt{(j_1+m_1)(j_1-m_1+1)}$$
$$J_{1+}\,|\,j_1, m_1\rangle = \sqrt{(j_1-m_1)(j_1+m_1+1)}$$

$$J_{1-}\,J_{2+}\,|\,1,1\,;\,j_2, m-1\rangle$$

$$= \sqrt{2}\,\sqrt{(j_2-m+1)(j_2+m)}\;\;|\,1,0\,;\,j_2\,m\rangle$$

$$J_{2-}\,J_{1+}\,|\,1,0\,;\,j_2, m\rangle$$
$$= \sqrt{2}\,\sqrt{(j_2+m)(j_2-m+1)}\;\;|\,1,1\,;\,j_2\,m-1\rangle$$

$$J_{2+}\,J_{1-}\,|\,1,0\,;\,j_2, m\rangle$$
$$= \sqrt{2}\,\sqrt{(j_2-m)(j_2+m+1)}\;\;|\,1,-1\,;\,j_2, m+1\rangle$$

$$J_{1+}\,J_{2-}\,|\,1,-1\,;\,j_2, m+1\rangle$$
$$= \sqrt{2}\,\sqrt{(j_2+m+1)(j_2-m)}\;\;|\,1,0\,;\,j_2, m\rangle$$

This leads to

$$J^2\,|\,j, m\rangle$$
$$= C_1\,[\,2 + j_2(j_2+1) + 2(m-1)\,]\;|\,1,1\,;\,j_2, m-1\rangle$$
$$+ \sqrt{2}\,C_1\,\sqrt{(j_2-m+1)(j_2+m)}\;\;|\,1,0\,;\,j_2, m\rangle$$
$$+ C_2\,[\,2 + j_2(j_2+1)\,]\;|\,1,0\,;\,j_2, m\rangle$$
$$+ \sqrt{2}\,C_2\,\sqrt{(j_2+m)(j_2-m+1)}\;\;|\,1,1\,;\,j_2, m-1\rangle$$
$$+ \sqrt{2}\,C_2\,\sqrt{(j_2-m)(j_2+m+1)}\;\;|\,1,-1\,;\,j_2, m+1\rangle$$
$$+ C_3\,[\,2 + j_2(j_2+1) - 2(m+1)\,]\;|\,1,-1\,;\,j_2, m+1\rangle$$
$$+ \sqrt{2}\,C_3\,\sqrt{(j_2+m+1)(j_2-m)}\;\;|\,1,0\,;\,j_2, m\rangle$$

This should be

$$j(j+1)\,|j,m\rangle$$

This therefore requires

$$c_1\left(2 + j_2(j_2+1) + 2(m-1)\right) + \sqrt{2}\,c_2\sqrt{(j_2+m)(j_2-m+1)} = j(j+1)\,c_1$$

$$c_2\left(2 + j_2(j_2+1)\right) + \sqrt{2}\,c_1\sqrt{(j_2-m+1)(j_2+m)} + \sqrt{2}\,c_3\sqrt{(j_2+m+1)(j_2-m)} = j(j+1)\,c_2$$

$$c_3\left(2 + j_2(j_2+1) - 2(m+1)\right) + \sqrt{2}\,c_2\sqrt{(j_2-m)(j_2+m+1)} = j(j+1)\,c_3$$

$\Rightarrow$

$$\left[j(j+1) - j_2(j_2+1) - 2m\right] c_1 = \sqrt{2(j_2+m)(j_2-m+1)}\;c_2$$

$$\left[j(j+1) - j_2(j_2+1) + 2m\right] c_3 = \sqrt{2(j_2-m)(j_2+m+1)}\;c_2$$

$$\left[j(j+1) - j_2(j_2+1) - 2\right] c_2 = \sqrt{2(j_2+m)(j_2-m+1)}\;c_1 + \sqrt{2(j_2-m+1)(j_2-m)}\;c_3$$

The first 2 equations imply 3rd automatically

Consider now $j = j_2 + 1$

$$j(j+1) - j_2(j_2+1) - 2m$$
$$= (j_2+1)(j_2+2) - j_2(j_2+1) - 2m$$
$$= 2(j_2+1) - 2m = 2[j_2 - m + 1].$$

$$j(j+1) - j_2(j_2+1) + 2m$$
$$= 2[j_2 + m + 1]$$

Then

$$\frac{c_1}{c_2} = \frac{\sqrt{2(j_2+m)(j_2-m+1)}}{2(j_2-m+1)}$$

$$= \sqrt{\frac{j_2+m}{2(j_2-m+1)}}$$

$$\frac{c_3}{c_2} = \frac{\sqrt{2(j_2-m)(j_2+m+1)}}{2(j_2+m+1)}$$

$$= \sqrt{\frac{j_2-m}{2(j_2+m+1)}}$$

This means the state vector is

$$|j_2+1, m\rangle = A\left[\sqrt{\frac{j_2+m}{2(j_2-m+1)}}\,|1,1;j_2,m-1\rangle + |1,0;j_2,m\rangle + \sqrt{\frac{j_2-m}{2(j_2+m+1)}}\,|1,-1;j_2,m+1\rangle\right]$$

