Physics 828 Sketch of Solution to Set 2

Shankar 12.3.7

(1) The two-dimensional harmonic oscillator is obviously invariant under rotations about the z-axis: the magnitude of the position and momenta are unaltered by rotations around the z-axis. Therefore, the Hamiltonian commutes with the generator of rotations about the z-axis, L_z.

(2) So we write $\psi(\rho,\phi) = e^{im\phi} R_{Em}(\rho)$ where m is an integer, positive or negative. In two dimensions since we know the Laplacian we have

$$-\frac{\hbar^2}{2\mu}\left(R''_{Em} + \frac{1}{\rho}R'_{Em}\right) + \left(\frac{\hbar^2 m^2}{2\mu\rho^2} + \frac{1}{2}\mu\omega^2\rho^2\right) R_{Em}(\rho) = E R_{Em}\,.$$

For small ρ assuming that $R_{Em}(\rho) \propto \rho^k$ (derivatives decrease the power by unity increasing its importance for small ρ etc.) we can neglect the potential energy term and the constant term on the right-hand side. Thus we have

$$R''_{Em} + \frac{R'_{Em}}{\rho} \propto \frac{m^2}{\rho^2} R_{Em} \;\Rightarrow\; [k(k-1)+k] = m^2 \;\Rightarrow\; k^2 = m^2\,.$$

For the wave function to be normalizable, for $\int_0^\infty d\rho\,\rho\, R^2_{Em}(\rho)$ to be finite at the lower limit $k \geq 0$. Thus we have $R_{Em}(\rho) \overset{\rho\to 0}{\longrightarrow} \rho^{|m|}$.

(3) For large ρ, terms with inverse powers of ρ including the centrifugal term and also the constant term on the right-hand side can be neglected. So we have

$$R''_{Em} = \frac{\mu^2\omega^2\rho^2}{\hbar^2} R_{Em}\,;$$

this is identical to the one-dimensional oscillator equation. See the careful analysis on page 191. Up to powers of ρ the solution is[1]

$$R_{Em}(\rho) \overset{\rho\to\infty}{\longrightarrow} e^{-\frac{\mu\omega}{2\hbar}\rho^2}\,.$$

So we write as instructed

[1] $$R'_{Em} = -\frac{\mu\omega}{\hbar}\,\rho\, e^{-\frac{\mu\omega}{2\hbar}\rho^2} \text{ and } R''_{Em} = \frac{\mu\omega}{\hbar}\left(-1 + \frac{\mu\omega}{\hbar}\rho^2\right) e^{-\frac{\mu\omega}{2\hbar}\rho^2}\,.$$
Thus we can neglect the constant term in R'' and recover the solution. One can use this to check that terms we have neglected are indeed small compared to the terms we have retained. When you neglect terms it is a good idea to substitute the solution you have obtained and check that the terms you have neglected are indeed smaller.

$$R_{Em}(\rho) \,=\, U_m(\rho)\,\rho^{|m|}\,e^{-\frac{\mu\omega}{2\hbar}\rho^2}\,.$$

(4) Use the dimensionless variable $\epsilon \,=\, \frac{E}{\hbar\omega}$ and $y^2 \,=\, \frac{\mu\omega}{\hbar}\rho^2$. Dividing the radial equation by $\hbar\omega$ we have dropping the annoying subscripts

$$\left[-\frac{1}{2}\left(\frac{d^2}{dy^2} \,+\, \frac{1}{y}\frac{d}{dy} \,-\, \frac{m^2}{y^2}\right) \,+\, \frac{1}{2}y^2\right] R(y) \,=\, \epsilon\, R(y)\,.$$

(5) We do the substitution and do elementary calculus and obtain the result given:

$$U'' \,+\, \left[\left(\frac{2|m|+1}{y}\right) \,-\, 2y\right] U' \,+\, (2\epsilon - 2|m| - 2)\,U \,=\, 0\,.$$

(6) We substitute $U(y) \,=\, \sum_{r=0}^{\infty} C_r y^r$ and collect the coefficient of y^r. The second derivative reduces the power by two and so we use the C_{r+2} term etc.

$$(r+2)(r+1)C_{r+2} \,+\, (2|m|+1)(r+2)C_{r+2} - 2rC_r \,+\, (2\epsilon - 2|m| - 2)C_r \,=\, 0$$

yielding a two-term recursion relation.

(7) We write this as

$$\frac{C_{r+2}}{C_r} \,=\, -\frac{(2(\epsilon - |m| - r - 1)}{(r+2)(2|m|+r+2)}\,.$$

First if C_0 is given C_2 and the other even terms can be computed. As $r \to \infty$ we have

$$\frac{C_{r+2}}{C_r} \,\to\, \frac{2}{r}\,.$$

This implies that $U(y)$ grows as e^{y^2} which overwhelms the $e^{-y^2/2}$ in R pushing it out of the Hilbert space. So the series must terminate. Thus the boundary condition at infinity leads as usual to energy quantization.

