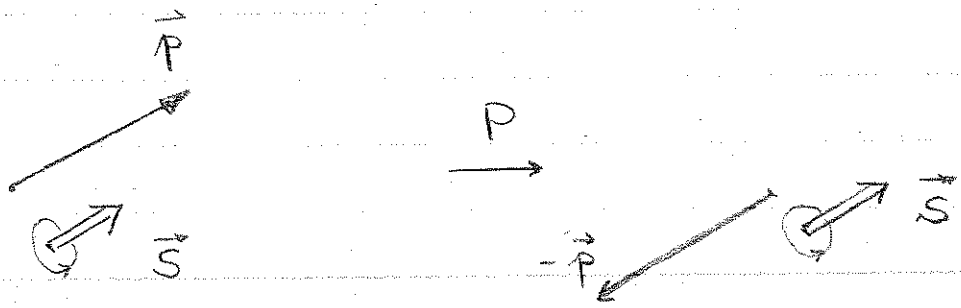
# Homework Set # 12

S 11.4.3




P = parity  $\vec{p}$ = momentum

$\vec{S}$ = spin angular momentum

Under parity $\vec{r} \to -\vec{r}$, $t \to t$

$$\vec{p} = m \frac{d\vec{r}}{dt} \to -\vec{p}$$

$$\vec{L} = m\, \vec{r} \times \frac{d\vec{r}}{dt} \to \vec{L}$$

$$\vec{S} \to \vec{S} \quad (\text{like } \vec{L})$$

If parity is conserved, the state with $\vec{S}$ anti-parallel to $\vec{p}$ is equally likely than the state with $\vec{S}$ parallel to $\vec{p}$ (assuming that the rest of the experiment is parity symmetric).

S11.4.4

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

under reflection in a mirror located in the $(x, y)$ plane

$$\begin{pmatrix} x \\ y \\ -z \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta\, x + \sin\theta\, y \\ -\sin\theta\, x + \cos\theta\, y \\ -z \end{pmatrix}$$

under rotation by angle $\theta$ around the $z$ axis

For $\theta = 180^\circ$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ -z \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = -\vec{r}$$

11.4.4. parity $\vec{r} \to -\vec{r}$ ∴ parity transformation

$$\overline{\Pi} = -\underline{I}_3$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \therefore \quad R_z(\overline{\Pi}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \quad R_z(\theta)\,\overline{\Pi} = \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} = \overline{\Pi}\, R_z(\theta) \qquad R_z(\theta)\,\overline{\Pi}\,\vec{r} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix} \checkmark$$

S12.3.3

$$\psi(\rho,\varphi) = A e^{-\rho^2/2\Delta^2} \cos^2\varphi$$

$$= R(\rho)\left( e^{2i\varphi} + 2 + e^{-2i\varphi} \right)$$

$$= \sqrt{2\pi}\, R(\rho) \left[ \psi_{l_z=2\hbar}(\varphi) + 2\psi_{l_z=0}(\varphi) + \psi_{l_z=-2\hbar}(\varphi) \right]$$

where: $\psi_{l_z=m\hbar}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$

The probabilities to find the particle in the states of angular momentum

$l_z = \quad 2\hbar \quad , \quad 0 \quad , \quad -2\hbar$

are in the ration

$$1 : 2^2 : 1$$

The sum of probabilities must be one.

Hence the probabilities are

$$1/6 \; , \; 4/6 = 2/3 \; , \; 1/6$$

S 12.3.6

$$L = \frac{\mu}{2} a^2 \dot{\varphi}^2 = L(\dot{\varphi})$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \mu a^2 \dot{\varphi} = l_z$$

$$H(p_\varphi) = \dot{\varphi} p_\varphi - L = \frac{1}{2\mu a^2} p_\varphi^2$$

$$= \frac{1}{2\mu a^2} l_z^2$$

$$H = \frac{1}{2\mu a^2} L_z^2$$

We know that

$$L_z |l_z\rangle = l_z |l_z\rangle$$

requires $l_z = m\hbar \qquad m = 0, \pm 1, \pm 2$

Thus $H|m\rangle = E_m |m\rangle$

with $|m\rangle = |l_z = m\hbar\rangle$, $E_m = \frac{m^2\hbar^2}{2\mu a^2}$

$E_m = E_{-m}$ because, for a given magnitude of $L_z$, the energy is the same whether the particle moves clockwise or counterclockwise.

