PHY 615-2012
PROBLEM SET #6

Problem 1:

$$V(x) = \frac{\lambda}{4}\left(x^4 + ax^3 - \frac{1}{2}a^2x^2\right)$$

$$V'(x) = \frac{\lambda}{4}\left(4x^3 + 3ax^2 - a^2x\right) = 0$$

$$V''(x) = \frac{\lambda}{4}\left(12x^2 + 6ax - a^2\right) = 0$$

Extrema:

$x = 0$ $\qquad 4x^2 + 3ax - a^2 = 0$

$$(4x - a)(x + a) = 0$$

ie $\quad x = 0,\ a/4,\ -a.$

$$V''(0) = -\frac{\lambda a^2}{4} < 0 \quad \Rightarrow \text{maxima}$$

$$V''(-a) = \frac{\lambda}{4}\left(12a^2 - 6a^2 - a^2\right) = \frac{5\lambda a^2}{4} > 0$$

$$V''(a/4) = \frac{\lambda}{4}\,\frac{5a^2}{4} > 0$$

$$V(-a) = -\frac{\lambda a^4}{8} \qquad V\left(\frac{a}{4}\right) = -\frac{3\lambda a^4}{1024}$$

Thus $V(-a) < V(a/4)$.

$\Rightarrow$ Global minimum is $x = -a$

The trial wavefunction is

$$\psi(x) = \left(\frac{\beta}{\pi}\right)^{1/4} \exp\left[-\beta(x+a)^2/2\right]$$

$$\left\langle \frac{P^2}{2m} \right\rangle = -\frac{1}{2m}\left(\frac{\beta}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} dx\, e^{-\beta(x+a)^2/2}\,\frac{d^2}{dx^2}e^{-\beta(x+a)^2/2}$$

In integral shift $x \to x-a$

$$\left\langle \frac{P^2}{2m} \right\rangle = -\frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-\beta x^2/2}\,\frac{d^2}{dx^2}e^{-\beta x^2/2}$$

$$= \frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx \left(\frac{d}{dx}e^{-\beta x^2/2}\right)^2$$

$$= \frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\,\beta^2\int_{-\infty}^{\infty} dx\, x^2 e^{-\beta x^2}$$

$$= \frac{1}{2m}\sqrt{\frac{\beta}{\pi}}\,\beta^2\,\frac{1}{\beta^{3/2}}\int_{-\infty}^{\infty} dy\cdot y^2 e^{-y^2}$$

$$= \frac{\beta}{2m}\,\frac{1}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2} = \frac{\beta}{4m}$$

For the potential

$$\langle V \rangle = \frac{\lambda}{4}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-\beta(x+a)^2}\left[x^4 + ax^3 - \frac{1}{2}a^2x^2\right]$$

$$= \frac{\lambda}{4}\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-\beta x^2}\left[(x-a)^4 + a(x-a)^3 - \frac{1}{2}a^2(x-a)^2\right]$$

$$= \frac{\lambda}{4}\sqrt{\frac{\beta}{\pi}}\sqrt{\frac{\pi}{\beta}}\left[\frac{3}{4\beta^2} + \frac{5a^2}{4\beta} - \frac{1}{2}a^4\right]$$

Thus

$$\langle E \rangle = \frac{\beta}{4m} + \frac{\lambda}{4}\left(\frac{3}{4\beta^2} + \frac{5a^2}{4\beta} - \frac{1}{2}a^4\right)$$

$$= \frac{\beta}{4m} + \frac{\lambda}{16\beta^2}\left(3 + 5a^2\beta - 2\beta^2 a^4\right)$$

Thus a natural variable is $\beta a^2 = y$

$$\langle E \rangle = \frac{1}{4ma^2}\left[y + \frac{\lambda a^6 m}{4y^2}\left(3 + 5y - 2y^2\right)\right]$$