What about the odd terms? A series only with odd terms (set $C_0 = 0$ so that all even terms vanish) is inconsistent since then $U(y) \,\sim\, y$ for small y and thus $R(\rho) \,\sim\, \rho^{|m|+1}$ inconsistent with our earlier result in (2). This appears to be suggested as an argument. What if one starts with C_0 and C_1 non-zero? Substituting into the equation for U we find that the $(1/y)(dU/dy)$ leads to the term C_1/y and there is no other source of y^{-1} terms. Thus $C_1 \,=\, 0$ and therefore, all odd terms vanish.

Therefore, we have $r = 2k$ and the termination of the series condition yields

$$\epsilon \,=\, 1 \,+\, r \,+\, |m| \,=\, |m| \,+\, 2k \,+\, 1 \,\equiv\, (n+1)$$

with $k = 0, 1, , 2, \cdots$.

(8) Since $|m| \,=\, n \,-\, 2k$ for a given n (i.e., for a given energy) the maximum value of m is n which occurs for $k = 0$, The azimuthal quantum number m decreases by steps of 2 until m reaches the value $-n$. It is easy to se that there are $n+1$ allowed values of m yielding a degeneracy of $n+1$. In Cartesian coordinates the energy is $n_x + n_y + 1$ in units of $\hbar\omega$. So the degeneracy corresponds to the number of ways in which we can choose two non-negative integers to add up to n. We can choose n_x to be any integer from 0 to n and $n_y = n - n_x$. This yields the same degeneracy.

Shankar 12.6.1

(1) Since there is no angular dependence $\ell = 0$.

(b) We have $R(r) \,\propto\, e^{-r/a_0}$ and for large r (retaining only the dominant terms)

$$-\frac{\hbar^2}{2m} R'' \;=\; E R(r)\,.$$

Substituting the given form we obtain $\boxed{E \,=\, -\frac{\hbar^2}{2ma_0^2}}$.

(c) Clearly the R'' term and the energy term cancel for all r. If the equation is valid for all r we must have

$$-\frac{\hbar^2}{2m}\frac{2}{r}\, R' \,+\, V(r)\, R(r) \,=\, 0\,.$$

Substituting $R(r) \,=\, e^{-r/a_0}$ we obtain

$$\frac{\hbar^2}{ma_0 r} \,+\, V(r) \,=\, 0 \;\;\Rightarrow\;\; V(r) \,=\, -\frac{\hbar^2}{ma_0 r}\,.$$

Shankar 13.1.1 and 13.1.3 You should be able to fill in the steps. Here are some steps dropping some subscripts for notational simplicty.

$$v \,=\, \sum_{k=0}^{\infty} C_k\, \rho^{k+\ell+1}\,.$$

Substituting into

$$v'' \,-\, 2v' \,+\, \left(\frac{e^2\lambda}{\rho} \,-\, \frac{\ell(\ell+1)}{\rho^2}\right) v \;=\; 0 \tag{1}$$

we extract the coefficient of ρ^{k+l} carefully. Since two derivatives reduce the power of ρ by two we should start from the term with $\rho^{k+\ell+2}$ with coefficient C_{k+1} for the first term and similarly for the last term. For the second and third terms which reduce the power of ρ by unity we can start with the $\rho^{k+\ell+1}$ term with coefficient C_k. Thus we have

$$(k+\ell+2)(k+\ell+1)C_{k+1} \,-\, 2(k+\ell+1)C_k \,+\, e^2\lambda C_k \,-\, \ell(\ell+1)C_{k+1} \;=\; 0 \tag{2}$$

which yields

$$\frac{C_{k+1}}{C_k} \;=\; \frac{-q^2\lambda \,+\, 2(k+\ell+1)}{(k+\ell+2)(k+\ell+1) \,-\, \ell(\ell+1)} \tag{3}$$

Since $\lambda^2 \,=\, \frac{2m}{\hbar^2 W}$ from 13.1.9. For the numerator to vanish we have

$$e^4\lambda^2 \;=\; -\frac{2me^2}{\hbar^2 E} \;=\; 4(k+\ell+1)^2\,.$$

This yields 13.1.14.

