Solution Set 9

(1.) Let us solve the differential equation (1.)

$$[\nabla^2 + k^2]\, G(\vec{r}) = \delta^3(\vec{r})$$

We see that except at the origin the above equation is the same as the equation one would solve for the wave function of free particles!

$$[\nabla^2 + k^2]\, G(\vec{r}) = 0 \qquad \text{for } |\vec{r}| \neq 0 .$$

We know that the general solution to this equation (in spherical polar coordinates) is given by

$$G(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A_l j_l(kr) + B_l n_l(kr)]\, Y_l^m(\theta, \phi)$$

where $j_l(kr)$ and $n_l(kr)$ are spherical Bessel and spherical Neumann functions.

Now for $r \to \infty$ we want the "radial" part of the Green's function to be of the form $\frac{e^{ikr}}{r}$.

We know for $r \to \infty$

$$j_l(kr) \longrightarrow \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right)$$

$$n_l(kr) \longrightarrow -\frac{1}{kr} \cos\left(kr - \frac{l\pi}{2}\right).$$

Thus we need $A_l = -B_l$ to cancel the $\frac{e^{-ikr}}{r}$ part. Thus

$$G(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l [j_l(kr) - n_l(kr)]\, Y_l^m(\theta, \phi) .$$

Now $\delta^3(\vec{r})$ has an integrable singularity as $r \to 0$! which means $\delta^3(\vec{r}) \to \frac{1}{r^2}$ as $r \to 0$! This means the coefficient of $n_\ell(kr)$ must be zero for all $\ell > 0$!

Remember

$$n_\ell(kr) \sim \frac{1}{(kr)^{\ell+1}}$$

and $$\nabla^2 n_\ell(kr) \sim \frac{1}{(kr)^{\ell+2}}$$

Hence

$$G(\vec{r}) = A_0 \frac{e^{ikr}}{kr}$$

To determine A_0 we integrate the original eq. over V (a small sphere of radius ε) i.e...

$$\int_V d^3r \, [\nabla^2 + k^2] G(\vec{r}) = \int_V d^3\vec{r} \, \delta^3(\vec{r})$$

Substituting we get

$$\int_V d^3r \, \vec{\nabla} \cdot \vec{\nabla} G(\vec{r}) + k^2 \int_V d^3r \, G(\vec{r}) = 1 .$$

$\int_V d^3r \, G(\vec{r}) \to 0$ (taking $\varepsilon \to 0$)

$$= \int_S d\Omega \, \hat{e}_r \cdot \vec{\nabla} G(\vec{r}) = 1$$

S ↑ Surface of the sphere

$$\Rightarrow \quad A_0 = -\frac{k}{4\pi} \quad \Rightarrow \quad \boxed{G(\vec{r}) = -\frac{1}{4\pi} \frac{e^{ikr}}{r}}$$

The Born approximation is valid as long as

$$\frac{2\mu}{\hbar^2 k}\left|\int_0^\infty e^{ikr'}\sin kr'\,V(r)\,dr'\right| << 1 .$$

for a spherically symmetric potential. For the Yukawa potential this condition becomes

$$\frac{2\mu|V_0|}{\hbar^2 k\alpha}\left|\int_0^\infty e^{ikr'}\sin kr'\,\frac{e^{-\alpha r'}}{r'}\,dr'\right| << 1$$

When $\frac{k}{\alpha} << 1$ we can write the above condition as

$$\frac{2\mu V_0}{\hbar^2 k\alpha}\left|\int_0^\infty e^{i\frac{k}{\alpha}t}\sin\frac{k}{\alpha}t\;e^{-t}\,\frac{dt}{t}\right| << 1$$

where we substituted $\alpha r' = t$

which then leads to

$$\boxed{\frac{2\mu|V_0|}{\hbar^2\alpha^2} << 1} \quad \text{(Low energies)} .$$

When $\frac{k}{\alpha} >> 1$ we can the write the condition as

$$\frac{2\mu|V_0|}{\hbar^2 k\alpha}\left|\int_0^\infty e^{it}\,\frac{\sin t}{t}\;e^{-\frac{\alpha}{k}t}\,dt\right|$$

Now

$$\int_0^\infty e^{it}\,\frac{\sin t}{t}\,e^{-\frac{\alpha}{k}t}\,dt = \int_0^\infty \left\{\frac{1}{2}\,\frac{\sin 2t}{t}\,e^{-\frac{\alpha}{k}t} + i\,\frac{\sin^2 t}{t}\,e^{-\frac{\alpha}{k}t}\right\} dt$$

