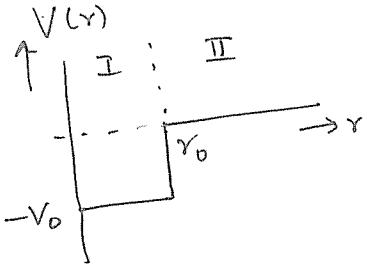
Solution set 11

Problem 19.5.4 (Shankar):

For S-ware scattering we need to solve.

$$\left\{ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left[E - V(r)\right] \right\} U(r) = 0 .$$



We have $V(r) = -V_0 \quad r < r_0$

$= 0 \quad r > r_0$

Thus if we define $\frac{2m(E+V_0)}{\hbar^2} = k'^2$, $\frac{2mE}{\hbar^2} = k^2$

we see that for

$r < r_0$ $U_I(r)$ obeys

$$\left(\frac{d^2}{dr^2} + k'^2 \right) U_I(r) = 0 .$$

$r > r_0$ $U(r)$ obeys

$$\left(\frac{d^2}{dr^2} + k^2 \right) U(r) = 0 .$$

$\therefore$ Region I

$$U(r) = A' \sin k'r + B' \cos k'r$$

Region II

$$U(r) = A \sin kr + B \cos kr$$

$$= R \sin(kr + \delta_0) .$$

→ Wave function must vanish at $r=0$. $\Rightarrow$ $B' = 0$.

→ Wave function must be continuous at $r = r_0$

$$A \sin k' r_0 = R \sin(k r_0 + \delta_0)$$

→ First derivative of the wave function must be continuous at r_0.

$$A k' \cos k' r_0 = R k \cos(k r_0 + \delta_0)$$

∴ We see that-

$$k \tan k' r_0 = k' \tan(k r_0 + \delta_0)$$

$$\therefore \quad \delta_0 = -k r_0 + \tan^{-1}\left(\frac{k}{k'} \tan k' r_0\right).$$

Neglecting $k r_0$ we see that-

$$\delta_0 = \tan^{-1}\left(\frac{k}{k'} \tan k' r_0\right).$$

When $k' \simeq k'_n = (2n+1)\pi/2r_0$

then $\tan k'_n r_0 = \tan\left((2n+1)\pi/2\right) = \infty$.

Thus $\tan(k' r_0) \simeq \dfrac{1}{(k'_n - k')r_0} = \dfrac{k'_n + k'}{(k'^2_n - k'^2) r_0}$.

$$= \frac{\frac{\hbar^2}{2m r_0}(k'_n + k')}{(E_n - E)} \simeq \frac{\frac{\hbar^2}{m r_0} k'_n}{(E_n - E)} = \frac{\Gamma/2}{(E_R - E)}$$

$$\therefore \quad \frac{\Gamma}{2} = \frac{\hbar^2}{m r_0} k'_n .$$

Let us now consider bound states with energy E. Then we can find a condition.

if the bound states are of energy $E = 0$. Then we must have

$$A \sin k' r_0 = R e^{-k r_0}$$

$$A k' \cos k' r_0 = -R k e^{-k r_0}$$

$$\therefore \quad \tan k' r_0 = \frac{-k'}{k}$$

Now,

$$\begin{cases} \frac{\hbar^2 k^2}{2m} = |E| \\ \frac{\hbar^2 k'^2}{2m} = (V_0 - |E|) \end{cases}$$

If ($E \to 0$ or) $k \to 0$ then $k' r_0 = (2n+1)\pi/2 = k'_n r_0$

$\therefore \quad k' = k'_n$ is the point where we get bound states of energy $\neq 0$!

Problem 19.5.6 (Shankar).