Thus eqn determining y is

Define $\alpha = \lambda m a^6$

$$1 - \frac{6\alpha}{4y^3} - \frac{5\alpha}{4y^2} = 0$$

$$4y^3 - 6\alpha - 5\alpha y = 0$$

For $\alpha = 1$ $\quad (2y-3)(2y^2+3y+2) = 0$

Solutions of $2x^2+3x+2=0$ are complex. $(3^2 - 4.4 < 0)$

$\Rightarrow$ only real solution is $y = 3/2$

For this y

$$E_0 = \frac{1}{2ma^2}\left(\frac{13}{12}\right)$$

Problem 2: (16.2.1)

$$W(E) = \int_{x'}^{x} dx'' \sqrt{2M(E - V(x''))}$$

$$\frac{\partial W(E)}{\partial E} = \int_{x'}^{x} dx'' \sqrt{\frac{M}{2}} \frac{1}{\sqrt{E - V(x'')}}$$

$$= \int_{x'}^{x} dx'' \frac{dt}{dx''} = \int_{t'}^{t} dt$$

$$= (t - t')$$

Problem 3:

(i) Consider a trial wave function

$$\psi_\alpha(x) = \left(\frac{\alpha^2}{2\pi}\right)^{1/4} \exp\left[-\frac{\alpha^2 x^2}{4}\right]$$

Then

$$\langle \psi_\alpha | \frac{\hat{p}^2}{2m} | \psi_\alpha \rangle = \frac{\hbar^2 \alpha^2}{8m} \geq 0$$

Now consider

$$\langle \psi_\alpha | V(\hat{x}) | \psi_\alpha \rangle = -\frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |V(x)| e^{-\alpha^2 x^2/2} \, dx$$

Since $V(\hat{x})$ is negative everywhere.

If $\int_{-\infty}^{\infty} dx\, V(x)$ converges, $\langle \psi_\alpha | V | \psi_\alpha \rangle$ has a finite limit as $\alpha \to 0$. In this case,

$$\langle E \rangle = \frac{\hbar^2 \alpha^2}{8m} - \alpha c$$

where c is some $+ve$ number.

$$\frac{\partial \langle E \rangle}{\partial \alpha} = 0 \Rightarrow \frac{\hbar^2 \alpha_0}{4m} = c \qquad \alpha_0 = \frac{4mc}{\hbar^2}$$

$$\Rightarrow \langle E(\alpha_0) = -\frac{2mc^2}{\hbar^2} < 0.$$

Since $E_0 \leq \langle E \rangle(\alpha)$ (E_0 is ground state energy)
This shows $E_0 < 0$. Thus there is at least one bound state

If $\int_{-\infty}^{\infty} dx\, V(x)$ is divergent, clearly $\langle \psi_\alpha | H | \psi_\alpha \rangle < 0$, so that $E_0 < 0$ as well

(ii) In 3d, consider once again the trial wave fn.

$$\psi_\alpha = \left(\frac{\alpha^2}{2\pi}\right)^{3/4} \exp\left[-\frac{\alpha^2 r^2}{4}\right]$$

Then

$$\langle \psi_\alpha | V(x) | \psi_\alpha \rangle \sim -\alpha^3$$

If the integral of V is now finite

$$\langle E \rangle = \frac{\hbar^2 \alpha^2}{8M} - \alpha^3 C$$

and now the value of $\langle E \rangle$ at extremum is zero or +ve — so we cannot use this argument to show there is a bound state