From the "table of integrals" (Gradshteyn & Ryzhik) we see that-

$$\int_0^\infty \frac{\sin 2t}{t}\,e^{-\frac{\alpha}{k}t} = \tan^{-1}\left(\frac{2k}{\alpha}\right)$$

$$\int_0^\infty \frac{\sin^2 t}{t}\,e^{-\frac{\alpha}{k}t} = \frac{1}{4}\ln\left(\frac{(2k/k)^2+4}{(\frac{\alpha}{k})^2}\right)$$

In the $\frac{\alpha}{k} \to 0$ limit we see that-

$$\int_0^\infty e^{it}\,\frac{\sin t}{t}\,e^{-\frac{\alpha}{k}t}\,dt \simeq -\frac{i}{2}\ln\left(\frac{\alpha}{k}\right) \simeq \frac{i}{2}\ln\left(\frac{k}{\alpha}\right)$$

* Also see the end of Problem

∴ The condition is given by.

$$\frac{\mu |V_0|}{\hbar^2 k \alpha}\ln\left(\frac{k}{\alpha}\right) \ll 1$$

{ You can roughly estimate the above integral as follows:

$$\int_0^\infty \frac{e^{it}\sin t}{t}\,e^{-\frac{\alpha}{k}t}\,dt \simeq \int_0^{k/\alpha} \frac{e^{it}\sin t}{t}\,dt \xrightarrow[k/\alpha \to \infty]{\text{diverges for}} \text{as}\left(\ln\frac{k}{\alpha}\right) \}.$$

When there are bound states

$$\frac{2m|V_0|}{\hbar^2\alpha^2} \geq 2.7 .$$

Thus the Born approximation is violated at

necessarily
low energies but not at high energies! This is

reasonable since the bound states affect only low energies.

Thus the sign of V_0 should not matter at high energies!

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* We can also estimate the integral $\int_0^\infty e^{it}\frac{\sin t}{t}e^{-\frac{\alpha}{k}t}dt$ more easily! since the exponential is simply cutting off the t integral we can approx it by

$$\simeq \int_0^{k/\alpha} e^{it}\frac{\sin t}{t}dt \approx \int_0^{k/\alpha}\frac{e^{2it}}{2it} - \int_0^{k/\alpha}\frac{1}{2it}dt$$

We only want the singular part as $k/\alpha \to \infty$. this is obtained from the 2nd integral (first one is convergent at ave k/α due to oscillating part)
The second integral is $\frac{i}{2}\ln k/\alpha$ as expected!

(3) The Lippmann-Schwinger formalism in 1-d can be formulated as follows:

We want to solve

$$\hat{H}|\psi_{E_k}\rangle = E_k|\psi_{E_k}\rangle$$

where $\hat{H} = \hat{H}_0 + \hat{V}$

we know $\hat{H}_0|\phi_k\rangle = E_k|\phi_k\rangle$. (Free problem) .

We want to solve

$$|\psi_{E_k}\rangle = |\phi_{E_k}\rangle + |\psi_{sc}\rangle .$$

$$\therefore \cancel{H_0|\phi_{E_k}\rangle} + V|\phi_{E_k}\rangle + H_0|\psi_{sc}\rangle + V|\psi_{sc}\rangle = \cancel{E_k|\phi_{E_k}\rangle} + E_k|\psi_{sc}\rangle .$$

$$(E_k - H_0)|\psi_{sc}\rangle = V|\psi_{E_k}\rangle .$$

$$\therefore |\psi_{sc}\rangle = \frac{1}{E_k - H_0} V|\psi_{E_k}\rangle .$$

$$\therefore \boxed{|\psi_{E_k}\rangle = |\phi_k\rangle + \frac{1}{E_k - H_0} V|\psi_{E_k}\rangle} .$$

Thus we need to find the prescription for handling the singularity in $\frac{1}{E_k - H_0}$.

