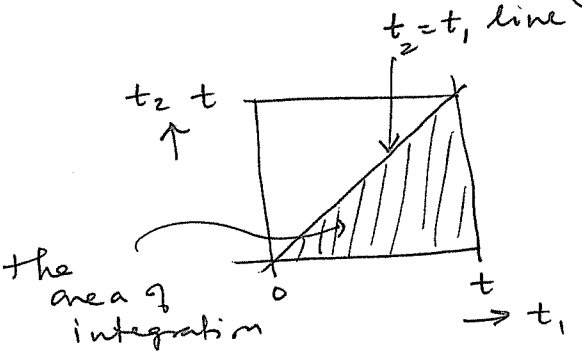
Solution Set 7

(1) Consider the integral

$$I = \int_0^t dt_1 \int_0^{t_1} dt_2 \; f(t_1, t_2)$$



where $f(t_1, t_2)$ is some quantity. The same integral can be written as

$$I = \int_0^t dt_2 \int_{t_2}^t dt_1 \; f(t_1, t_2)$$

(If you don't understand this change of integration stop and think about it. It is important to understand that!).

Since t_1 and t_2 are dummy variables we can relabel

$$I = \int_0^t dt_1 \int_{t_1}^t dt_2 \; f(t_2, t_1) .$$

Now

$$I = \frac{1}{2} \left\{ \int_0^t dt_1 \int_0^{t_1} dt_2 \, f(t_1, t_2) + \int_0^t dt_1 \int_{t_1}^t dt_2 \, f(t_2, t_1) \right\}$$

Now if we choose

$$f(t_1, t_2) = H(t_1) H(t_2)$$

we get

$$I = \frac{1}{2} \left\{ \int_0^t dt_1 \int_0^{t_1} dt_2 \, H(t_1) H(t_2) + \int_0^t dt_1 \int_{t_1}^{t_2} dt_2 \, H(t_2) H(t_1) \right\} .$$

Now since t_1 and t_2 always appear in a time ordered way we can replace

$$H(t_1)\,H(t_2) \longrightarrow \hat{T}\{H(t_1)\,H(t_2)\} \quad \text{in } 1^{st} \text{ integral}$$

$$H(t_2)\,H(t_1) \longrightarrow \hat{T}\{H(t_1)\,H(t_2)\} \quad \text{in } 2^{nd} \text{ integral}$$

and thus.

$$I = \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2) = \frac{1}{2}\left\{\int_0^t dt_1 \int_0^{t_1} dt_2\, \hat{T}\{H(t_1)H(t_2)\} + \int_0^t dt_1 \int_{t_1}^t dt_2\, \hat{T}\{H(t_1)H(t_2)\}\right\}$$

$$= \frac{1}{2}\hat{T}\left\{\int_0^t dt_1\, \hat{H}(t_1) \int_0^t dt_2\, \hat{H}(t_2)\right\}$$

$$= \hat{T}\left\{\frac{1}{2}\left(\int_0^t dt_1\, \hat{H}(t_1)\right)^2\right\}.$$

18.2.2 We want to calculate

$$A_{i\to f} = \left(\frac{-i}{\hbar}\right)\langle 2\ell m|-e\mathcal{E}\hat{Z}|100\rangle \underbrace{\int_{-\infty}^{\infty} dt\, e^{i(\omega_{21}t)}\, e^{-t^2/\tau^2}}_{\sqrt{\pi\tau^2}\, e^{-\tau^2\omega_{21}^2/4}}.$$

only $\ell=1$ $m=0$ contributes.

$$\therefore P(n=2) = \frac{e^2\mathcal{E}^2}{\hbar^2}\underbrace{|\langle 210|\hat{Z}|100\rangle|^2}_{\frac{2^{15}}{3^{10}}a_0^2 \text{ (see prob 17.2.6 in Set 6)}}\pi\tau^2\, e^{-\tau^2\omega_{21}^2/2}$$

where

$$\omega_{21} = \frac{E_{210}-E_{100}}{\hbar}.$$

18.2.3 The velocity of the particle is

$$v = \frac{\pi\hbar}{mL} \Rightarrow T = \frac{L}{v} = \frac{mL^2}{\pi\hbar}$$

If the box expands symmetrically to twice its size the sudden approximation can be used if $\tau \ll T$.