(1) We know that the full wavefunction of scattering is

$$\psi_k(\vec{r}) = e^{ikz} + f(\theta, \phi) \frac{e^{ikr}}{r}$$

$$= e^{ikr\cos\theta} + f(\theta, \phi) \frac{e^{ikr}}{r}.$$

Now the probability current is

$$\vec{J} = \frac{\hbar}{2mi} \left[\psi^* \vec{\nabla} \psi - (\vec{\nabla} \psi^*) \psi \right]$$

$$= \frac{\hbar}{m} \mathrm{Im} \left[\psi^* \vec{\nabla} \psi \right].$$

$$\lim_{r \to \infty} \vec{J} = \frac{\hbar}{m} \Big[\underbrace{k \hat{z}}_{J_{inc}} + \underbrace{|f|^2 \frac{k \hat{r}}{r^2}}_{\vec{J}_{sc}} + \underbrace{\frac{1}{r} \mathrm{Im} \{ ik e^{ikr(\cos\theta - 1)} f^* \hat{z} + ik e^{-ikr(\cos\theta - 1)} f \hat{r} \}}_{\vec{J}_{interference}} \Big]$$

$$\lim_{r\to\infty} \vec{J}_{int} = \frac{\hbar k}{m} \mathrm{Im}\left\{ i \frac{e^{ikr(\cos\theta - 1)}}{r} f^*\hat{z} + e^{-ikr(\cos\theta-1)} \frac{f}{r}\hat{r} \right\}.$$

↑ interference term

The radial component is given by

$$\lim_{r\to\infty} j_{int,r} = \frac{\hbar k}{m}\frac{1}{r} \mathrm{Im}\left[i e^{ikr(\cos\theta-1)} f^*(\theta)\cos\theta + i e^{-ikr(\cos\theta-1)} f(\theta) \right].$$

(2) When we integrate over a small solid angle at $\theta \neq 0$ we see $(\cos\theta - 1) \neq 0$. Hence $e^{ikr(\cos\theta-1)}$ oscillates rapidly (even) over the small integration region and gives zero!

(3) When $\theta \approx 0$ then $(\cos\theta - 1) \approx -\frac{\theta^2}{2}$. Then the integral survives! We now see that $\cos\theta \approx 1$ $\sin\theta\, d\theta \approx \theta\, d\theta$

$$\int_0^\theta e^{-ikr\theta^2/2} f^*(\theta)\, \theta\, d\theta = \frac{f^*(0)}{kr}$$

$$\int_0^\theta e^{ikr\theta^2/2} f(\theta)\, \theta\, d\theta = -\frac{f(0)}{kr}.$$

$$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = \frac{\hbar k}{\mu}\frac{1}{r}\mathrm{Im}\left[i\frac{f^*(0)}{kr} - \frac{i f(0)}{kr}\right] 2\pi r^2$$

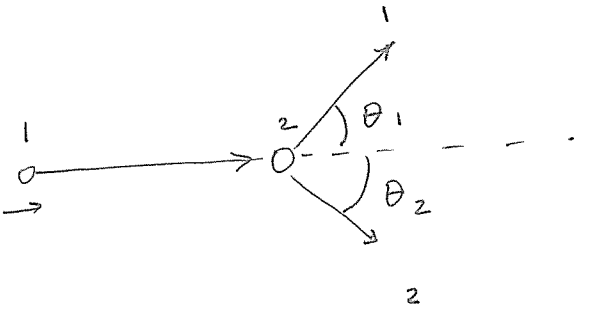
$$= -\frac{\hbar k}{\mu}\left(\frac{4\pi}{k}\right). \mathrm{Im}\, f(0).$$

(2) · We can see that- $\theta_1 \leq \pi/2$ from energy - momentum conservation.

To see this note that-

$$mv = mv_1 \cos\theta_1 + mv_2 \cos\theta_2$$

$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2$$



clearly we can assume without loss of generality that- $\theta_1 > 0$ and $\theta_2 > 0$. (because the reverse situation $\theta_1 < 0$ and $\theta_2 < 0$ is a symmetric situation).

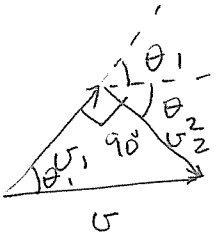
Thus,

$$v = v_1 \cos\theta_1 + v_2 \cos\theta_2$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 .$$

and Energy conservation $\Rightarrow$ $v^2 = v_1^2 + v_2^2$.

Geometrically



$\therefore$ v_1 v_2 and v form sides of a right triangle.

$\Rightarrow$ $\theta_1 + \theta_2 = \pi/2$. $\Longrightarrow$ Along with $\theta_1 > 0$ $\theta_2 > 0$ means $\underline{\theta_1 \leq \pi/2}$

$$\therefore \quad \frac{\int_{\text{forward cone}} j_r^{\text{int}} r^2 d\Omega}{j_{\text{inc}}} = -\left(\frac{4\pi}{k}\right) \text{Im } f(0).$$

$-$ ↑ deficit (depletion).

