(1) Two State Problem

Let us write the states as $|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then
$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \frac{E_1+E_2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{E_1-E_2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Similarly
$$H_e = H_0 + \gamma \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

$$H = \frac{E_1+E_2}{2}\mathbb{1} + \gamma \cos\omega t\, \sigma_1 - \gamma \sin\omega t\, \sigma_2 + \frac{E_1-E_2}{2}\sigma_3$$

$$H = e^{\frac{i\omega t}{2}\sigma_3}\left[\frac{E_1+E_2}{2} + \gamma\sigma_1 + \frac{E_1-E_2}{2}\sigma_3\right] e^{-\frac{i\omega t}{2}\sigma_3}$$

We used the "trick" discussed in the class. If

$|\psi(t)\rangle$ is the state that satisfies the equation

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle.$$

We have shown $|\psi'(t)\rangle = e^{-\frac{i\omega t}{2}\sigma_3}|\psi(t)\rangle$ satisfies the equation

$$i\hbar \frac{d}{dt}|\psi'(t)\rangle = H'\,|\psi'(t)\rangle$$

where
$$H' = \frac{E_1+E_2}{2} + \gamma\sigma_1 + \frac{E_1-E_2}{2}\sigma_3 + \frac{\hbar\omega}{2}\sigma_3$$

$$H_\Lambda = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}_\Lambda\right)^2 + q\phi_\Lambda$$

$$= e^{\frac{-iq}{\hbar c}\Lambda}\left[\frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + q\phi + \frac{q}{c}\frac{\partial \Lambda}{\partial t}\right]e^{\frac{iq}{\hbar c}\Lambda}$$

$$= e^{\frac{-iq}{\hbar c}\Lambda}\left[H + \frac{q}{c}\frac{\partial \Lambda}{\partial t}\right]e^{\frac{iq}{\hbar c}\Lambda}$$

(iii) If $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$

and $i\hbar \frac{d}{dt}|\psi(t)\rangle_\Lambda = \hat{H}_\Lambda\,|\psi(t)\rangle_\Lambda$

$$\Rightarrow \quad |\psi(t)\rangle_\Lambda = e^{\frac{-iq}{\hbar c}\Lambda}\,|\psi(t)\rangle$$

Substituting we see

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_\Lambda = \frac{q}{c}\frac{\partial \Lambda}{\partial t}\,|\psi(t)\rangle_\Lambda + e^{\frac{-iq}{\hbar c}\Lambda}\hat{H}\,e^{\frac{iq}{\hbar c}\Lambda}|\psi$$

$$= \hat{H}_\Lambda\,|\psi(t)\rangle_\Lambda$$

Let $E_1 - E_2 = \omega_{12}\hbar$, $\gamma = \omega_\gamma \hbar$, $\frac{E_1+E_2}{2} = \omega_0 \hbar$

Then we see that

$$H' = \hbar\left(\omega_0 + \omega_\gamma \sigma_1 + \frac{1}{2}(\omega_{12}+\omega)\sigma_3\right)$$

Let us define $\Omega = \left[\omega_\gamma^2 + (\omega_{12}+\omega)^2/4\right]^{1/2}$

$$n_1 = \frac{\omega_\gamma}{\Omega} \quad n_2 = 0 \quad n_3 = \frac{1}{2}\frac{(\omega_{12}+\omega)}{\Omega}$$

Then $$H' = \hbar\left(\omega_0 + \Omega \hat{n}\cdot\vec{\sigma}\right).$$

Now it is easy to solve for $|\psi'(t)\rangle$, we have

$$|\psi'(t)\rangle = \underbrace{\exp\left(\frac{-i}{\hbar}H't\right)}\ |\psi'(0)\rangle$$

$$e^{-i\omega_0 t}\left[\cos\Omega t - i\,\hat{n}\cdot\vec{\sigma}\sin\Omega t\right]$$

Initial state $|\psi'(0)\rangle = |\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

