(1) [12.4.2]

Using simpler notation, we're asked to show:

$$\hat{R}_y(-\varepsilon_y)\hat{R}_x(-\varepsilon_x)\hat{R}_y(\varepsilon_y)\hat{R}_x(\varepsilon_x) = \hat{R}_z(-\varepsilon_x\varepsilon_y)$$

$$\hat{R}_x(\varepsilon_x) \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2) \quad ; \quad \hat{R}_y(\varepsilon_y) \Leftrightarrow \begin{pmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{pmatrix}$$

$$\hat{R}_z(\varepsilon_z) \Leftrightarrow \begin{pmatrix} 1 & -\varepsilon_z & 0 \\ \varepsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}_y(\varepsilon_y)\hat{R}_x(\varepsilon_x) \overset{1^{st}\text{ order}}{\Leftrightarrow} \begin{pmatrix} 1 & \varepsilon_x\varepsilon_y & \varepsilon_y \\ 0 & 1 & -\varepsilon_x \\ -\varepsilon_y & \varepsilon_x & 1 \end{pmatrix} \equiv \hat{B}$$

$$\hat{R}_y(-\varepsilon_y)\hat{R}_x(-\varepsilon_x) \Leftrightarrow \begin{pmatrix} 1 & \varepsilon_x\varepsilon_y & -\varepsilon_y \\ 0 & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{pmatrix} \equiv \hat{A}$$

$$\hat{A}\hat{B} = \begin{pmatrix} (1+\varepsilon_y^2) & \varepsilon_x\varepsilon_y & -\varepsilon_x^2\varepsilon_y \\ -\varepsilon_x\varepsilon_y & (1+\varepsilon_x^2) & 0 \\ 0 & \varepsilon_x\varepsilon_y^2 & 1+\varepsilon_x^2+\varepsilon_y^2 \end{pmatrix}$$

Zero terms of order ε_x^2, ε_y^2:

$$= \begin{pmatrix} 1 & \varepsilon_x\varepsilon_y & 0 \\ -\varepsilon_x\varepsilon_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

QED

(2) [12.5.3]

1) $\langle \hat{J}_x \rangle = \frac{1}{2}\langle \hat{J}_+ + \hat{J}_- \rangle = \frac{1}{2}\left[\langle \hat{J}_+ \rangle + \langle \hat{J}_- \rangle\right]$

$= \frac{1}{2}\left[\langle jm|\hat{J}_+|jm\rangle + \langle jm|\hat{J}_-|jm\rangle\right]$

$= \frac{1}{2}\left[C^+_{jm}\langle jm|j\,m+1\rangle + C^-_{jm}\langle jm|j\,m-1\rangle\right]$ (see Eq. 12.5.20 for values of $C^\pm_{jm}$)

→ Note that even if $m = \pm j$ (and either $|j\,m+1\rangle$ or $|j\,m-1\rangle$ doesn't exist) then it still vanishes since $\hat{J}_+$ or $\hat{J}_-$ annihilates the state — i.e. $C^\pm_{jm}$ is zero.

Similarly, $\langle \hat{J}_y \rangle = \frac{1}{2i}\left[\langle \hat{J}_+ \rangle - \langle \hat{J}_- \rangle\right] = 0$.

2) $\langle \hat{J}_x^2 \rangle = \langle \hat{J}^2 - \hat{J}_y^2 - \hat{J}_z^2 \rangle$ But $\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle$ by symmetry,

so $2\langle \hat{J}_x^2 \rangle = \langle \hat{J}^2 - \hat{J}_z^2 \rangle$

$= \hbar^2 j(j+1) - \hbar^2 m^2$

$\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]$.

3) $\Delta J_x = \Delta J_y = \sqrt{\frac{\hbar^2}{2}\left[j(j+1)-m^2\right]} = \frac{\hbar}{\sqrt{2}}\sqrt{j^2+j-m^2}$

$(\Delta J_x)^2(\Delta J_y)^2 = \frac{\hbar^4}{4}\left[j(j+1)-m^2\right]^2 \overset{?}{\geq} |\langle \hat{J}_x \hat{J}_y \rangle|^2$

$$\langle \hat{J}_x \hat{J}_y \rangle = \frac{-i}{4} \left\langle (\hat{J}_+ + \hat{J}_-)(\hat{J}_+ - \hat{J}_-) \right\rangle$$

$$= \frac{-i}{4} \left\langle \hat{J}_+^2 + \hat{J}_-^2 + [\hat{J}_-, \hat{J}_+] \right\rangle$$

$$= \frac{-i}{4} \left\langle \hat{J}_+^2 + \hat{J}_-^2 - 2\hbar \hat{J}_z \right\rangle$$

$$= \frac{-i}{4} \left[C^+_{jm+1} C^+_{jm} \langle jm | j\,m+2 \rangle + C^-_{jm-1} C^-_{jm} \langle jm | j\,m-2 \rangle - 2\hbar^2 m \right]$$

$$= \frac{i}{2}\hbar^2 m \qquad \text{(Note that } \hat{J}_x \hat{J}_y \text{ is not Hermitian!)}$$

$$\text{so} \quad \left| \langle \hat{J}_x \hat{J}_y \rangle \right|^2 = \frac{\hbar^4}{4} m^2$$

