Physics 711 January 15, 2010

Problem Set 8

Problem 1: Exercise 7.3.1

Plug the power series expansion $\psi = \sum_{n=0}^{\infty} c_n y^n$ into the equation $\psi'' + (2\varepsilon - y^2)\psi = 0$ to get

$$\sum_{n=0}^{\infty} c_n \left[n(n-1)y^{n-2} + (2\varepsilon - y^2)y^n \right] = 0.$$

Shift $n \to n+2$ in the first term, and $n \to n-2$ in the third term to get

$$\sum_{n=0}^{\infty} y^n \left[(n+2)(n+1)c_{n+2} + 2\varepsilon c_n - c_{n-2} \right] = 0$$

with the convention that

$$c_{-2} = c_{-1} = 0.$$

This implies

$$c_{n+2} = \frac{c_{n-2}}{(n+1)(n+2)} - \frac{2\varepsilon c_n}{(n+1)(n+2)}$$

for all $n \geq 0$.

Problem 3: Exercise 7.3.7

In the momentum basis $|\psi\rangle \to \psi(p)$, $P \to p$, and $X \to i\hbar(d/dp)$, so the energy eigenvalue equation

$$\left(\frac{1}{2m} P^2 + \frac{m\omega^2}{2} X^2 \right) |E\rangle = E|E\rangle$$

becomes

$$\frac{1}{2m} p^2 \psi(p) - \frac{m\omega^2}{2} \hbar^2 \psi''(p) = E\psi(p).$$

Compare this to the position-basis equation

$$\frac{m\omega^2}{2} x^2 \psi(x) - \frac{1}{2m} \hbar^2 \psi''(x) = E\psi(x).$$

These are the same equations with the substitutions $x \leftrightarrow p$ and $m \leftrightarrow 1/(m\omega^2)$.

Problem 2: Exercise 7.3.5

$$\begin{aligned}
\langle n|X|n\rangle &= \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\cdot x\cdot\psi_n(x) \;=\; \int_{-\infty}^{\infty} dx\, x\cdot\psi_n^2(x) \quad \text{since } \psi_n(x) \text{ is real} \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } x \text{ is odd and } \psi_n^2(x) \text{ is even.}
\end{aligned}$$

$$\begin{aligned}
\langle n|P|n\rangle &= \int_{-\infty}^{\infty} dx\, \psi_n^*(x)(-i\hbar)\frac{d}{dx}\psi_n(x) \;=\; (-i\hbar)\int_{-\infty}^{\infty} dx\, \psi_n\psi_n' \;=\; -\frac{i\hbar}{2}\int_{-\infty}^{\infty} dx\, (\psi_n^2)' \\
&= -\frac{i\hbar}{2}\,\psi_n^2\big|_{-\infty}^{\infty} \;=\; 0 \qquad\qquad\qquad\qquad \text{since } \psi_n \to 0 \text{ as } |x| \to \infty.
\end{aligned}$$

$$\begin{aligned}
\langle 1|X^2|1\rangle &= \int_{-\infty}^{\infty} dx\, \psi_1^* x^2 \psi_1 \;=\; \left(\frac{m\omega}{4\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx\, x^2\left[2x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]^2 e^{-m\omega x^2/\hbar} \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, x^4 e^{-m\omega x^2/\hbar} \;=\; \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\frac{3\sqrt{\pi}}{4}\left(\frac{m\omega}{\hbar}\right)^{-5/2} \\
&= \frac{3\hbar}{2m\omega}.
\end{aligned}$$

$$\begin{aligned}
\langle 1|P^2|1\rangle &= \int_{-\infty}^{\infty} dx\, \psi_1^*(-i\hbar)^2\psi_1'' \;=\; -\hbar^2\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, xe^{-m\omega x^2/2\hbar}\left(xe^{-m\omega x^2/2\hbar}\right)'' \\
&= -\hbar^2\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, x^2\left(\frac{m\omega}{\hbar}\right)\left[\left(\frac{m\omega}{\hbar}\right)x^2 - 3\right]e^{-m\omega x^2/\hbar} \\
&= -\hbar^2\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{5/2}\cdot(-1)\frac{3\sqrt{\pi}}{4}\left(\frac{m\omega}{\hbar}\right)^{-3/2} \;=\; \frac{3m\omega\hbar}{2}.
\end{aligned}$$

$$\Delta X^2 \;=\; \langle 0|X^2|0\rangle \;=\; \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} dx\, x^2 e^{-m\omega x^2/\hbar} \;=\; \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{\sqrt{\pi}}{2}\left(\frac{m\omega}{\hbar}\right)^{-3/2} \;=\; \frac{\hbar}{2m\omega}.$$

$$\begin{aligned}
\Delta P^2 \;=\; \langle 0|P^2|0\rangle &= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-m\omega x^2/2\hbar}\left(e^{-m\omega x^2/2\hbar}\right)'' \\
&= -\frac{\hbar^2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} dx\, \left(\frac{m\omega}{\hbar}x^2 - 1\right)e^{-m\omega x^2/\hbar} \\
&= -\frac{\hbar^2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\cdot(-1)\frac{\sqrt{\pi}}{2}\left(\frac{m\omega}{\hbar}\right)^{-1/2} \;=\; \frac{m\omega\hbar}{2}.
\end{aligned}$$

$$\therefore \Delta X\,\Delta P \;=\; \sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{m\omega\hbar}{2}} \;=\; \frac{\hbar}{2}.$$