Physics 710-712 May 14, 2010

Problem Set 21

Problem 17.2.1: With $H^0 = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2$ and $H^1 = \lambda X^4$, and using that $X = \sqrt{\hbar/(2m\omega)}(a + a^\dagger)$, we have:

(1):

$$\begin{aligned}
E_n^1 &= \langle n|H^1|n\rangle = \frac{\lambda\hbar^2}{4m^2\omega^2}\langle n|(a+a^\dagger)^4|n\rangle \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\langle n|(a^2a^{\dagger 2} + aa^\dagger aa^\dagger + a^\dagger a^2 a^\dagger + aa^{\dagger 2}a + a^\dagger aa^\dagger a + a^{\dagger 2}a^2)|n\rangle \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\langle n|(6a^2a^{\dagger 2} - 12aa^\dagger + 3)|n\rangle \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\left\{6\langle n+2|(n+1)(n+2)|n+2\rangle - 12\langle n+1|(n+1)|n+1\rangle + 3\langle n|n\rangle\right\} \\
&= \frac{\lambda\hbar^2}{4m^2\omega^2}\left\{6(n+1)(n+2) - 12(n+1) + 3\right\} = \frac{3\hbar^2\lambda}{4m^2\omega^2}(2n^2+2n+1),
\end{aligned}$$

where in the first line we dropped the zero superscripts from the unperturbed eigenstates; in the second line we kept only terms with equal numbers of a's and $a^\dagger$'s; in the third line we used $[a, a^\dagger] = 1$; and in the fourth line we used $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(2): For any finite value of λ, as n gets large

$$\frac{E_n^1}{\Delta E} \sim \frac{\hbar^2\lambda n^2/(m^2\omega^2)}{\hbar\omega} \sim n^2\frac{\hbar\lambda}{m^2\omega^3} \gg 1.$$

Physically, at large x, no matter how small λ is,

$$V(x) = \frac{m\omega^2}{2}x^2 + \lambda x^4 \sim \lambda x^4$$

for $\lambda \neq 0$.

Problem 17.2.2: $H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = H^0 + H^1$ with $H^0 = -\gamma B_0 S_z$ and $H^1 = -\gamma B S_x$. The H^0 eigenvalues are $E_\pm^0 = \mp\gamma B_0\hbar/2$ with eigenstates $|\pm\rangle^0$, the S_z eigenstates. Then

$$E_\pm^1 = \langle\pm|^0 H^1|\pm\rangle^0 = -\gamma B\langle\pm|^0 S_x|\pm\rangle^0 = -\frac{\gamma B}{2}\langle\pm|^0(S_+ + S_-)|\pm\rangle^0 = 0,$$

and

$$\begin{aligned}
E_\pm^2 &= \sum_m{}' \frac{|\langle\pm|^0 H^1|m\rangle^0|^2}{E_\pm^0 - E_m^0} = \frac{|\langle\pm|^0 H^1|\mp\rangle^0|^2}{E_\pm^0 - E_\mp^0} = \frac{\gamma^2B^2}{4}\frac{|\langle\pm|^0(S_+ + S_-)|\mp\rangle^0|^2}{\mp\gamma B_0\hbar} \\
&= \mp\frac{\gamma B^2}{4B_0\hbar}|\langle\pm|^0 S_\pm|\mp\rangle^0|^2 = \mp\frac{\gamma B^2}{4B_0\hbar}|\langle\pm|^0\hbar\sqrt{(\tfrac{1}{2}\mp(\mp\tfrac{1}{2}))(\tfrac{3}{2}\pm(\mp\tfrac{1}{2}))}|\mp\rangle^0|^2 = \mp\frac{\gamma\hbar B^2}{4B_0},
\end{aligned}$$

and

$$|\pm\rangle' = \sum_m{}' \frac{|m\rangle^0\langle m|^0 H^1|\pm\rangle^0}{E^0_\pm - E^0_m} = \frac{|\mp\rangle^0\langle\mp|^0 H^1|\pm\rangle^0}{E^0_\pm - E^0_\mp} = -\frac{\gamma B}{2}\frac{|\mp\rangle^0\langle\mp|^0 S_\mp|\pm\rangle^0}{\mp\gamma B_0\hbar} = \pm\frac{B}{2B_0}|\mp\rangle^0,$$

where we used that $\langle\mp|^0 S_\mp|\pm\rangle^0 = \hbar$, as computed in the previous equation. Therefore

$$E_\pm = E^0_\pm + E^1_\pm + E^2_\pm + \mathcal{O}(B^3) = \mp\frac{\gamma B_0\hbar}{2} \mp \frac{\gamma B^2\hbar}{4B_0} = \mp\frac{\gamma B_0\hbar}{2}\left(1 + \frac{B^2}{2B_0^2}\right),$$

and

$$|\pm\rangle = |\pm\rangle^0 + |\pm\rangle^1 + \mathcal{O}(B^2) = |\pm\rangle^0 \pm \frac{B}{2B_0}|\mp\rangle^0.$$

