Solution Set 4 (Supplemental)

(1) t_{ij} $\left.\begin{matrix} i=1,2,3 \\ j=1,2,3 \end{matrix}\right\}$ form 9- Components of a Cartesian tensor.

Remembering that- t_{ij} transforms like $x_i x_j$ we can easily construct spherical tensors.

$t_{11} + t_{22} + t_{33}$ is like $x_1^2 + x_2^2 + x_3^2$ which is a scalar.

So $T_0^0 = t_{11} + t_{22} + t_{33}$.

The three components

$V_3 = t_{12} - t_{21}$, $V_2 = t_{31} - t_{13}$, $V_1 = t_{23} - t_{32}$

transform like a Cartesian vector. Thus

$$T_1^0 = t_{12} - t_{21}$$

$$T_1^1 = -\frac{1}{\sqrt{2}}(t_{23} - t_{32} + i t_{31} - i t_{13})$$

$$T_1^{-1} = +\frac{1}{\sqrt{2}}(t_{23} - t_{32} - i t_{31} + i t_{13})$$

Now Y_2^m transform like spherical tensor of rank 2.

$$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta\, e^{\pm i 2\phi} = \left(\frac{15}{32\pi}\right)^{1/2} \frac{(x \pm iy)^2}{r^2}$$

$$Y_2^{\pm 1} = \mp \left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta\, e^{\pm i\phi} = -\left(\frac{15}{8\pi}\right)^{1/2} \frac{z(x \pm iy)}{r^2}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1) = \left(\frac{5}{16\pi}\right)^{1/2} \frac{(3z^2 - r^2)}{r^2}$$

This means

$$T_2^{\pm 2} = \sqrt{\frac{3}{4}} \left(t_{11} - t_{22} \pm i (t_{12} + t_{21}) \right)$$

$$T_2^{\pm 1} = -\frac{\sqrt{3}}{2} \left(t_{31} \pm i t_{32}, + t_{13} \pm i t_{23} \right)$$

$$T_2^{0} = \sqrt{\frac{1}{2}} \left(2 t_{33} - t_{11} - t_{22} \right)$$

Problem 2 : We want to construct $D^1_{mm'}(R)$ explicitly.

First let us note that-

$$D^{(1)}(R) = \exp\left(-\frac{i}{\hbar} \Theta J_x^{(1)} \right)$$

where $J_x^{(1)}$ is the angular momentum operator for spin-1.

$$J_+^{(1)} |1m\rangle = \hbar \sqrt{(2+m)(1-m)} \; |1(m+1)\rangle$$

$$J_-^{(1)} |1m\rangle = \hbar \sqrt{(2-m)(1+m)} \; |1(m-1)\rangle .$$

$$J_x^{(1)} = \frac{J_+^{(1)} + J_-^{(1)}}{2}$$ which means explicitly.

$$J_x^{(1)} \equiv \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(J_x^{(1)}\right)^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\left(J_x^{(1)}\right)^3 = \hbar^2 J_x^{(1)}$$

where $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \equiv |1\,1\rangle$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \equiv |1\,0\rangle$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv |1\,{-1}\rangle$

Hence we can show

$$D^{(1)}(R) = I + (\cos\theta - 1)\left(\frac{J_x^{(1)2}}{\hbar^2}\right) \cdots -i\sin\theta\left(\frac{J_x^{(1)}}{\hbar}\right)$$

Explicitly this means

$$D^{(1)}(R) = \begin{pmatrix} \frac{(\cos\theta+1)}{2} & \frac{-i}{\sqrt{2}}\sin\theta & \frac{\cos\theta-1}{2} \\ \frac{-i}{\sqrt{2}}\sin\theta & \cos\theta & \frac{-i}{\sqrt{2}}\sin\theta \\ \frac{\cos\theta-1}{2} & \frac{-i}{\sqrt{2}}\sin\theta & \frac{(\cos\theta+1)}{2} \end{pmatrix}$$

12.5.4 (1) One can rotate $J_x^{(j)}$ and $J_y^{(j)}$ and for that matter $\hat{\theta}.\vec{J}^{(j)}$ into $J_z^{(j)}$. ie... the choice of $J_z^{(j)}$ is only a convention any component of angular momentum can be chosen as the "z" component! Hence the eigenvalues will be the same.