Now σ_{tot} = total scattering cross section.

$$\therefore \quad \sigma_{tot} = \left(\frac{4\pi}{k}\right) \text{Im } f(0) \quad \Longleftarrow \text{ Optical theorem.}$$

Problem 19.6.1 (Shankar).

(1) We know that

$$\left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right) \cdot \left(\frac{d\Omega_{CM}}{d\Omega_L}\right)$$

Now $d\Omega_{CM} = \sin\theta_{CM}\, d\theta_{CM}\, d\phi_{CM}$.

$d\Omega_L = \sin\theta_L\, d\theta_L\, d\phi_L$.

From eq. (19.6.16) / 6.17 we see that $2\theta_L = \theta_{CM}$ $\quad \phi_L = \phi_{CM}$

$$\therefore \quad \left(\frac{d\Omega_{CM}}{d\Omega_L}\right) = \frac{\sin 2\theta_L}{\sin\theta_L} \underbrace{\frac{d\theta_{CM}}{d\theta_L}}_{2} \underbrace{\frac{d\phi_{CM}}{d\phi_L}}_{1}$$

$$= 4\cos\theta_L$$

$$\therefore \quad \left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right)_{\theta_{CM}=2\theta_L} 4\cos\theta_L$$

(3) For unequal mass eqs. (19.6.14), (19.6.15) change as follows

$$\tan\theta = \frac{P_\perp'}{P_z'} \qquad \tan\phi = \frac{P_y'}{P_x'} \qquad \text{(same as } m_1 = m_2\text{)}$$

$$\tan\theta_L = \frac{P_\perp'}{\left(P_z' + \frac{m_1}{m_2}P\right)} \qquad \tan\phi_L = \frac{P_y'}{P_x'} \; . \longrightarrow$$

$$\therefore \quad \phi_L = \phi \quad \text{but} \quad \tan\theta_L = \frac{P_\perp'/P}{\left(\frac{P_z'}{P} + \frac{m_1}{m_2}\right)} \; .$$

Now $P_\perp'/P = \sin\theta \qquad P_z'/P = \cos\theta$.

$$\therefore \quad \tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}} \; .$$

Problem 19.6.2 (Shankar).

We start with eq. (19.3.16) for $f_c(\theta)$.

$$f_c(\theta) = \frac{-\gamma}{2k(\sin\theta/2)^2} e^{-i\gamma \ln \sin^2\theta/2 + iA} \qquad (A = \text{some const. purely real}) \; .$$

Now $\frac{d\sigma}{d\Omega} = |f(\theta,\phi) + f(\pi-\theta, \phi+\pi)|^2$

$$f_c(\pi-\theta) = \frac{-\gamma}{2k(\cos\theta/2)^2} e^{-i\gamma \ln \cos^2\theta/2 + A} \; .$$

$$|f_c(\theta)|^2 = \frac{\gamma^2}{4k^2 \sin^4\theta/2} \; . \qquad |f_c(\pi-\theta)|^2 = \frac{\gamma^2}{4k^2\cos^4\theta/2}$$

$$f_c^*(\theta) f_c(\pi-\theta) = \frac{\gamma^2}{4k^2 \sin^2\theta/2 \cos^2\theta/2} \cdot e^{i\gamma \ln \tan^2\theta/2}$$

$$f_c(\theta) f_c^*(\pi-\theta) = \frac{\gamma^2}{4k^2 \sin^2\theta/2 \cos^2\theta/2} e^{-i\gamma \ln \tan^2\theta/2}$$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{\gamma^2}{4k^2} \left[\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} + \frac{2\cos(\gamma \ln \tan^2\theta/2)}{\sin^2\theta/2 \cos^2\theta/2} \right]$$

$$\text{If } \gamma = \frac{e^2\mu}{\hbar^2 k} \qquad \frac{\gamma}{k} = \frac{e^2}{2E} \cdot \qquad \frac{\gamma^2}{4k^2} = \left(\frac{e^2}{4E}\right)^2$$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4E}\right)^2 \left[\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} + \frac{2\cos(\gamma \ln \tan^2\theta/2)}{\sin^2\theta/2 \cos^2\theta/2} \right]$$