Final state at time t $|\psi(t)\rangle = \exp\left(\frac{i\omega t}{2}\sigma_3\right)|\psi'(t)\rangle$

$$|\psi(t)\rangle = e^{\frac{i\omega t}{2}\sigma_3}\, e^{-i\omega_0 t}\left[\cos\Omega t - i\,\hat{n}\cdot\vec{\sigma}\sin\Omega t\right]|\psi(0)\rangle$$

Amplitude to find the state in $|2\rangle$ is $\langle 2|\psi(t)\rangle$ so

$$A = \langle 2|\psi(t)\rangle = (0\ 1)\, e^{\frac{i\omega t}{2}\sigma_3}\, e^{-i\omega_0 t}\left(\cos\Omega t - i\,\hat{n}\cdot\vec{\sigma}\sin\Omega t\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= e^{\frac{-i\omega t}{2} - i\omega_0 t}\,(-i n_1 + n_2)\sin\Omega t$$

Thus the probability

$$P_{|1\rangle\to|2\rangle}(t) = |A|^2 = n_1^2 \sin^2\Omega t$$

$$= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega+\omega_{12})^2/4}\sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega+\omega_{12})^2}{4}\right]^{1/2} t\right\}$$

Problem 14.4.4

The Hamiltonian of an electron in a magnetic field

$$H = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = -\frac{e}{2mc}\cdot 2\vec{S}$$

(2 ↑ g-factor of an electron.)

$$H = -\frac{e\hbar}{mc}\vec{\sigma}\cdot\vec{B}$$

Now $\vec{B} = B\hat{i}$, which means $H = -\frac{e\hbar}{mc}B\,\sigma_1$.

Thus

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\frac{e\hbar B}{mc}t\,\sigma_1\right]|\psi(0)\rangle$$

Let $\omega = \frac{eB}{mc}$, then the state at time t

$$|\psi(t)\rangle = (\cos\omega t - i\sigma_1 \sin\omega t)\,|\psi(0)\rangle$$

We are given $|\psi(0)\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}$, we want $|\psi(t)\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}e^{i\alpha}$

We see that $\omega t = \pi \Rightarrow t = \frac{\pi}{\omega}$.

$$t = \frac{\pi mc}{eB}$$

we know $\frac{e\hbar}{2mc} \simeq 0.6\times10^{-8}\,eV/G$.

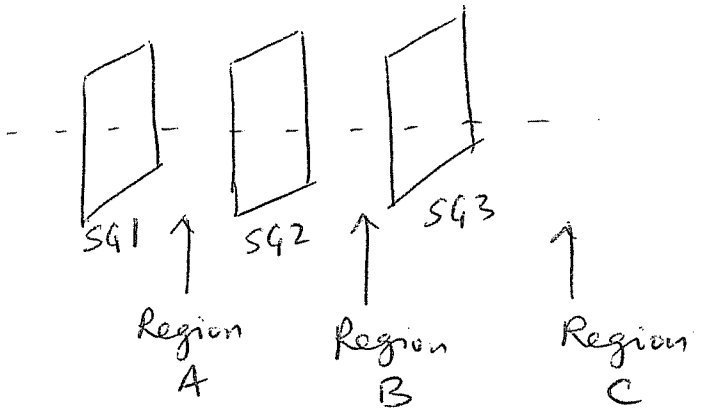
$B = 100\,G$.

$\hbar = 6.6\times10^{-16}\,eV\cdot s$

$$t = \frac{\pi}{2\times 0.6\times10^{-8}}\cdot\frac{6.6\times10^{-16}}{100}$$

$$t = \frac{\pi\hbar}{1.2\times10^{-6}}\,sec \simeq 1.73\times10^{-9}\,sec$$

Problem 14.5.3



Region A has particles in state $|\hat{z},+\rangle$.

$$|\hat{z},+\rangle = \frac{1}{\sqrt{2}}\left[|\hat{x},+\rangle + |\hat{x},-\rangle\right]$$

(i) If the second filter allows only $|\hat{x},+\rangle$ states to pass. Then 50% of particles in region A will come out of region B.