(2) Any state in the space spanned by $|jm\rangle$ $m=-j\cdots j$ can be expanded in this basis. This basis can be chosen to be eigenvector of J_z.

Then

$$|\psi\rangle = \sum_{m=-j}^{j} \psi_j |jm\rangle .$$

But then

$$(J-j\hbar)(J-(j-1)\hbar)\cdots(J+j\hbar)|\psi\rangle = 0$$ as can be checked easily. Hence an operator acting on all states giving 0 is a "zero" operator.

(3) Clearly

$$J^{2j+1} = \alpha_0 + \alpha_1 J + \alpha_2 J^2 + \cdots + \alpha_{2j} J^{2j}$$

Hence

$$J^{2j+2} = \alpha J + \alpha_1 J^2 + \alpha_3 J^3 + \cdots + \alpha_{2j} \underbrace{J^{2j+1}}_{\downarrow}$$

Can be expanded in $J^0 \ldots J^{2j}$

$$\therefore \quad J^{2j+2} = \alpha_0' + \alpha_1' J + \alpha_2' J^2 + \cdots + \alpha_{2j}' J^{2j}$$

and - similarly we can show any power of J of the form J^{2j+k} can be written as a linear combination of $J^0, J^1 \ldots J^{2j}$.

Exercise 15.3.3

(1) Using $\langle jj; j1 | jj, 10\rangle = \left(\frac{j}{j+1}\right)^{1/2}$ we have to show

$$\langle \alpha j || J_1 || \alpha' j' \rangle = \delta_{\alpha\alpha'} \delta_{jj'} \hbar \sqrt{j(j+1)}$$

Now we know

$$J_1^q \equiv \quad J_1^0 = J_z$$

$$J_1^1 = -\frac{(J_x + iJ_y)}{\sqrt{2}} = -\frac{J_+}{\sqrt{2}}$$

$$J_1^{-1} = \frac{(J_x - iJ_y)}{\sqrt{2}} = \frac{J_-}{\sqrt{2}} .$$

Consider

$$\langle \alpha jj | J_z | \alpha' j' j' \rangle = j\hbar \delta_{\alpha\alpha'} \delta_{jj'}$$

From Wigner-Eckart theorem

$$\langle \alpha jj | J_z | \alpha' j' j' \rangle = \langle \alpha jj | J_1^q | \alpha' j' j' \rangle$$

$$= \langle \alpha j || J_1 || \alpha' j' \rangle \quad \langle jj; j'1 | j'j', 10 \rangle$$

$\therefore$ We see that

$$\langle \alpha j || J_1 | \alpha' j' \rangle \langle jj; j'1 | j'j', 10 \rangle = j\hbar \delta_{\alpha\alpha'} \delta_{jj'}$$

Setting $j = j'$ we see

$$\langle \alpha j || J_1 || \alpha' j' \rangle = \hbar \sqrt{j(j+1)} \; \delta_{\alpha\alpha'} \delta_{jj'}$$

(2) $\vec{\mathbb{J}}\cdot\vec{A}$ is a scalar operator, which means

$$\langle \alpha' j m' | \vec{\mathbb{J}}\cdot\vec{A} | \alpha\, j m\rangle = \langle \alpha' j \| \vec{\mathbb{J}}\cdot\vec{A} \| \alpha\, j\rangle \underbrace{\langle j m'; j\,0 | j m, 00\rangle}_{\delta_{mm'}}$$

$\therefore$ The left hand side is non-zero only if $m = m'$. Further $\langle \alpha' j \| \vec{\mathbb{J}}\cdot\vec{A} \| \alpha\, j\rangle$ does not depend on m, m'. Hence we can set $m' = m = j$ to evaluate it. We see

$$\langle \alpha' j \| \vec{\mathbb{J}}\cdot\vec{A} \| \alpha\, j\rangle = \langle \alpha' jj | \vec{\mathbb{J}}\cdot\vec{A} | \alpha\, jj\rangle .$$

$$= \langle \alpha' jj | J_z A_z + \tfrac{1}{2}(J_- A_+ + J_+ A_-) | \alpha\, jj\rangle$$