In 1-d we have

$$\psi_{E_k}(x) = \phi_{E_k}(x) + \langle x| \frac{1}{E_k - H_0} \hat{V}|\psi_{E_k}\rangle .$$

Let us assume that the prescription $E_k \to E_k + i\varepsilon$ works and find out if the Green's function

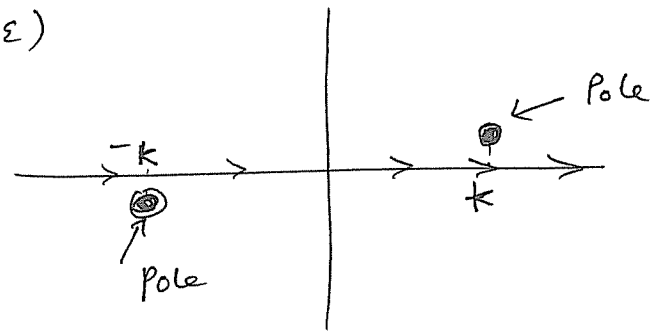
$$G^{(+)}(x,x') = \frac{\hbar^2}{2m} \langle x| \frac{1}{E_k - H_0 + i\varepsilon} |x'\rangle .$$

has the right boundary condition.

Now $G^{(+)}(x,x') = \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{\frac{i}{\hbar}p(x-x')} \frac{1}{\left(E_k - \frac{p^2}{2m} + i\varepsilon\right)}$

Let us substitute $\frac{p}{\hbar} = q$ and $E_k = \frac{\hbar^2 k^2}{2m}$. Then we get

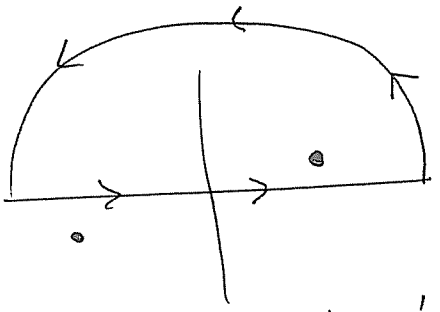
$$G^{(+)}(x,x') = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iq(x-x')}}{(q^2 - k^2 - i\varepsilon)}$$

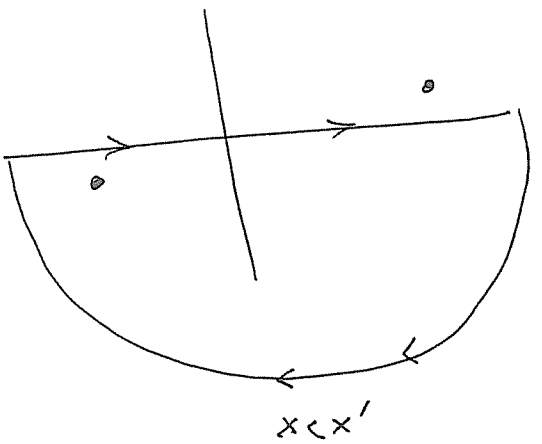


The poles are at

$$q = \pm (k + i\varepsilon')$$

Now if $x > x'$ we can close the contour in the upper half plane and when $x < x'$ we can close the contour in the lower half plane.





Thus

$$G^{(+)}(x,x') = -\oint \frac{dq}{2\pi} \frac{e^{iq(x-x')}}{(q-k-i\varepsilon')(q+k+i\varepsilon')}$$

$$= \begin{cases} -\frac{i}{2k} e^{ik(x-x')} & x > x' \\ -\frac{i}{2k} e^{-ik(x-x')} & x < x' . \end{cases}$$

Thus $G^{(+)}(x,x') = \frac{-1}{2k} e^{ik|x-x'|}$

Thus the integral equation for $\psi_k^+(x)$ is

$$\psi_k^+(x) = \psi_{inc}(x) + \int dx' G(x,x') V(x') \psi_k(x') .$$

(b) Now we are given $V(x)$, so

$$\psi_k^+(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{2m}{\hbar^2}\int dx' \left(-\frac{i}{2k} e^{ik|x-x'|}\right)\left(-\frac{\gamma\hbar^2}{2m}\right)\delta(x')\,\psi_k(x')$$

$$\therefore \psi_k^+(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k} e^{ik|x|}\psi_k^+(0) .$$

Now substituting $x = 0$ we see

$$\psi_k^+(0)\left[1 - \frac{i\gamma}{2k}\right] = \frac{1}{\sqrt{2\pi}}$$

$$\therefore \psi_k^+(0) = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{i\gamma}{2k}\right)^{-1} .$$