Mathematical aside: Here is a different representation using classical mathematical physics. Note that the function $L(\rho) \,\equiv\, \sum_{k=0}^{\infty} C_k\,\rho^k$ obeys the equation

$$\rho L''(\rho) \,+\, [\,2(\ell+1) \,-\, 2\rho\,]L'(\rho) \,+\, (q^2\lambda \,-\, 2(\ell \,+\, 1))L(\rho) \;=\; 0 \quad .$$

Substituting $q^2\lambda \,=\, 2n$ (where n is an integer eventually) we have

$$\frac{1}{2}\rho L'' \,+\, [\,(\ell+1) \,-\, \rho\,]L' \,+\, (n \,-\, (\ell \,+\, 1))L \;=\; 0 \quad .$$

Let $z = 2\rho$ we have

$$z\frac{d^2L}{dz^2} \,+\, [2(\ell+1) \,-\, z]\frac{dL}{dz} \,-\, (\ell+1-n)L \;=\; 0\,.$$

We know (with a good mathematical methods course) that the general solution to

$$zw'' \,+\, (c-z)\,w' \,-\, aw \;=\; 0$$

is given by the confluent hypergeometric function $w \,=\, {}_1F_1(a;c;z)$. Thus we find that the solution $L(\rho)$ is ${}_1F_1(\ell+1-n;\, 2\ell+2;\, 2\rho)$. The particular terminating (for integer n)

confluent hypergeometric function can be related to the *associated Laguerre polynomial.*

Shankar 13.3

We are considering the case $n = 2$ and $\ell = 1$ and thus $k = 0$ in the notation of the text. So we have $W = \frac{me^4}{8\hbar^2}$ from 13.1.4 and form 13.1.6

$$\rho = \sqrt{\frac{2mW}{\hbar^2}}\, r = \frac{me^2}{2\hbar^2}\, r = \frac{r}{2a_0}$$

using 13.1.24. From 13.1.10 since $k = 0$ and $\ell = 1$ we have $v = C_0\rho^2$ and thus

$$R(r) = \frac{U(r)}{r} = C\, e^{-\frac{r}{2a_0}}\, r$$

where C is an overall constant. We know that $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$ from 12.5.39 which is normalized. So all we need is that $Ce^{-r/(2a_0)}r$ is notmalized when integrated over the radial coordinate. We have

$$C^2 \int_0^\infty dr\, r^2\, r^2\, e^{-\frac{r}{a_0}} = C^2\, 24\, a_0^5\,.$$

Thus $C = \sqrt{\frac{1}{24a_0^3}} \times 1/a_0$ and including the normalization from the spherical harmonic yields the quoted answer.

Shankar 13.5

Since we are asked to compute $\langle\Omega\rangle$ for stationary states $|n\ell m\rangle$ its time derivative vanishes. Thus we have $\langle[\Omega,\, H]\rangle = 0$ by Ehrenfests' theorem. So we compute the commutator for $\Omega = \vec{R}\cdot\vec{P}$ as ordered. We calculate (using the summation convention)

$$\left[R_jP_j, \frac{P_iP_i}{2m}\right] = \left[R_j, \frac{P_iP_i}{2m}\right] P_j = 2i\hbar\, \frac{\vec{P}\cdot\vec{P}}{2m} = 2i\hbar\, T\,.$$

We have used

$$[R_j,\, P_iP_i] = P_i[R_j,\, P_i] + [R_j,\, P_i]P_i = 2i\hbar\, P_j\,.$$

We consider the potential energy term next:

$$[R_jP_j,\, V(R)] = R_j[P_j,\, V(R)] = R_j\left(-i\hbar\frac{d}{dR_j}V(R)\right)\,.$$

Note that $[P_i,\, V(R)]$ can be evaluated in the coordinate representation (in Cartesian coordinates) by acting on a function $f(r)$:

$$[P_i,\, V(r)]f(r) \,=\, -i\hbar\left(\frac{\partial}{\partial r_i}V(r)f(r) \,-\, V(r)\,\frac{\partial}{\partial r_i}V(r)f(r)\right] \,=\, -i\hbar\,\frac{\partial}{\partial r_i}V(r) \,=\, -i\hbar\vec{R}\cdot\vec{\nabla}V.$$

Thus we can write formally $[P_i,\, V(R)] \,=\, -i\hbar\partial V(R)/\partial R_i$. So we need to evaluate $\vec{R}\cdot\vec{\nabla}V(R)$. In the coordinate representation using spherical coordinates for central potentials this is just $rV'(r)$. For the Coulomb potential we obtain $-V(r)$ and including the factor of $-i\hbar$ we obtain $i\hbar\,V(r)$. Substituting into the basic relation we have

$$\langle 2T \,+\, V\rangle \,=\, 0$$

as asserted. If $V(R) \,\propto\, R^n$, $rV'(r) \,=\, nV$ and thus we obtain $\langle T\rangle \,=\, \frac{n}{2}\,\langle V\rangle$.