$$\begin{cases} \langle \alpha' jj | J_z = j\hbar \langle \alpha' jj | \\ \langle \alpha' jj | J_- = 0 \\ \langle \alpha' jj | J_+ = \hbar\sqrt{2j}\, \langle \alpha' j(j-1) | \end{cases}$$

$$\therefore \quad \langle \alpha' j \| \vec{\mathbb{J}}\cdot\vec{A} \| \alpha\, j\rangle = \hbar j \langle \alpha' jj | A_z | \alpha\, jj\rangle + \hbar\sqrt{\frac{j}{2}} \langle \alpha' j(j-1) | A_- | \alpha\, jj\rangle$$

$$= \hbar j \langle \alpha' jj | A_1^0 | \alpha\, jj\rangle + \hbar\sqrt{\frac{j}{2}} \langle \alpha' j(j-1) | \sqrt{2}\, A_1^{-1} | \alpha\, jj\rangle .$$

$$\langle \alpha' j \| \vec{J}\cdot\vec{A} \| \alpha j \rangle = \left[\hbar j \langle jj | jj, 10\rangle + \hbar\sqrt{j}\langle j(j-1)|jj\,1-1\rangle \right] \cdot \langle \alpha' j \| A_1 \| \alpha j\rangle$$

$\therefore$ substituting

$$\langle jj|jj,10\rangle = \sqrt{\frac{j}{j+1}} \quad \text{(given)}.$$

$$\langle j(j-1)|jj\,1-1\rangle = \sqrt{\frac{1}{j+1}} \quad \text{(proved below)}.$$

we get

$$\langle \alpha' j \| \vec{J}\cdot\vec{A} \| \alpha j\rangle = \hbar\sqrt{j(j+1)}\ \langle \alpha' j \| A_1 \| \alpha j\rangle$$

$\therefore$

$$\langle \alpha' j m' | \vec{J}\cdot\vec{A} | \alpha j m\rangle = \hbar\sqrt{j(j+1)}\ \langle \alpha' j \| A_1 \| \alpha j\rangle\ \delta_{mm'}$$

Let us now show $\langle j\, j-1 | jj\, 1-1\rangle = \sqrt{\frac{1}{j+1}}$.

Consider.

$$\langle jm | J_1^q | jm'\rangle = \langle j \| J_1 \| j_1\rangle \langle jm | j m', 1q\rangle .$$

Now when $m=j$ $m'=j$ $q=0$ we see L.H.S $= \hbar j$

$$\therefore \hbar j = \langle j \| J_1 \| j_0\rangle \cdot \underbrace{\langle jj | jj, 10\rangle}_{\sqrt{\frac{j}{j+1}}\ \text{(given)}} \quad -(1)$$

when $m=j=1$ $m'=j$ $q=-1$ we get L.H.S $= \langle j\, j-1 | \underbrace{J_1^{-1}}_{\frac{J_-}{\sqrt{2}}} | jj\rangle$

$\Rightarrow$ L.H.S $= \hbar\sqrt{j}$

$$\hbar\sqrt{j} = \langle j \| J_1 \| j\rangle \langle j\, j-1 | jj, 1^{-1}\rangle \quad -(2)$$

using eq. (1) and (2) we get the result.

(3) Now $\langle \alpha' jm' | A^q_1 | \alpha jm \rangle = \langle \alpha' j || A_1 || \alpha j \rangle \cdot \langle jm' | jm, 1q \rangle$.

using the result from (2) we see R.H.S

$$= \frac{\langle \alpha' jm | \vec{J} \cdot \vec{A} | \alpha jm \rangle}{\hbar \sqrt{j(j+1)}} \cdot \langle jm' | jm, 1q \rangle$$

Now

$$\langle jm' | J^q_1 | jm \rangle = \underbrace{\frac{\langle j | J_1 | j \rangle}{\hbar \sqrt{j(j+1)}}} \langle jm' | jm, 1q \rangle$$

R.H.S

$$= \frac{\langle \alpha' jm | \vec{J} \cdot \vec{A} | \alpha jm \rangle}{\hbar \sqrt{j(j+1)}} \cdot \frac{\langle jm' | J^q_1 | jm \rangle}{\hbar \sqrt{j(j+1)}}$$

$$= \frac{\langle \alpha' jm | \vec{J} \cdot \vec{A} | \alpha jm \rangle}{\hbar^2 j(j+1)} \langle jm' | J^q_1 | jm \rangle$$

15.3.4 We are given.

$$\vec{\mu} = \gamma_1 \vec{J}_1 + \gamma_2 \vec{J}_2$$

The state $|jm;j_1j_2\rangle$ is the total angular momentum eigenstates of $\vec{J}=\vec{J}_1+\vec{J}_2$.