Hence we see

$$\psi_k^+(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k}\frac{1}{\sqrt{2\pi}}\cdot\left(1 - \frac{i\gamma}{2k}\right)^{-1} e^{ik|x|}$$

for $x \to +\infty$ we see

$$\psi_k^+(x) = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{\frac{i\gamma}{2k}}{1 - \frac{i\gamma}{2k}}\right) e^{ikx}$$

for $x \to -\infty$ we see

$$\psi_k^+(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{1}{\sqrt{2\pi}} \frac{\frac{i\gamma}{2k}}{\left(1 - \frac{i\gamma}{2k}\right)} e^{-ikx}$$

$$\therefore \quad R = \left| \frac{\frac{i\gamma}{2k}}{\left(1 - \frac{i\gamma}{2k}\right)} \right|^2 = \left(\frac{\gamma^2}{4k^2 + \gamma^2} \right)$$

and $$T = \left| \frac{1}{1 - \frac{i\gamma}{2k}} \right|^2 = \left(\frac{4k^2}{4k^2 + \gamma^2} \right) .$$

Exercise 19.5.6 (Optical Theorem)

(1) The wavefunction is given by

$$\psi(\vec{r}) = \psi_{inc} + \psi_{sc}$$

where
$$\left.\begin{aligned} \psi_{inc} &= e^{ikr\cos\theta} \\ \psi_{sc} &= \frac{e^{ikr}}{r} f(\theta) . \end{aligned}\right\} \text{ for } r \to \infty .$$

The current density due to the interference term is given by

$$(\vec{J})^{int} = \frac{\hbar}{2\mu i}\left[\psi_{inc}^* \vec{\nabla}\psi_{sc} + \psi_{sc}^* \vec{\nabla}\psi_{inc} - \vec{\nabla}\psi_{inc}^* \psi_{sc} - \vec{\nabla}\psi_{sc}^* \psi_{inc} \right] .$$

$$= \frac{\hbar}{\mu} Im\left[\psi_{inc}^* \vec{\nabla}\psi_{sc} + \psi_{sc}^* \vec{\nabla}\psi_{inc} \right] .$$

If we look at the radial part at $r \to \infty$ we just consider

$$\vec{\nabla}\psi_{sc} \to \frac{\partial}{\partial r}\psi_{sc} \quad \text{and} \quad \vec{\nabla}\psi_{inc} \to \frac{\partial}{\partial r}\psi_{inc}$$

Substituting we get

$$j_r^{int} \underset{r\to\infty}{\sim} \frac{\hbar k}{\mu r} Im\left[i e^{ikr(1-\cos\theta)} f(\theta) + e^{ikr(\cos\theta - 1)} f^*(\theta)\cos\theta \right]$$

(2) As long as $\theta \neq 0$ we see that $(1-\cos\theta)$ is not zero.

This means j_r^{int} oscillates wildly over any small change in the angle (when $r \to \infty$) and hence integrates to zero!

(3) When $\theta \simeq 0$ $\quad (1-\cos\theta) \simeq \frac{\theta^2}{2}$. This means

$$\int_{\substack{\text{forward} \\ \text{cone}}} j_r^{int}\, r^2 d\Omega \simeq \left(\frac{\hbar k}{\mu}\right) \cdot r \cdot \mathrm{Im}\left[i \int_0^{2\pi}\int_0^{\theta} e^{-ikr\theta^2/2}\, \theta\, d\theta\, d\phi\, f^*(0) + i \int_0^{2\pi}\int_0^{\theta} e^{ikr\theta^2/2}\, \theta\, d\theta\, d\phi\, f(0) \right]$$

$\theta = 0$ to θ (small)
$\phi = 0$ to 2π

$$\simeq \left(\frac{\hbar k}{\mu}\right) r\, 2\pi\, \mathrm{Im}\left[i f^*(0) \left(\frac{e^{-ikr\theta^2/2}}{ikr}\right)_0^{\theta} + i f(0) \left(\frac{e^{ikr\theta^2/2}}{ikr}\right)_0^{\theta} \right]$$

Now in the $r \to \infty$ limit we can set $e^{\pm ikr\theta^2/2} \to 0$ (using argument in Shankar page 66 on evaluating rapidly oscillating functions).

$$\simeq -\left(\frac{\hbar k}{\mu}\right) \cdot \frac{4\pi}{k}\, \mathrm{Im}\, f(0)$$

Exercise 19.6.1 ① We know from (19.6.16) and (19.6.17) that

$$\theta_L = \theta_{CM}/2 \qquad \phi_L = \phi_{CM}.$$

Further

$$\left(\frac{d\sigma}{d\Omega_L}\right) \sin\theta_L \, d\theta_L \, d\phi_L = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \sin\theta_{CM} \, d\theta_{CM} \, d\phi_{CM}.$$