S12.3.7 $H = \frac{1}{2\mu}(P_x^2 + P_y^2) + \frac{1}{2}\mu\omega^2(x^2+y^2)$

i.

$$[L_z, x^2+y^2] = 2[L_z, x]x + 2[L_z, y]y$$
$$= 2i\hbar(yx - xy) = 0$$

similarly $[L_z, P_x^2 + P_y^2] = 0$

$\therefore \quad [L_z, H] = 0$

$$H|E,m\rangle = E|E,m\rangle$$
$$L_z|E,m\rangle = \hbar m|E,m\rangle \qquad m = 0, \pm 1, \pm 2 \ldots$$

$$\langle \rho, \varphi | E,m\rangle = \psi_{E,m}(\rho,\varphi)$$
$$= R_{E,m}(\rho)\frac{1}{\sqrt{2\pi}}e^{im\varphi}$$

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega^2(x^2+y^2)\right]\psi_{E,m} = E\psi_{E,m}$$

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial}{\partial\varphi^2}$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right)R_{E,m} + \frac{\mu\omega^2}{2}\rho^2 R_{E,m} = E R_{E,m}$$

2.

ii) Let $R_{E,m} \sim \rho^p + o(\rho^{p+1})$ as $\rho \to 0$

$$-\frac{\hbar^2}{2\mu}\left[p(p-1) + p - m^2\right]\rho^{p-2} + o(\rho^{p-1})$$

$$+ \frac{\mu\omega^2}{2}\rho^{p+2} + o(\rho^{p+1}) = E\rho^p + o(\rho^{p+1})$$

$$\therefore \quad p^2 - p + p - m^2 = 0$$

$$\therefore \quad p^2 = m^2 \qquad \therefore p = \pm m$$

For $m = 0$ $\quad R_{E,m} \sim$ constant as $\rho \to 0$

For $m = \pm 1, \pm 2$

$$R_{E,m} \sim \rho^{\pm|m|}$$

However, the wavefunction must be normalizable

$$\therefore \int_0^\infty \rho\, d\rho\, |R_{E,m}|^2 < \infty$$

$$\therefore \quad p > -1 \qquad (p = -1, -2, \ldots \text{ are excluded})$$

$$\therefore \quad R_{E,m} \sim \rho^{|m|} \quad \text{for any } m.$$

iii) For $\rho \to \infty$, $E R_{E,m}$ and $\frac{\hbar^2}{2\mu}\frac{m^2}{\rho^2} R_{E,m}$

are negligible compared to $\frac{\mu\omega^2}{2}\rho^2 R_{E,m}$.

Thus $R_{E,m}(\rho) \sim R_E^{\infty}(\rho)$ as $\rho \to \infty$:

with $$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right) R_E^{\infty} + \frac{\mu\omega^2}{2}\rho^2 R_E^{\infty} = 0$$

Let $$R_E^{\infty} = e^{\varphi}$$

$$-\frac{\hbar^2}{2\mu}\left(\varphi'' + \varphi'^2 + \frac{1}{\rho}\varphi'\right) + \frac{\mu\omega^2}{2}\rho^2 = 0$$

We need $\varphi \to -\infty$ as $\rho \to \infty$

in order that

$$\int^{\infty} \rho\, d\rho\, |R_E^{\infty}|^2 < \infty$$

Therefore let $\varphi \sim -c\rho^p$ $\quad c>0$, $p>0$

$$\varphi'' \sim \rho^{p-2}, \quad \varphi'^2 \sim \rho^{2(p-1)}$$

$$\frac{1}{\rho}\varphi' \sim \rho^{p-2}$$

Hence $\varphi'^2$ dominates.

$$+\frac{\hbar^2}{2\mu}\varphi'^2 = \frac{\mu\omega^2}{2}\rho^2 \qquad \text{as } \rho \to \infty$$

$$\varphi = -\frac{\mu\omega}{\hbar}\frac{1}{2}\rho^2 \qquad \left(R \sim e^{+\frac{1}{2}\frac{\mu\omega}{\hbar}\rho^2} \text{ is not allowed}\right)$$

$$R_E^{\infty} = e^{-\frac{\mu\omega}{\hbar}\frac{1}{2}\rho^2}$$

iv $\qquad \epsilon = \dfrac{E}{\hbar\omega} \qquad y = \left(\dfrac{\mu\omega}{\hbar}\right)^{1/2}\rho$

$$-\frac{1}{2}\left(\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{m^2}{y^2}\right)R + \frac{1}{2}y^2 R = \epsilon R$$

v. $\qquad R_{E,m} = \Omega e^{-\frac{1}{2}y^2} = U\, y^{|m|} e^{-\frac{1}{2}y^2}$

$$R' = (-y\Omega + \Omega')e^{-\frac{1}{2}y^2}$$

$$R'' = \left[(-1+y^2)\Omega - 2y\Omega' + \Omega''\right]e^{-\frac{1}{2}y^2}$$

$$-\frac{1}{2}\left[(-1+y^2)\Omega - 2y\Omega' + \Omega'' - \Omega + \frac{1}{y}\Omega' - \frac{m^2}{y^2}\Omega\right] + \frac{1}{2}y^2\Omega = \epsilon\Omega$$