Back to inequality:

$$\frac{\hbar^4}{4} \left([j(j+1) - m^2] \right)^2 \overset{?}{\geq} \frac{\hbar^4}{4} m^2$$

$$j^2(j+1)^2 - 2m^2 j(j+1) + m^4 \overset{?}{\geq} m^2$$

Recall that $j \geq 0$ and $|m| \leq j$

so $j(j+1) \geq m^2$ and $j^2(j+1)^2 \geq m^4$

4)

If $j = \pm m$, $j^2 = m^2$:

$$j^2(j^2+1)^2 - 2j^3(j+1) + j^4 \overset{?}{=} j^2$$

$$j^4 + 2j^3 + j^2 - j^2 - 2j^4 - 2j^3 + j^4 \overset{?}{=} 0 \quad \checkmark$$

(3) [12.6.1]

1) No angular dependence: $l = m = C$.

2) Assume $V(r)$ vanishes at large r (as stated).
With $l=0$, radial equation is $\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}\psi_E(r) = E\psi_E(r) - V(r)\psi$

$$\Rightarrow A\left(\frac{-1}{a_0}\right)^2\left(\frac{\hbar^2}{2\mu}e^{-r/a_0}\right) = EAe^{-r/a_0}$$

$$\Rightarrow E = \frac{-\hbar^2}{2\mu a_0^2}$$

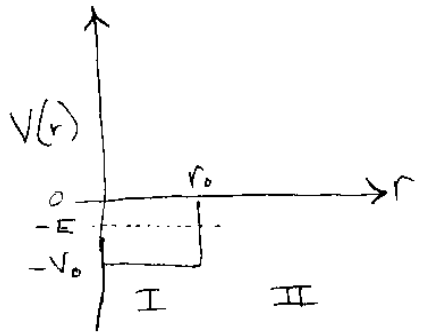
3) Back to radial Schr. eqn:

$$E\psi(r) + \frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\psi = V(r)\psi$$

$$\frac{\hbar^2}{2\mu}\cdot\frac{2}{r}\frac{d}{dr}\psi \qquad = V(r)\psi \qquad = \frac{\hbar^2}{\mu r}\left(\frac{-1}{a_0}\right)\psi$$

$$V(r) = \frac{-\hbar^2}{\mu a_0 r}$$

(4) [12.6.9] — A little Bessel function practice.



Region I: $u\psi = j_0(k'r) = \frac{\sin k'r}{k'r}$

where $k' = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

Region II (classically forbidden)

$$u\psi = \frac{e^{-\kappa r}}{\kappa r} \quad \text{where } \kappa = \frac{\sqrt{2mE}}{\hbar}$$

Match at r_0: $\frac{1}{j_0(k'r_0)} \frac{d}{dr}\Big|_{r_0} j_0(k'r_0) = \frac{-\kappa r}{e^{-\kappa r}} \frac{d}{dr}\Big|_{r_0} \frac{-e^{-\kappa r}}{\kappa r}$

$$\frac{k'r_0}{\sin k'r_0}\left(\frac{k'r_0}{\sin k'r_0} - \frac{\sin k'r_0}{k'r_0^2}\right) \qquad = \frac{-\kappa r_0}{e^{-\kappa r_0}}\left(-\kappa - \frac{1}{r_0}\right)\left(\frac{-e^{-\kappa r_0}}{\kappa r_0}\right)$$

$$= k' \cot k'r_0 - \frac{1}{r_0} \qquad = -\kappa - \frac{1}{r_0}$$

so $k' \cot k'r_0 - \frac{1}{r_0} = -\kappa - \frac{1}{r_0}$

$$\frac{k'}{\kappa} = -\tan k'r_0$$

This is the same situation as 5.2.6 except the wave fn must vanish at $r \to 0$ so "even" solutions from 5.2.6 don't exist. At critical $V_0 = \frac{\pi^2\hbar^2}{8mr_0^2}$, solution has $E \to 0$: not bound.