(ii) If the 3rd filter allows only in $|\hat{z},+\rangle$ then

Since $$|\hat{x},+\rangle = \frac{1}{\sqrt{2}}\left[|\hat{z},+\rangle + |\hat{z},-\rangle\right]$$

only 50% of region B particles come into region C.

ie... overall 25% of region A particles come to region C.

(iii) If the 2nd filter allows both spins up/down (no blocking) then Region B contains $|\hat{z},+\rangle$ and then filter 3 transmits only $|\hat{z},-\rangle$ then <u>all</u> particles will be blocked and 0% will reach region C.

Problem 15.1.2

If we take into account the electron and the proton spin, the Coulomb Hamiltonian ground state $E_0 = -Ry$ is four-fold degenerate!

(1) A Hyper fine interaction

$$H_{hf} = A\,\vec{S}_1 \cdot \vec{S}_2$$

$$= \frac{A}{2}\left[\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right]$$

splits these levels. Since $\vec{S}_1$ and $\vec{S}_2$ are both spin-$\frac{1}{2}$, $\vec{S}$ can be spin-1 or spin 0.

$$E_{hf} = \frac{A}{2}\left[2\hbar^2 - \frac{3}{2}\hbar^2\right] = \frac{\hbar^2 A}{4} \quad \text{triplet}$$

$$= \frac{A}{2}\left[0 - \frac{3}{2}\hbar^2\right] = \frac{-3\hbar^2 A}{4} \quad \text{singlet.}$$

$$\therefore \quad E_+ = -Ry + \frac{\hbar^2 A}{4} \quad \text{triplet}$$

$$E_- = -Ry - \frac{3\hbar^2 A}{4} \quad \text{singlet.}$$

(2) To estimate the frequency emitted we equate

$$h\nu = \Delta E = A\hbar^2$$

In order to express it as wave length we use

$$\frac{hc}{\lambda} = \Delta E$$

$$\Rightarrow \quad \lambda = \frac{hc}{\Delta E}$$

Now $\Delta E = A\hbar^2 = \frac{1}{2}\frac{e^2\hbar^2}{a_0^3}\cdot\frac{5.6}{Mmc^2}$

$a_0 = \frac{\hbar^2}{me^2}$ and $\frac{me^4}{2\hbar^2} \simeq 13.3\,eV$

$$\Delta E = \frac{5.6}{2}\frac{m^2e^6}{\hbar^4}\frac{e^2\hbar^2}{Mmc^2}$$

$$= \frac{5.6}{2}\left(\frac{me^4}{2\hbar^2}\right)^2\cdot 4^2\cdot\frac{1}{Mc^2}$$

$$\simeq 2\cdot 5.6\cdot\frac{(13.3)^2 eV^2}{1\,GeV} = 1.98\times 10^{-6}\,eV.$$

$$\lambda = \frac{hc}{\Delta E} \simeq 62\,cms.$$

(3) The probability ratio $P(\text{triplet})/P(\text{singlet}) = 3\,\frac{\exp(-E_{\text{triplet}}/kT)}{\exp(-E_{\text{singlet}}/kT)}$ (3 ← degeneracy)

$$\simeq 3\,e^{-\Delta E/kT}$$

$kT \simeq \frac{1}{40}\,eV$ at $T = 300k$ (room temperature).

Since $\Delta E \ll kT$ we see that

$P(\text{triplet})/P(\text{singlet}) \simeq 3$.

