Physics 710-712 April 23, 2010

Problem Set 19

Problem 15.2.2:

(1): $\frac{1}{2}\otimes 1 = \frac{3}{2}\oplus\frac{1}{2}$, so $j_1=\frac{1}{2}$ and $j_2=1$ implies $j\in\{\frac{3}{2},\frac{1}{2}\}$. Since there is a unique state with $m=\frac{3}{2}$ we must have

$$(|jm\rangle \equiv)\ \ \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \left|\tfrac{1}{2}\tfrac{1}{2},11\right\rangle\ \ (\equiv |j_1m_1,j_2m_2\rangle). \tag{1}$$

Then, $J_-\left|\frac{3}{2}\frac{3}{2}\right\rangle = \hbar\sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{3}{2}+1)}\left|\frac{3}{2}\frac{1}{2}\right\rangle = \hbar\sqrt{3}\left|\frac{3}{2}\frac{1}{2}\right\rangle$, so

$$\begin{aligned}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \frac{1}{\hbar\sqrt{3}}J_-\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \frac{1}{\hbar\sqrt{3}}(J_{1-}+J_{2-})\left|\tfrac{1}{2}\tfrac{1}{2},11\right\rangle\\ &= \frac{1}{\sqrt{3}}\left(\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\left|\tfrac{1}{2}-\tfrac{1}{2},11\right\rangle + \sqrt{(1+1)(1-1+1)}\left|\tfrac{1}{2}\tfrac{1}{2},10\right\rangle\right)\\ &= \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}-\tfrac{1}{2},11\right\rangle + \frac{\sqrt{2}}{\sqrt{3}}\left|\tfrac{1}{2}\tfrac{1}{2},10\right\rangle.\end{aligned} \tag{2}$$

Then $\left|\frac{1}{2}\frac{1}{2}\right\rangle$ is the state orthogonal to $\left|\frac{3}{2}\frac{1}{2}\right\rangle$, so

$$\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}\tfrac{1}{2},10\right\rangle - \frac{\sqrt{2}}{\sqrt{3}}\left|\tfrac{1}{2}-\tfrac{1}{2},11\right\rangle, \tag{3}$$

where the sign is chosen to follow rule (4) on p. 413 of the text. From eqns. (1), (2), and (3) we read off the Clebsch-Gordon coefficients

$$\begin{aligned}\langle\tfrac{1}{2}\tfrac{1}{2},11|\tfrac{3}{2}\tfrac{3}{2}\rangle &= 1,\\ \langle\tfrac{1}{2}-\tfrac{1}{2},11|\tfrac{3}{2}\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}, \qquad \langle\tfrac{1}{2}-\tfrac{1}{2},11|\tfrac{1}{2}\tfrac{1}{2}\rangle = -\sqrt{\tfrac{2}{3}},\\ \langle\tfrac{1}{2}\tfrac{1}{2},10|\tfrac{3}{2}\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}, \qquad \langle\tfrac{1}{2}\tfrac{1}{2},10|\tfrac{1}{2}\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}.\end{aligned}$$

Then rule (5) on p. 413 gives the rest:

$$\begin{aligned}\langle\tfrac{1}{2}-\tfrac{1}{2},1-1|\tfrac{3}{2}-\tfrac{3}{2}\rangle &= 1,\\ \langle\tfrac{1}{2}\tfrac{1}{2},1-1|\tfrac{3}{2}-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}, \qquad \langle\tfrac{1}{2}\tfrac{1}{2},1-1|\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}},\\ \langle\tfrac{1}{2}-\tfrac{1}{2},10|\tfrac{3}{2}-\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}, \qquad \langle\tfrac{1}{2}-\tfrac{1}{2},10|\tfrac{1}{2}-\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{3}}.\end{aligned}$$

All others are zero.