$$+\Omega'' + \left(\frac{1}{y} - 2y\right)\Omega' - \left(+2 + \frac{m^2}{y^2}\right)\Omega = -2\epsilon\Omega$$

$$\Omega' = y^{|m|}\left(\frac{|m|}{y}U + U'\right)$$

$$\Omega'' = y^{|m|}\left[\frac{m^2 - |m|}{y^2}U + \frac{2|m|}{y}U' + U''\right]$$

$$U'' + U'\left(\frac{2|m|}{y} + \frac{1}{y} - 2y\right)$$

$$+ U\left(\frac{m^2 - |m|}{y^2} + \left(\frac{1}{y} - 2y\right)\frac{|m|}{y} - 2 - \frac{m^2}{y^2}\right)$$

$$= -2\epsilon U$$

$$U'' + \left(\frac{2|m|+1}{y} - 2y\right)U' + \left(2\epsilon - 2 - 2|m|\right)U = 0$$

vi. $-2$ $\quad -2$ $\quad 0$ $\quad 0$

The differential operator on the LHS of the equation shifts a power of $y$ either by $-2$ or $0$.

vii $$v = \sum_{r=0}^{\infty} c_r y^r$$

$$\sum_{r=1}^{\infty} c_r y^{r-2} \left[ r(r-1) + 2|m|+1 \right]$$

$$+ \sum_{r=0}^{\infty} c_r y^r \left[ -2r + 2\epsilon - 2 - 2|m| \right] = 0$$

$$c_{r+2} = - c_r \frac{2\left[\epsilon - 1 - |m| - r\right]}{(r+2)(r+1) + 2|m|+1}$$

$c_1 = 0$ because the first term in the first sum

$$c_1 \frac{1}{y} (2|m|+1)$$

must vanish by itself

Hence $c_3 = c_5 = c_7 = \cdots = 0$.

The series is even.

If the series does not terminate ($\epsilon \neq 1 + |m| + r$ for all even $r$),

$c_{r+2} \sim \frac{2}{r+2} c_r$ for large $r$, and

$$U \sim \sum_{p=0}^{\infty} \frac{1}{p!} y^{2p} = e^{y^2}$$

which is the rejected solution raising its ugly head.

We need to have

$$\epsilon = 1 + |m| + 2k \quad \text{for some positive integer } k$$

$$E = (2k + |m| + 1)\hbar\omega$$

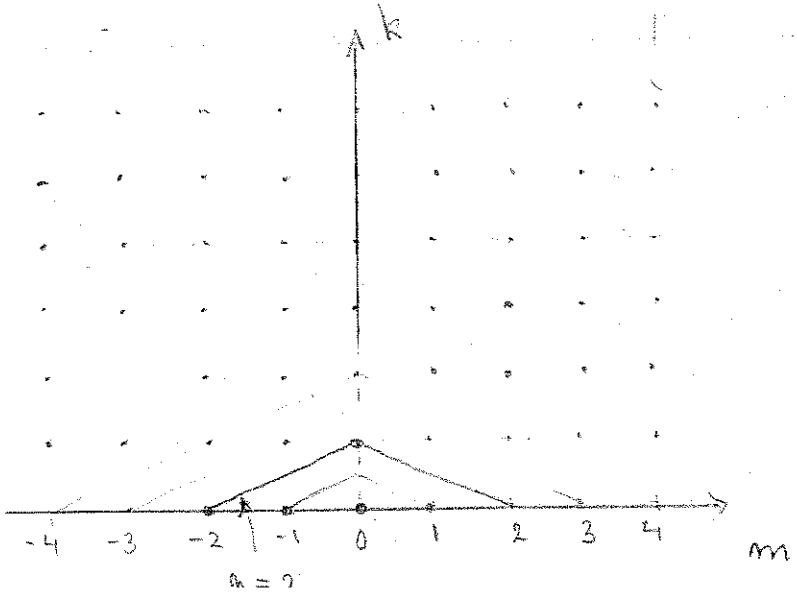
$$= (n+1)\hbar\omega$$

with $n = |m| + 2k$

$m = 0, \pm 1, \pm 2, \ldots$

$k = 0, 1, 2, 3 \ldots$

viii



The degeneracy of the $n^{th}$ level is $n+1$. Check:

- for $n$ even: $(k, m) = (0, \pm n)$
$= (1, \pm(n-2))$
$\vdots$
$= \left(\frac{n}{2} - 1, \pm 2\right)$
$= \left(\frac{1}{2}n, 0\right)$

$$\text{degeneracy} = \frac{n}{2} \cdot 2 + 1 = n + 1$$

- for $n$ odd: $(k, m) = (0, \pm n)$
$= (1, \pm(n-2))$
$\vdots$
$= \left(\frac{n-1}{2}, \pm 1\right)$

$$\text{degeneracy} = \frac{n+1}{2} \cdot 2 = n + 1$$

In Cartesian coordinates:

$$E = \hbar\omega\left(n_1 + \frac{1}{2} + n_2 + \frac{1}{2}\right)$$

$$= k\omega\,(n+1)$$

$n = n_1 + n_2 \qquad n_1, n_2 = 0, +1, +2 \ldots$

For given $n$ , $(n_1, n_2) = (0,n), (1,n-1) \ldots (n,0)$

$\therefore$ the degeneracy of the $n^{th}$ level is $n+1$

ix $\quad n = 0 \qquad m = 0 \qquad E = \hbar\omega$

$$U'' + \left(\frac{1}{\gamma} - 2\gamma\right) U' = 0$$

$U$ = constant or $U = e^{\gamma^2}$ (reject)

$$\psi_{E=\hbar\omega} = N e^{-\frac{\mu\omega}{2\hbar}\rho^2}$$

$$|N|^2 \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\, e^{-\frac{\mu\omega}{\hbar}\rho^2}$$

$$= |N|^2 \, 2\pi \, \frac{\hbar}{2\mu\omega} = 1$$

$$\psi_{E=\hbar\omega} = \sqrt{\frac{\mu\omega}{\pi\hbar}}\, e^{-\frac{\mu\omega}{2\hbar}\rho^2}$$

$n = 1 \qquad m = \pm 1 \qquad E = 2\hbar\omega$

$$U'' + \left(\frac{3}{y} - 2y\right) U' = 0$$

$U = \text{constant}$ or $U' = \frac{1}{y^3} e^{y^2}$

$U \sim e^{y^2}$ (reject)

$$\psi_{E=2\hbar\omega,\, m=\pm 1} = N e^{\pm i\varphi} \rho e^{-\frac{\mu\omega}{2\hbar}\rho^2}$$

$$|N|^2 \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\, \rho^2 e^{-\frac{\mu\omega}{\hbar}\rho^2}$$

$$= |N|^2 \, 2\pi \left(-\frac{d}{d\alpha}\right) \frac{1}{2\alpha} \Bigg|_{\alpha = \frac{\mu\omega}{\hbar}}$$

$$= |N|^2 \left(\frac{\hbar}{\mu\omega}\right)^2 \pi = 1$$

$$\psi_{E=2\hbar\omega,\, m=\pm 1} = \left(\frac{\mu\omega}{\hbar}\right) \frac{1}{\sqrt{\pi}} \rho e^{\pm i\varphi} e^{-\frac{\mu\omega}{2\hbar}\rho^2}$$

x. The $E = \hbar\omega$ state is unique.

Therefore

$$\psi_{E=\hbar\omega}(\rho,\varphi) = \psi_0(x)\,\psi_0(y)$$

$$= \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{\mu\omega}{2\hbar}(x^2+y^2)} \quad \checkmark$$

The two $E = 2\hbar\omega$ states are linear combinations of

$$\psi_1(x)\,\psi_0(y) = \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/2} \frac{1}{\sqrt{2}}\, e^{-\frac{\mu\omega}{2\hbar}(x^2+y^2)}\, 2\left(\frac{\mu\omega}{\hbar}\right)^{1/2} x$$

$$= \left(\frac{\mu\omega}{\hbar}\right)\sqrt{\frac{2}{\pi}}\,\rho\, e^{-\frac{\mu\omega}{2\hbar}\rho^2}\cos\varphi$$

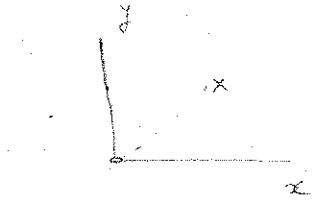
and

$$\psi_0(x)\,\psi_1(y) = \left(\frac{\mu\omega}{\hbar}\right)\sqrt{\frac{2}{\pi}}\,\rho\, e^{-\frac{\mu\omega}{2\hbar}\rho^2}\sin\varphi$$

We have

$$|n=1, m=\pm 1\rangle = \frac{1}{\sqrt{2}}\left(|n_1=1, n_2=0\rangle \pm i\,|n_1=0, n_2=1\rangle\right)$$

Under parity




$$\rho \to \rho$$

$$\varphi \to \varphi + \pi$$

$$e^{im\varphi} \to e^{im\pi} e^{im\varphi} = (-1)^m e^{im\varphi}$$

$$\psi_{E,m}(\rho,\varphi) = N R_{E,m}(\rho) e^{im\varphi}$$

$$\hookrightarrow (-1)^m \psi_{E,m}(\rho,\varphi)$$

But $$(-1)^m = (-1)^{m-2k} = (-1)^m$$

Thus

$$\psi_{E,m}(\rho,\varphi) \to (-1)^m \psi_{E,m}(\rho,\varphi)$$