Exercise 15.2.2

(1) $j_1 = 1 \quad j_2 = \frac{1}{2} \Rightarrow j = \frac{3}{2}$ or $\frac{1}{2}$

$\downarrow m$ / $\rightarrow j$	$j = \frac{3}{2}$	$j = \frac{1}{2}$
	$\left\|\frac{3}{2}\ \frac{3}{2}\right\rangle$	
	$\left\|\frac{3}{2}\ \frac{1}{2}\right\rangle$	$\left\|\frac{1}{2}\ \frac{1}{2}\right\rangle$
	$\left\|\frac{3}{2}\ -\frac{1}{2}\right\rangle$	$\left\|\frac{1}{2}\ -\frac{1}{2}\right\rangle$
	$\left\|\frac{3}{2}\ -\frac{3}{2}\right\rangle$	

$$\rightarrow \left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = |1\ 1\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle .$$

Applying J_- on both sides we get-

$$\sqrt{3}\hbar \left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{2}\hbar\, |1\,0\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \hbar\, |1\ 1\rangle \otimes \left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle$$

$$\rightarrow \left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, \left|1\,0, \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \left|1\,1, \tfrac{1}{2} - \tfrac{1}{2}\right\rangle$$

$$\rightarrow \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, \left|1\,0, \tfrac{1}{2} - \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \left|1\,{-1}, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$\rightarrow \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \left|1\,{-1}, \tfrac{1}{2} - \tfrac{1}{2}\right\rangle .$$

$$\rightarrow \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, \left|1\,1, \tfrac{1}{2} - \tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\, \left|1\,0, \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

$$\rightarrow \left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\, \left|1\,0, \tfrac{1}{2} - \tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\, \left|1\,{-1}, \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

One can read off the clebsch-Gordan coefficients.

(2) $j_1 = 1 \quad j_2 = 1 \Rightarrow j = 2, 1, 0$

m ↓ / j →	$j=2$	$j=1$	$j=0$
	$\|2,2\rangle$		
	$\|2,1\rangle$	$\|1,1\rangle$	
	$\|2,0\rangle$	$\|1,0\rangle$	$\|0,0\rangle$
	$\|2,-1\rangle$	$\|1,-1\rangle$	
	$\|2,-2\rangle$		

$\rightarrow \quad |2\,2\rangle = |11,11\rangle$

Applying J_- on both sides:

$$2\hbar\,|2\,1\rangle = \hbar\sqrt{2}\,|10,11\rangle + \hbar\sqrt{2}\,|11,10\rangle$$

$$\rightarrow \quad |2\,1\rangle = \frac{1}{\sqrt{2}}|10,11\rangle + \frac{1}{\sqrt{2}}|11,10\rangle$$

$$\sqrt{3}\cdot\sqrt{2}\hbar\,|2\,0\rangle = \frac{\hbar\sqrt{2}}{\sqrt{2}}|1\text{-}1,11\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|11,1\text{-}1\rangle$$

$$\rightarrow \quad |2\,0\rangle = \frac{1}{\sqrt{6}}|1\text{-}1,11\rangle + \sqrt{\frac{2}{3}}|10,10\rangle + \frac{1}{\sqrt{6}}|11,1\text{-}1\rangle$$

$$\rightarrow \quad |2\,\text{-}1\rangle = \frac{1}{\sqrt{2}}|10,1\text{-}1\rangle + \frac{1}{\sqrt{2}}|1\text{-}1,10\rangle$$

$$\rightarrow \quad |2\,\text{-}2\rangle = |1\text{-}1,1\text{-}1\rangle$$

$$\rightarrow \quad |1\,1\rangle = \frac{1}{\sqrt{2}}|11,10\rangle - \frac{1}{\sqrt{2}}|10,11\rangle$$

$$\hbar\sqrt{2}\,|1\,0\rangle = \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle + \frac{\hbar\sqrt{2}}{\sqrt{2}}|11,1\text{-}1\rangle - \frac{\hbar\sqrt{2}}{\sqrt{2}}|1\text{-}1,11\rangle - \frac{\hbar\sqrt{2}}{\sqrt{2}}|10,10\rangle$$

$$\rightarrow \quad |1\,0\rangle = \frac{1}{\sqrt{2}}|11,1\text{-}1\rangle - \frac{1}{\sqrt{2}}|1\text{-}1,11\rangle$$

$$\rightarrow \quad |1,-1\rangle = \frac{1}{\sqrt{2}} |10, 1,-1\rangle - \frac{1}{\sqrt{2}} |1,-1, 10\rangle$$