Thus.

$$\psi_{init} = \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} \qquad -\frac{L}{2} \le x \le \frac{L}{2}$$

$$= 0 \qquad \text{outside.}$$

$$\psi_{final} = \psi_{init}.$$

The wave fn of the new ground state

$$\psi_{init} = \sqrt{\frac{1}{L}} \cos\frac{\pi x}{2L} \qquad -L \le x \le L$$

$$= 0 \qquad \text{outside.}$$

$$\therefore \quad Prob = \left\{ \int_{-L/2}^{L/2} dx \, \frac{\sqrt{2}}{L} \underbrace{\left(\cos\frac{\pi x}{2L} \cos\frac{\pi x}{L} \right)}_{\frac{1}{2}\left(\cos\frac{3\pi x}{2L} + \cos\frac{\pi x}{2L}\right)} \right\}^2$$

$$= \left\{ \frac{1}{\sqrt{2}L} \int_{-L/2}^{L/2} dx \left(\cos\frac{3\pi x}{2L} + \cos\frac{\pi x}{2L} \right) \right\}^2$$

$$= \left\{ \frac{1}{\sqrt{2}L} \left\{ \frac{2L}{3\pi} \left(\sin\frac{3\pi x}{2L} \right)_{-L/2}^{L/2} + \frac{2L}{\pi} \left(\sin\frac{\pi x}{2L} \right)_{-L/2}^{L/2} \right\} \right\}^2$$

$$= \left[\frac{\sqrt{2}}{\pi} \left\{ \frac{2}{3} + 2 \right\} \sin\frac{\pi}{4} \right]^2 = \left(\frac{8}{3\pi} \right)^2.$$

18.2.4 In the β-decay of H^3

$$H^3 \longrightarrow He^3 + e^-$$

The K.E of e^- is 16 keV.

Mass of the electron 511 keV

The size of H^3 $a_0 = \frac{\hbar^2}{me^2}$ $\qquad \frac{v}{c} \simeq \sqrt{\frac{16}{511}}$

∴ Time for the β-particle to leave H^3 is

$$\tau = \frac{\hbar^2}{me^2} \cdot \frac{1}{v} = \frac{\hbar^2}{me^2} \sqrt{\frac{511}{16}} \frac{1}{c}$$

Time period of the electron in H^3

$$T = \frac{\hbar}{me^2} \frac{1}{\alpha c}$$

$$\therefore \quad \frac{\tau}{T} = \sqrt{\frac{511}{16}} \cdot \frac{1}{137} \simeq 0.04 \ll 1.$$

So we can use the sudden approx.

The Hamiltonian of the Helium atom is

$$H_{He} = \frac{p^2}{2\mu} - \frac{Ze^2}{r} \qquad \text{where } Z = 2.$$

ground state of He: $\psi_{1,00} = \frac{Z^{3/2}}{(\pi a_0^3)^{1/2}} e^{-rZ/a_0}$

ground state of H^3: $\psi_{1,00} = \frac{1}{(\pi a_0^3)^{1/2}} e^{-r/a_0}$.

$$A_{i\to f} \text{ (amplitude)} = \int d^3r \, \frac{Z^{3/2}}{(\pi a_0^3)} \, e^{-r(Z+1)/a_0}$$

$$= \frac{4\pi Z^{3/2}}{\pi a_0^3} \cdot \int_0^\infty dr \, r^2 \, e^{-r(Z+1)/a_0}$$

$$= \frac{8 Z^{3/2}}{(Z+1)^3} = \frac{8 \cdot 2\sqrt{2}}{3^3} .$$

$\therefore$ Amplitude to be in the ground state $= 16\sqrt{2}/27$.

Probability to be in some other state. $|n\ell m\rangle$ is

$$\psi_{n\ell m} = R_{n\ell}(r, Z) \, Y_{\ell m}(\theta, \phi) .$$

So probability to be in $|n=16, \ell=3, m=0\rangle$ state of He^{3+} is

$$\langle n=16, \ell=3, m=0 |_{He} \; | n=1, \ell=0, m=0 \rangle_{H^3} = 0$$

The $Y_{\ell m}(\theta, \phi)$ integration will give zero!

18.2.6 If $H_1(t) = H_1 \delta(t)$, then we see that-

$$|\psi(t=\tfrac{\varepsilon}{2})\rangle = |\psi(t=-\tfrac{\varepsilon}{2})\rangle + \left(\frac{-i}{\hbar}\right) \int_{-\varepsilon/2}^{\varepsilon/2} dt \; H_1 \delta(t) \; |\psi(t=-\tfrac{\varepsilon}{2})\rangle$$

$$|\psi(t=\tfrac{\varepsilon}{2})\rangle = |i^0\rangle + \left(\frac{-i}{\hbar}\right) H_1 |i^0\rangle$$

$\therefore$ amplitude to go to $|f^0\rangle$ $f \neq i$ is

$$A_{i\to f} = \left(\frac{-i}{\hbar}\right) \langle f^0 | H_1 | i^0 \rangle .$$