The normalization constant is determined by

$$|A|^2\left(\frac{j_2+m}{2(j_2-m+1)} + 1 + \frac{j_2-m}{2(j_2+m+1)}\right) = 1$$

$$\Rightarrow |A|^2\,\frac{(2j+1)(j+1)}{(j_2-m+1)(j_2+m+1)} = 1$$

$$\Rightarrow A = \sqrt{\frac{(j_2-m+1)(j_2+m+1)}{(2j+1)(j+1)}}$$

$$\Rightarrow |j_2+1, m\rangle = \frac{1}{\sqrt{(2j+1)(j+1)}}\left\{\frac{1}{\sqrt{2}}\sqrt{(j_2+m)(j_2+m+1)}\,|1,1;j_2,m-1\rangle + \sqrt{(j_2-m+1)(j_2+m+1)}\,|1,0;j_2,m\rangle + \frac{1}{\sqrt{2}}\sqrt{(j_2-m)(j_2-m+1)}\,|1,-1;j_2,m+1\rangle\right\}$$

For $j = j_2$

$$\frac{c_1}{c_2} = \frac{\sqrt{2(j_2+m)(j_2-m+1)}}{-2m}$$

$$\frac{c_3}{c_2} = \frac{\sqrt{2(j_2-m)(j_2+m+1)}}{2m}$$

Thus state

$$|j_2, m\rangle = B\left[\frac{\sqrt{(j_2+m)(j_2-m+1)}}{-\sqrt{2}m}|1,1;j_2,m-1\rangle + |1,0;j_2,m\rangle + \frac{\sqrt{(j_2-m)(j_2+m+1)}}{\sqrt{2}m}|1,-1;j_2,m+1\rangle\right]$$

The normalization determined by

$$|B|^2\left[\frac{(j_2+m)(j_2-m+1)}{2m^2} + 1 + \frac{(j_2-m)(j_2+m+1)}{2m^2}\right] = 1$$

$$|B|^2 \frac{j_2(j_2+1)}{m^2} = 1 \Rightarrow B = \frac{m}{\sqrt{j_2(j_2+1)}}$$

$$|j_2, m\rangle = \frac{1}{\sqrt{j_2(j_2+1)}}\left\{-\frac{1}{\sqrt{2}}\sqrt{(j_2+m)(j_2-m+1)}|1,1;j_2,m-1\rangle + m|1,0;j_2,m\rangle + \frac{1}{\sqrt{2}}\sqrt{(j_2-m)(j_2+m+1)}|1,-1;j_2,m+1\rangle\right\}$$

For $j = j_2 - 1$

$$j(j+1) - j_2(j_2+1) \pm 2m$$
$$= -2j_2 \pm 2m$$
$$\Rightarrow$$
$$\frac{c_1}{c_2} = \frac{\sqrt{2(j_2+m)(j_2-m+1)}}{-2(j_2+m)}$$
$$= -\sqrt{\frac{j_2-m+1}{2(j_2+m)}}$$
$$\frac{c_3}{c_2} = -\sqrt{\frac{j_2+m+1}{2(j_2-m)}}$$
$$\Rightarrow |j_2-1, m\rangle$$
$$= C\Big[\sqrt{\frac{j_2-m+1}{j_2+m}}\,|1,1;j_2,m-1\rangle - |1,0;j_2,m\rangle \sqrt{\frac{j_2+m+1}{j_2-m}}\,|1,-1;j_2\,m+1\rangle\Big]$$

Normalization $\Rightarrow$ $C = \sqrt{\frac{j_2^2-m^2}{j_2(2j_2+1)}}$

$$|j_2-1,m\rangle = \frac{1}{\sqrt{j_2(2j_2+1)}}\Big[\frac{1}{\sqrt{2}}\sqrt{(j_2-m)(j_2-m+1)}\,|1,1;j_2,m-1\rangle - \sqrt{(j_2-m)(j_2+m)}\,|1,0;j_2,m\rangle - \frac{1}{\sqrt{2}}\sqrt{(j_2+m)(j_2+m+1)}\,|1,-1;j_2,m+1\rangle\Big]$$