We next want $|0,0\rangle$ which has the form

$$|00\rangle = \alpha |11, 1,-1\rangle + \beta |10, 10\rangle + \gamma |1,-1, 11\rangle$$

We want $|00\rangle$ to be orthogonal to $|10\rangle$ and $|20\rangle$.

$$\langle 10|00\rangle = 0 \implies \alpha = \gamma$$

$$\langle 20|00\rangle = 0 \implies \alpha + \gamma + 2\beta = 0$$

which means: $\alpha = \gamma = -\beta = \frac{1}{\sqrt{3}}$

$$|00\rangle = \frac{1}{\sqrt{3}} |11, 1,-1\rangle - \frac{1}{\sqrt{3}} |10, 10\rangle + \frac{1}{\sqrt{3}} |1,-1, 11\rangle$$

15.2.5

(1) First notice that $\vec{S}_1 = \vec{\sigma}_1 \frac{\hbar}{2}$ and $\vec{S}_2 = \vec{\sigma}_2 \frac{\hbar}{2}$.

$$\mathbb{P}_1 = \frac{3}{4} I + \frac{\vec{S}_1 \cdot \vec{S}_2}{\hbar^2} = \left(\frac{3}{4} I + \frac{1}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right)$$

$$\mathbb{P}_1 \mathbb{P}_1 = \left(\frac{9}{16} I + \frac{6}{16} \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{1}{16} (\vec{\sigma}_1 \cdot \vec{\sigma}_2)^2 \right)$$

Now $(\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) = \underbrace{(\vec{\sigma}_1 \cdot \vec{\sigma}_1)}_{3} + i \underbrace{(\vec{\sigma}_1 \times \vec{\sigma}_1)}_{2i\vec{\sigma}_1} \cdot \vec{\sigma}_2$

$$= \left(3I - 2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right).$$

$$\therefore \quad \mathbb{P}_1^2 = \left(\frac{3}{4} I + \frac{6}{16} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{2}{16} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) = \mathbb{P}_1.$$

$$\mathbb{P}_2 = \left(\frac{1}{4} I - \frac{1}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right)$$

$$\mathbb{P}_2 \mathbb{P}_2 = \left(\frac{1}{16} I - \frac{2}{16} \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{1}{16} \left(3I - 2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \right) = \mathbb{P}_2$$

$$\mathbb{P}_1\,\mathbb{P}_2 = \left(\frac{3}{4}I + \frac{1}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2\right)\left(\frac{1}{4}I - \frac{1}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2\right)$$

$$= \left(\frac{3}{16}I - \frac{2}{16}(\vec{\sigma}_1\cdot\vec{\sigma}_2) - \frac{1}{16}\left(3I - 2\vec{\sigma}_1\cdot\vec{\sigma}_2\right)\right)$$

$$= 0$$

(2) Now let us write $\vec{S} = \vec{S}_1 + \vec{S}_2$; $\vec{S}_1^2 = \frac{3}{4}I\hbar^2 = \vec{S}_2^2$

$$\therefore \frac{(\vec{S}_1\cdot\vec{S}_2)}{\hbar^2} = \frac{1}{2\hbar^2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right) = \frac{\vec{S}^2}{2\hbar^2} - \frac{3}{4}I .$$

$$\therefore \mathbb{P}_1 = \frac{\vec{S}^2}{2\hbar^2}$$

Now if $\mathbb{P}_1$ acts on $|11\rangle$ $|10\rangle$ or $|1{-1}\rangle$ we get 1.

If it acts on $|00\rangle$ we get 0.

$$\mathbb{P}_2 = I - \frac{\vec{S}^2}{2\hbar^2}$$

If $\mathbb{P}_2$ acts on $|11\rangle$, $|10\rangle$, $|1{-1}\rangle$ we get a 0 and when it acts on $|00\rangle$ we get 1.

Thus $\mathbb{P}_1$ projects into the spin-1 space and $\mathbb{P}_2$ projects into spin-0.