Now compare to the exact answer. $H = -\gamma(B_0 S_z + B S_x) = -\gamma\sqrt{B_0^2 + B^2}\,\widehat{n}\cdot\vec{S}$, where $\widehat{n} = (B_0\widehat{k} + B\widehat{i})/\sqrt{B_0^2 + B^2}$. The eigenvalues of $\widehat{n}\cdot\vec{S}$ are $\pm\hbar/2$ with eigenvectors $|\pm\rangle$ with

$$\begin{aligned} |+\rangle &= \cos\tfrac{\theta}{2}\, e^{-i\phi/2}|+\rangle^0 + \sin\tfrac{\theta}{2}\, e^{i\phi/2}|-\rangle^0, \\ |-\rangle &= -\sin\tfrac{\theta}{2}\, e^{-i\phi/2}|+\rangle^0 + \cos\tfrac{\theta}{2}\, e^{i\phi/2}|-\rangle^0, \end{aligned} \tag{1}$$

where $|\pm\rangle^0$ are the S_z eigenvectors. Eqn. (1) can be rewritten as

$$|\pm\rangle = \cos\tfrac{\theta}{2}\, e^{\mp i\phi/2}|\pm\rangle^0 \pm \sin\tfrac{\theta}{2}\, e^{\pm i\phi/2}|\mp\rangle^0,$$

and

$$\begin{aligned} \cos\tfrac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+n_z}{2}} = \sqrt{\frac{\sqrt{B_0^2+B^2}+B_0}{2\sqrt{B_0^2+B^2}}} = 1 + \mathcal{O}(B^2), \\ \sin\tfrac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-n_z}{2}} = \sqrt{\frac{\sqrt{B_0^2+B^2}-B_0}{2\sqrt{B_0^2+B^2}}} = \frac{B}{2B_0} + \mathcal{O}(B^3), \end{aligned}$$

and $\phi = 0$, since $\widehat{n}$ is in the x-z plane. This implies

$$|\pm\rangle = |\pm\rangle^0 \pm \frac{B}{2B_0}|\mp\rangle^0 + \mathcal{O}(B^2),$$

in agreement with the perturbation theory result. Similarly, the exact eigenvalues are

$$E_\pm = \mp\frac{\gamma\hbar}{2}\sqrt{B_0^2 + B^2} = \mp\frac{\gamma\hbar B_0}{2}\left(1 + \frac{B^2}{B_0^2}\right)^{1/2} = \mp\frac{\gamma\hbar B_0}{2}\left(1 + \frac{B^2}{2B_0^2} + \mathcal{O}(B^4)\right),$$

again in agreement with the perturbation theory result.

Problem 17.3.2: $H = AS_z^2 + B(S_x^2 - S_y^2)$ on the spin-1 Hilbert space. In the S_z basis, $|m\rangle$ $(m = \pm 1, 0)$, we have

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(see p. 328 of the text). This implies

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}. \tag{2}$$

Clearly

$$|m\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is the eigenvector with $E = 0$. So we only need to look at the $|m\rangle = |\pm 1\rangle$ subspace where $H = H^0 + H^1$ with

$$H^0 = \hbar^2 A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H^1 = \hbar^2 B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{on the } |m\rangle = \{|\pm 1\rangle\} \text{ subspace.}$$

H^0 is degenerate: $H^0|m\rangle = \hbar^2 A|m\rangle$ for $m = \pm 1$. The basis stable under H^1 is the one which diagonalizes H^1. Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

the eigenvectors of H^1 are

$$\left|\overline{1}\right\rangle \equiv \frac{1}{\sqrt{2}} \left(|1\rangle + |-1\rangle\right) \qquad \text{and} \qquad \left|\overline{-1}\right\rangle \equiv \frac{1}{\sqrt{2}} \left(|1\rangle - |-1\rangle\right).$$

To order $\mathcal{O}(B)$, the energy shifts of $\left|\overline{1}\right\rangle$ and $\left|\overline{-1}\right\rangle$ are

$$\begin{aligned} E^1_{\overline{1}} &= \left\langle \overline{1}\right| H^1 \left|\overline{1}\right\rangle = \hbar^2 B \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \hbar^2 B, \\ E^1_{\overline{-1}} &= \left\langle \overline{-1}\right| H^1 \left|\overline{-1}\right\rangle = \hbar^2 B \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} = -\hbar^2 B. \end{aligned}$$

So, the eigenvalues, to $\mathcal{O}(B)$, are

$$E_{\overline{1}} = \hbar^2(A + B), \qquad E_{\overline{-1}} = \hbar^2(A - B), \qquad E_0 = 0.$$

Compare this to the exact eigenvalues of (2) given by $0 = \det(H - \lambda) = \lambda(\hbar^2(B - A) + \lambda)(\hbar^2(B + A) - \lambda)$, which implies $\lambda = \{0, \hbar^2(A - B), \hbar^2(A + B)\}$. So, the $\mathcal{O}(B)$ results are exact.