In fact generically there may or may not be bound states

Problem 4

(i) $$\left\langle \frac{p^2}{2m} \right\rangle = -\frac{\alpha}{2m} \int_{-\infty}^{\infty} dx\, e^{-\alpha|x|} \left(e^{-\alpha|x|}\right)''$$

$$\left(e^{-\alpha|x|}\right)'' = \left(-\alpha\left(\theta(x) - \theta(-x)\right) e^{-\alpha|x|}\right)'$$

$$= \left(-2\alpha\,\delta(x) + \alpha^2\right) e^{-\alpha|x|}$$

$$\left(\theta'(x) = \delta(x)\right)$$

Thus

$$\left\langle \frac{p^2}{2m} \right\rangle = -\frac{\alpha}{2m}\left[-2\alpha + \alpha^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha|x|}\right]$$

$$= -\frac{\alpha}{2m}\left[-2\alpha + 2\alpha^2 \int_{0}^{\infty} dx\, e^{-2\alpha x}\right]$$

$$= -\frac{\alpha}{2m}\left(-2\alpha + \alpha\right)^0$$

$$= \alpha^2/2m$$

The potential energy is

$$V = qE\left[\theta(x) - \theta(-x)\right] x$$

$$= qE|x|$$

Thus

$$\langle V \rangle = qE \int_{-\infty}^{\infty} dx\, |x| e^{-2\alpha|x|} = \frac{qE}{2\alpha}$$

$$\Rightarrow$$

$$\langle E \rangle = \frac{\alpha^2}{2m} + \frac{qE}{2\alpha}$$

$$\frac{\partial \langle E \rangle}{\partial \alpha} = \frac{\alpha}{m} - \frac{qE}{2\alpha^2} \Rightarrow \alpha = \left(\frac{qEm}{2}\right)^{1/3}$$

$$E_0 = \frac{3}{2^{5/3}} \left[\frac{(qE)^2}{m}\right]^{1/3}$$

(ii) Bohr Sommerfeld

Turning points are $x_0 = \pm \frac{e}{qE}$

where e is the energy

Thus

$$\int_{-e/qE}^{e/qE} dx \sqrt{2m(e - qE|x|)} = \left(n+\tfrac{1}{2}\right)\pi$$

$$\int_{-e/qE}^{0} dx \sqrt{2m(e+qEx)} + \int_{0}^{e/qE} dx \sqrt{2m(e-qEx)} = \left(n+\tfrac{1}{2}\right)\pi$$

$$\frac{2}{3}\frac{e^{3/2}\sqrt{2m}}{qE} + \frac{2}{3}\frac{e^{3/2}\sqrt{2m}}{qE} = \left(n+\tfrac{1}{2}\right)\pi$$

Thus gives for $n = 0$ (ground state)

$$e^{3/2} = \frac{3qE\pi}{8\sqrt{2m}} = \frac{3\pi}{8\sqrt{2}}\left(\frac{qE}{\sqrt{m}}\right)$$

$$e = \left(\frac{3\pi}{8\sqrt{2}}\right)^{2/3}\left(\frac{(qE)^2}{m}\right)^{1/3}$$

Problem #5

For energy E the turning points are $\pm x_0$ where

$$E = V(x_0) = \frac{1}{2} m\omega^2 x_0^2$$

$$\Rightarrow \quad x_0 = \sqrt{\frac{2E}{m\omega^2}}$$

(1) In the allowed region the WKB wave function is (for energy E)

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{-x_0}^{x} p(x')\,dx' - \frac{\pi}{4}\right)$$

where $\quad p(x) = \sqrt{2m\left(E - \frac{1}{2}m\omega^2 x^2\right)}$

$$= \sqrt{2m\,\frac{1}{2}m\omega^2\left(x_0^2 - x^2\right)}$$

$$= m\omega\sqrt{x_0^2 - x^2}$$

Quantization condition is

$$\int_{-x_0}^{x_0} dx'\,p(x') = \left(n + \frac{1}{2}\right)\hbar$$

$$\Rightarrow \quad m\omega x_0^2 \int_{-1}^{1} dx\,\sqrt{1-x^2} = \left(n + \frac{1}{2}\right)\hbar\pi$$

$$\frac{2E}{\omega}\cdot\frac{\pi}{2} = \left(n + \frac{1}{2}\right)\hbar$$

$$\Rightarrow \quad \boxed{E = \left(n + \frac{1}{2}\right)\hbar\omega}$$