(2): $1\otimes 1 = 2\oplus 1\oplus 0$, so $j_1=j_2=1$ implies $j\in\{2,1,0\}$. Since there is a unique state with $m=2$ we have

$$|22\rangle = |11,11\rangle. \tag{4}$$

Since $J_-|22\rangle = 2\hbar|21\rangle$,

$$|21\rangle = \tfrac{1}{2\hbar}J_-|22\rangle = \tfrac{1}{2\hbar}(J_{1-}+J_{2-})|11,11\rangle = \tfrac{1}{\sqrt{2}}|10,11\rangle + \tfrac{1}{\sqrt{2}}|11,10\rangle. \tag{5}$$

Since $J_-|21\rangle = \sqrt{6}\hbar|20\rangle$,

$$\begin{aligned}
|20\rangle &= \tfrac{1}{\sqrt{6}\hbar}(J_{1-}+J_{2-})(|10,11\rangle + |11,10\rangle)\tfrac{1}{\sqrt{2}} \\
&= \tfrac{1}{2\sqrt{3}\hbar}(J_{1-}|10,11\rangle + J_{2-}|10,11\rangle + J_{1-}|11,10\rangle + J_{2-}|11,10\rangle) \\
&= \tfrac{1}{2\sqrt{3}}(\sqrt{2}|1\,{-1},11\rangle + \sqrt{2}|10,10\rangle + \sqrt{2}|10,10\rangle + \sqrt{2}|11,1\,{-1}\rangle) \\
&= \tfrac{1}{\sqrt{6}}|1\,{-1},11\rangle + \sqrt{\tfrac{2}{3}}|10,10\rangle + \tfrac{1}{\sqrt{6}}|11,1\,{-1}\rangle.
\end{aligned} \tag{6}$$

$|11\rangle$ is the state orthogonal to $|21\rangle$, so

$$|11\rangle = \tfrac{1}{\sqrt{2}}|11,10\rangle - \tfrac{1}{\sqrt{2}}|10,11\rangle, \tag{7}$$

and $J_-|11\rangle = \sqrt{2}\hbar|10\rangle$ so

$$\begin{aligned}
|10\rangle &= \tfrac{1}{\sqrt{2}\hbar}(J_{1-}+J_{2-})(|11,10\rangle - |10,11\rangle)\tfrac{1}{\sqrt{2}} \\
&= \tfrac{1}{2\hbar}(J_{1-}|11,10\rangle + J_{2-}|11,10\rangle - J_{1-}|10,11\rangle - J_{2-}|10,11\rangle) \\
&= \tfrac{1}{2}(\sqrt{2}|10,10\rangle + \sqrt{2}|11,1\,{-1}\rangle - \sqrt{2}|1\,{-1},11\rangle - \sqrt{2}|10,10\rangle) \\
&= \tfrac{1}{\sqrt{2}}|11,1\,{-1}\rangle - \tfrac{1}{\sqrt{2}}|1\,{-1},11\rangle.
\end{aligned} \tag{8}$$

Finally, $|00\rangle$ is perpendicular to both $|20\rangle$ and $|10\rangle$. Say $|00\rangle = a|11,1\,{-1}\rangle + b|10,10\rangle + c|1\,{-1},11\rangle$. Then

$$\begin{aligned}
0 &= \langle 00|20\rangle = \tfrac{1}{\sqrt{6}}(a+2b+c), \\
0 &= \langle 00|10\rangle = \tfrac{1}{\sqrt{2}}(a-c),
\end{aligned}$$

which together imply $a = -b = c$. Then normalizing $|00\rangle$ implies $a = 1/\sqrt{3}$, so

$$|00\rangle = \tfrac{1}{\sqrt{3}}|11,1\,{-1}\rangle - \tfrac{1}{\sqrt{3}}|10,10\rangle + \tfrac{1}{\sqrt{3}}|1\,{-1},11\rangle. \tag{9}$$

From eqns. (4–9) by taking inner products and by using rule (5) on p. 413 we get all the Clebsch-Gordon coefficients.

Problem 15.2.3:

$$\begin{aligned}
\tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2} &= \tfrac{1}{2}\otimes(\tfrac{1}{2}\otimes\tfrac{1}{2}) = \tfrac{1}{2}\otimes(1\oplus 0) = (\tfrac{1}{2}\otimes 1)\oplus(\tfrac{1}{2}\otimes 0) = (\tfrac{3}{2}\oplus\tfrac{1}{2})\oplus\tfrac{1}{2} \\
&= \tfrac{3}{2}\oplus\tfrac{1}{2}\oplus\tfrac{1}{2}.
\end{aligned}$$