Problem 3. (16.1.2)

$$\psi(x) = (x-a)(x+a)$$

$$E(\psi) = \frac{\int dx\, \psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi(x)}{\int dx\, \psi^*(x)\,\psi(x)}$$

$$\int dx\, \psi^*\psi = \int_{-a}^{a} dx\,(x^2-a^2)^2 = 16a^5/15$$

$$\psi'(x) = 2x \qquad \psi'' = 2$$

$$E(\psi) = -\frac{\hbar^2}{m}\int_{-a}^{a} dx\,(x^2-a^2)\cdot\frac{15}{16a^5}$$

$$= -\frac{\hbar^2}{m}\left(-\frac{4a^3}{3}\right)\left(\frac{15}{16a^5}\right)$$

$$= \frac{5\hbar^2}{4ma^2}$$

Compare

$$E_0 = \frac{\hbar^2\pi^2}{2m(2a)^2} = \frac{\hbar^2\pi^2}{8ma^2}$$

$$\frac{E(\psi)}{E_0} = \frac{5\hbar^2}{4ma^2}\cdot\frac{8ma^2}{\hbar^2\pi^2} = \frac{10}{\pi^2}$$

Problem 4: (16.1.3)

Trial wave fn

$$\psi(x) = e^{-\alpha x^2}$$

$$\int dx |\psi|^2 = \sqrt{\frac{\pi}{2\alpha}}$$

$$\int dx\, \psi^* \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - aV_0\,\delta(x) \right] \psi$$

$$= -\frac{\hbar^2}{2m}\int dx\, \psi^*\psi'' - aV_0 |\psi(0)|^2$$

$$= -\frac{\hbar^2}{2m}\int dx\, e^{-2\alpha x^2}\left[2\alpha(2\alpha x^2 - 1) \right] - aV_0$$

$$= -\frac{\hbar^2}{2m}\left(-\sqrt{\frac{\pi\alpha}{2}} \right) - aV_0$$

$$E(\psi) = \sqrt{\frac{2\alpha}{\pi}}\left[\frac{\hbar^2}{2m}\sqrt{\frac{\pi\alpha}{2}} - aV_0 \right]$$

$$= \frac{\hbar^2}{2m}\left[\alpha - \frac{2maV_0}{\hbar^2}\sqrt{\frac{2\alpha}{\pi}} \right]$$

$$\frac{dE(\alpha)}{d\alpha} = \frac{\hbar^2}{2m}\left[1 - \frac{2maV_0}{\hbar^2}\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\,\frac{1}{\sqrt{\alpha}} \right]$$

Thus $\alpha_0 = \frac{(aV_0)^2}{2\pi}\cdot\left(\frac{2m}{\hbar^2}\right)^2$

$$E(\alpha_0) = \frac{\hbar^2}{2m}\left[\left(\frac{2m}{\hbar^2}\right)^2 \frac{(aV_0)^2}{2\pi} - \frac{2maV_0}{\hbar^2}\sqrt{\frac{2}{\pi}}\,\frac{aV_0}{\sqrt{2\pi}}\left(\frac{2m}{\hbar^2}\right) \right]$$

ie $$E(\alpha_0) = -\frac{2m}{\hbar^2}\frac{1}{2\pi}(aV_0)^2$$
$$= -\frac{ma^2V_0^2}{\pi\hbar^2}$$

Problem #5: (16.1.4)

$$\psi = (x^2-a^2)^2 \quad |x| \le a$$
$$= 0 \quad |x| > a$$

Then
$$\langle\psi|\psi\rangle = \int_{-a}^{a} dx\,(x^2-a^2)^4 = \frac{256}{315}a^9$$

$$\langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle = +\frac{\hbar^2}{2m}\cdot\frac{256}{105}a^7$$

$$\langle\psi|\frac{1}{2}m\omega^2\hat{x}^2|\psi\rangle = \frac{1}{2}m\omega^2\frac{256}{3465}a^{11}$$

Thus
$$\langle H\rangle = \frac{\frac{\hbar^2}{2m}\frac{256}{105}a^7 + \frac{1}{2}m\omega^2\frac{256}{3465}a^{11}}{\frac{256}{315}a^9}$$

$$E(a) = \frac{\hbar^2}{2m}\frac{3}{a^2} + \frac{1}{2}m\omega^2\frac{a^2}{33}$$

$$E'(a) = 0 \Rightarrow -\frac{6\hbar^2}{2ma_0^3} + \frac{m\omega^2 a_0}{33} = 0$$

This gives $a_0 = \left[\frac{\hbar^2}{M^2\omega^2}\left(\frac{33 \times 6}{2}\right)\right]^{1/4}$

$$a_0^2 = \frac{\hbar}{m\omega}(99)^{1/2}$$

$$E(a_0) = \frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{1}{(99)^{1/2}} + \frac{1}{2}\frac{m\omega^2}{33}\frac{\hbar}{m\omega}(99)^{1/2}$$

$$E(a_0) = \frac{1}{2}\hbar\omega\frac{1}{\sqrt{11}} + \frac{1}{2}\hbar\omega\sqrt{11}$$

$$E(a_0) = \frac{1}{2}\hbar\omega\sqrt{\frac{12}{11}}$$