Using Eq.(15.3.19) in Shankar we get

$$\langle jm;j_1j_2|\vec{J}_i|jm;j_1j_2\rangle$$

$$= \frac{\langle jm;j_1j_2|\vec{J}\cdot\vec{J}_i|jm;j_1j_2\rangle}{\hbar^2 j(j+1)} \langle jm;j_1j_2|\vec{J}|jm;j_1j_2\rangle$$

clearly $\langle J_x\rangle = \langle J_y\rangle = 0$.

$$\therefore \langle \mu_x\rangle = \langle \mu_y\rangle = 0$$

$$\langle \mu_z\rangle = \frac{\gamma_1}{\hbar^2 j(j+1)} \langle (\vec{J}_1)_z\rangle + \frac{\gamma_2}{\hbar^2 j(j+1)} \langle (\vec{J}_2)_z\rangle$$

$$= \frac{m\hbar}{\hbar^2 j(j+1)} \{ \gamma_1 \langle jm;j_1j_2|\vec{J}\cdot\vec{J}_1|jm;j_1j_2\rangle + \gamma_2 \langle jm;j_1j_2|\vec{J}\cdot\vec{J}_2|jm;j_1j_2\rangle \}.$$

$$\langle jm;j_1j_2|\underbrace{\vec{J}\cdot\vec{J}_1}_{\vec{J}_1^2+\vec{J}_2\cdot\vec{J}_1}|jm;j_1j_2\rangle = \hbar^2 j_1(j_1+1) + \frac{\hbar^2}{2}\big(j(j+1) - j_1(j_1+1) - j_2(j_2+1)\big)$$

$$\langle jm;j_1j_2|\underbrace{\vec{J}\cdot\vec{J}_2}_{(\vec{J}_2^2+\vec{J}_1\cdot\vec{J}_2)}|jm;j_1j_2\rangle = \hbar^2 j_2(j_2+1) + \frac{\hbar^2}{2}\big(j(j+1) - j_1(j_1+1) - j_2(j_2+1)\big)$$

$$\langle \mu_z \rangle = m_t \left[\frac{\gamma_1 + \gamma_2}{2} + \frac{(\gamma_1 - \gamma_2)}{2} \frac{\left(j_1(j_1+1) - j_2(j_2+1) \right)}{j(j+1)} \right]$$

(2) A proton in a $^2P_{1/2}$ state is a proton with $\vec{L} = 1$ $\vec{S} = \frac{1}{2}$ $\vec{J} = \frac{1}{2}$.

Thus.

$$\vec{\mu}_{proton} = \frac{ge}{2Mc} \vec{S} + \frac{e}{2Mc} \vec{L} \qquad (g = 5.6)$$

Using the formula obtained earlier

$$\langle \mu_z \rangle = \pm \frac{\hbar}{2} \cdot \frac{e}{2Mc} \left[\frac{(g+1)}{2} + \frac{(g-1)}{2} \frac{\left(\frac{3}{4} - 2\right)}{\frac{3}{4}} \right]$$

$$= \pm \frac{1}{4} \left[(g+1) - \frac{5}{3}(g-1) \right] \underbrace{\frac{\hbar e}{2Mc}}_{\text{Nuclear Magneton}}$$

Substituting $g = 5.6$ we get

$$|\langle \mu_z \rangle| \sim \left| \frac{1}{2} \left[\frac{4}{3} - \frac{9}{3} \right] \right| \text{Nuclear Magnetons}$$

$$\simeq 0.26 \text{ Nuclear Magnetons}$$

(3) For an electron in $^2P_{1/2}$ state we can use the same procedure as above and we get

$$\langle \mu_z \rangle \sim \frac{1}{2} \left[\frac{4-9}{3} \right] \text{ Bohr Magnetons where } g = 2$$

$$\simeq \frac{1}{3} \text{ Bohr magnetons}$$