Physics 710-712 April 9, 2010

Problem Set 17

Problem 14.3.2:

(1) $\vec{\sigma}\cdot\widehat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$ with

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

so

$$\vec{\sigma}\cdot\widehat{n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

So,

$$\vec{\sigma}\cdot\widehat{n}|\widehat{n}+\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} = |\widehat{n}+\rangle,$$

where I used the trig identities:

$$\begin{aligned} \cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2) &= \cos(\theta - \tfrac{1}{2}\theta) = \cos(\theta/2), \\ \sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2) &= \sin(\theta - \tfrac{1}{2}\theta) = \sin(\theta/2). \end{aligned}$$

Similarly,

$$\vec{\sigma}\cdot\widehat{n}|\widehat{n}-\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix} = \begin{pmatrix} \sin(\theta/2)e^{-i\phi/2} \\ -\cos(\theta/2)e^{i\phi/2} \end{pmatrix} = -|\widehat{n}-\rangle.$$

(2) $\langle\widehat{n}\pm|\vec{S}|\widehat{n}\pm\rangle = \langle\widehat{n}\pm|\frac{\hbar}{2}\vec{\sigma}|\widehat{n}\pm\rangle = \langle\widehat{n}\pm|\frac{\hbar}{2}(\sigma_x\widehat{\imath} + \sigma_y\widehat{\jmath} + \sigma_z\widehat{k})|\widehat{n}\pm\rangle$, so

$$\begin{aligned} \langle\widehat{n}+|\vec{S}|\widehat{n}+\rangle &= \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} \widehat{k} & \widehat{\imath} - i\widehat{\jmath} \\ \widehat{\imath} + i\widehat{\jmath} & -\widehat{k} \end{pmatrix}\begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix} \\ &= \frac{\hbar}{2}\Big[\widehat{\imath}\big(\cos(\theta/2)\sin(\theta/2)(e^{i\phi} + e^{-i\phi})\big) + \widehat{\jmath}\big(-i\cos(\theta/2)\sin(\theta/2)(e^{i\phi} - e^{-i\phi})\big) \\ &\qquad + \widehat{k}\left(\cos^2(\theta/2) - \sin^2(\theta/2)\right)\Big] = \frac{\hbar}{2}\left[\widehat{\imath}\sin\theta\cos\phi + \widehat{\jmath}\sin\theta\sin\phi + \widehat{k}\cos\theta\right], \end{aligned}$$

and similarly for $\langle\widehat{n}-|\vec{S}|\widehat{n}-\rangle$.

Problem 14.3.3: From

$$\begin{aligned} \sigma_i\sigma_j &= -\sigma_j\sigma_i \qquad (i \neq j) && (1) \\ \sigma_x\sigma_y &= i\sigma_z \qquad \text{(and cyclic)}, && (2) \end{aligned}$$

we have

$$\begin{aligned}\mathrm{Tr}(\sigma_z) &= \mathrm{Tr}(-i\sigma_x\sigma_y) = -i\mathrm{Tr}(\sigma_x\sigma_y)\\ &= -i\mathrm{Tr}(-\sigma_y\sigma_x) = i\mathrm{Tr}(\sigma_y\sigma_x)\\ &= i\mathrm{Tr}(\sigma_x\sigma_y)\\ &= i\mathrm{Tr}(i\sigma_z) = -\mathrm{Tr}(\sigma_z),\end{aligned}$$

and therefore $\mathrm{Tr}(\sigma_z)=0$, and similarly for σ_x and σ_y. In the above we used (1) to get the second line, (2) to get the first and fourth lines, and cyclicity of the trace to get the third line.