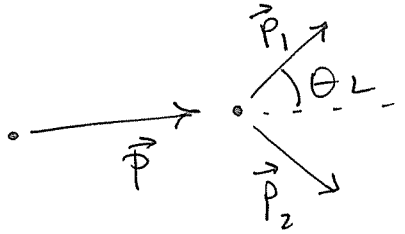
From the transformation above we see

$$\left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right) \cdot \frac{\sin\theta_{CM}}{\sin\theta_L} \cdot \left[\frac{(d\theta_{CM}\, d\phi_{CM})}{(d\theta_L\, d\phi_L)}\right]$$

↑ Jacobian of the transformation

$$\left(\frac{d\sigma}{d\Omega_L}\right) = \left(\frac{d\sigma}{d\Omega_{CM}}\right)\Bigg|_{\substack{\theta_{CM}=2\theta_L \\ \phi_{CM}=\phi_L}} \cdot \, 4\cos\theta_L$$

② Let $\vec{P}$ be the momentum of the projectile before collision.



Let $\vec{P}_1$ be the momentum of the projectile after collision and $\vec{P}_2$ be the momentum of the target after collision.

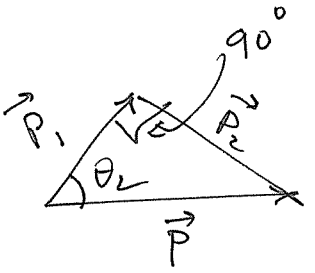
From momentum conservation

$$\vec{P} = \vec{P}_1 + \vec{P}_2.$$

From Energy (kinetic Energy) conservation (Non-relativistic)

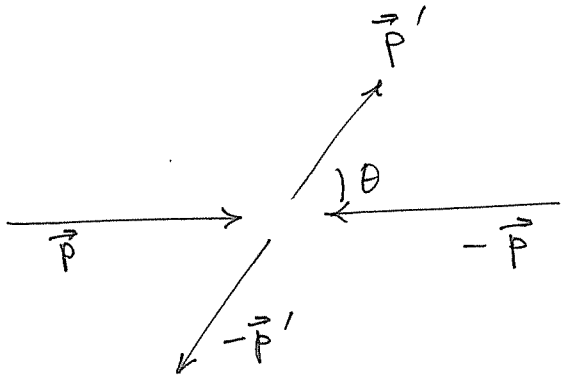
$$P^2 = P_1^2 + P_2^2.$$

Thus $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}$ form a right triangle as shown below



Clearly $0 \le \theta_L \le \pi/2$!

(3) For scatteng of unequal masses.



$$\tan\theta = \frac{\sqrt{P_x'^2 + P_y'^2}}{P_z'} \qquad \tan\phi = \frac{P_y'}{P_x'} \qquad P = \sqrt{P_x^2 + P_y^2 + P_z'^2}$$

The velocity of the target in CM frame $= -\frac{\vec{P}}{m_2}$.

In the Lab frame

$$\vec{P}_{Proj} = \vec{P} + \frac{m_1}{m_2}\vec{P}$$

$$\vec{P}_{target} = -\vec{P} + \frac{m_2}{m_2}\vec{P} = 0$$

$$\vec{P}'_{Proj} = \vec{P}' + \frac{m_1}{m_2}\vec{P}$$

$$\vec{P}'_{target} = -\vec{P}' + \vec{P}$$

$$\left\{ \begin{array}{l} (\vec{P}'_{proj})_x = P_x' \\ (\vec{P}'_{Proj})_y = P_y' \\ (\vec{P}'_{proj})_z = P_z' + \frac{m_1}{m_2}P \end{array} \right.$$

which means

$$\tan\theta_L = \frac{\sqrt{(\vec{P}'_{Proj})_x^2 + (\vec{P}'_{Proj})_y^2}}{(\vec{P}'_{Proj})_z}$$

$$\tan\theta_L = \frac{\sqrt{P_x'^2 + P_y'^2}}{P_z' + \frac{m_1}{m_2}P} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

Exercise 19.6.3 Consider two fermions (spin-$\frac{1}{2}$) in the triplet state scattering off of each other. Then the wave function will be antisymmetric in space. This means.

$$\frac{d\sigma}{d\Omega} = \left| f(\theta,\phi) - f(\pi-\theta, \phi+\pi) \right|^2$$

If $f(\theta,\phi) \simeq f(\theta)$ then.

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) - f(\pi-\theta) \right|^2$$

Clearly at $\theta = \pi/2$ $\left(\frac{d\sigma}{d\Omega}\right) = 0$!