Problem 14.3.4:

(1) $\sigma_i\sigma_j = \frac{1}{2}[\sigma_i,\sigma_j]_+ + \frac{1}{2}[\sigma_i,\sigma_j] = \frac{1}{2}2\delta_{ij}I + \frac{1}{2}2i\sum_k \epsilon_{ijk}\sigma_k = \delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k$, therefore

$$\begin{aligned}(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) &= \sum_{ij} A_iB_j\sigma_i\sigma_j = \sum_{ij} A_iB_j\left(\delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k\right)\\ &= \left(\sum_i A_iB_i\right)I + i\sum_k\left(\sum_{ij}\epsilon_{ijk}A_iB_j\right)\sigma_k\\ &= (\vec{A}\cdot\vec{B})I + i\sum_k(\vec{A}\times\vec{B})_k\sigma_k = (\vec{A}\cdot\vec{B})I + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}.\end{aligned}$$

(2) Eqns. (14.3.42) and (14.3.43) imply that $M = \frac{1}{2}\sum_\alpha \mathrm{Tr}(M\sigma_\alpha)\sigma_\alpha$ where $\sigma_\alpha = (\sigma_x,\sigma_y,\sigma_z,I)$. For $M=(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma})$, we have $\mathrm{Tr}(M\sigma_0) = \mathrm{Tr}(MI) = \mathrm{Tr}(M) = \sum_{ij} A_iB_j\mathrm{tr}\,(\sigma_i\sigma_j) = 2\sum_{ij} A_iB_j\delta_{ij} = 2\vec{A}\cdot\vec{B}$, and

$$\begin{aligned}\mathrm{Tr}(M\sigma_k) &= \mathrm{Tr}((\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma})\sigma_k) = \sum_{ij} A_iB_j\mathrm{Tr}(\sigma_i\sigma_j\sigma_k)\\ &= \sum_{ij}\tfrac{1}{2}A_iB_j\mathrm{Tr}(\sigma_k\sigma_i\sigma_j + \sigma_i\sigma_j\sigma_k) \qquad \text{(using cyclicity of the trace)}\\ &= \sum_{ij}\tfrac{1}{2}A_iB_j\mathrm{Tr}([\sigma_k,\sigma_i]\sigma_j + \sigma_i[\sigma_k,\sigma_j]_+)\\ &= \sum_{ij}\tfrac{1}{2}A_iB_j\mathrm{Tr}(2i\textstyle\sum_\ell \epsilon_{ki\ell}\sigma_\ell\sigma_j + 2\sigma_i\delta_{kj})\\ &= i\sum_{ij\ell} A_iB_j\epsilon_{ki\ell}\mathrm{Tr}(\sigma_\ell\sigma_j) \qquad \text{(since } \mathrm{Tr}(\sigma_i)=0)\\ &= i\sum_{ij\ell} A_iB_j\epsilon_{ki\ell}2\delta_{\ell j} = 2i\sum_{ij\ell} A_iB_j\epsilon_{ijk} \qquad \text{(using } \epsilon_{ijk}=\epsilon_{kij})\\ &= 2i(\vec{A}\times\vec{B})_k.\end{aligned}$$

So

$$\begin{aligned}(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) &= \tfrac{1}{2}\sum_\alpha \mathrm{Tr}(M\sigma_\alpha)\sigma_\alpha \;=\; \tfrac{1}{2}\mathrm{Tr}(M\sigma_0)I + \tfrac{1}{2}\sum_k \mathrm{Tr}(M\sigma_k)\sigma_k \\ &= \tfrac{1}{2}(2\vec{A}\cdot\vec{B})I + \tfrac{1}{2}\sum_k 2i(\vec{A}\times\vec{B})_k\sigma_k \;=\; (\vec{A}\cdot\vec{B})I + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}.\end{aligned}$$

Problem 14.3.5:

$$\begin{aligned}M &= \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\frac{\alpha+\delta}{2}+\frac{\alpha-\delta}{2} & \frac{\beta+\gamma}{2}-i\left(i\frac{\beta-\gamma}{2}\right)\\ \frac{\beta+\gamma}{2}+i\left(i\frac{\beta-\gamma}{2}\right) & \frac{\alpha+\delta}{2}-\frac{\alpha-\delta}{2}\end{pmatrix} \\ &= \left(\frac{\alpha+\delta}{2}\right)I + \left(\frac{\beta+\gamma}{2}\right)\sigma_x + i\left(\frac{\beta-\gamma}{2}\right)\sigma_y + \left(\frac{\alpha-\delta}{2}\right)\sigma_z.\end{aligned}$$