Problem Set # 15

S12.6.1

$$\psi_E(r,\theta,\varphi) = A\, e^{-r/a_0}$$

i. $l = 0$

ii. (vanishes)

$$\left(-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] + V(r) \right) e^{-r/a_0} = E\, e^{-r/a_0}$$

$$V(r) \to 0 \quad \text{as} \quad r \to \infty$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right) e^{-r/a_0} \to \frac{1}{a_0^2} e^{-r/a_0}$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{a_0^2} = E$$

iii. $$\left(-\frac{\hbar^2}{2\mu}\frac{2}{r}\frac{d}{dr} + V(r) \right) e^{-r/a_0} = 0$$

$$V(r) = -\frac{\hbar^2}{\mu a_0}\frac{1}{r} = -\frac{e^2}{r} \qquad \left(\text{for } a_0 = \frac{\hbar^2}{\mu e^2} \right)$$

§12.6.4

a.

$$\int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \; f(r,\theta,\varphi)\, \delta^3(\vec{r}-\vec{r}')$$

$$= \int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \; f(r,\theta,\varphi) \frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{r^2 \sin\theta}$$

$$= f(r',\theta',\varphi')$$

b.

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\frac{1}{r}\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{-1}{r^2}\right)\right) = 0 \qquad \text{for } r \neq 0$$

$$\int_{r<R} d^3x \; \nabla^2\left(\frac{1}{r}\right) = \int_{r<R} d^3x \; \vec{\nabla}\cdot\left(\vec{\nabla}\frac{1}{r}\right)$$

$$= \int_{r=R} d^2x \; \vec{n}\cdot\vec{\nabla}\frac{1}{r} = -4\pi R^2 \frac{1}{R^2} = -4\pi$$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(\vec{r}) .$$

12.6.5

$$D_\ell = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

Consider two solutions U_1 and U_2 with the same ℓ and E eigenvalues:

$$D_\ell U_1 = E U_1 \qquad \text{and}$$

$$D_\ell U_2 = E U_2 .$$

Then:

$$+\frac{\hbar^2}{2\mu}\left[U_2 \frac{d^2U_1}{dr^2} - U_1 \frac{d^2U_2}{dr^2}\right]$$

$$= \left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right](U_1U_2 - U_2U_1) = 0 .$$

Hence

$$\frac{d}{dr}\left[U_2 \frac{dU_1}{dr} - U_1 \frac{dU_2}{dr}\right] = 0$$

$$\therefore \quad U_2 \frac{dU_1}{dr} - U_1 \frac{dU_2}{dr} = C , \text{ a constant.}$$

$$U_{1,2} \to 0 \qquad \text{as} \qquad r \to 0$$

Let $\quad U_j \to c_j r^{\alpha_j} \qquad \text{as} \quad r \to 0$

$$j = 1, 2 \qquad \alpha_j > 0$$

$$c = \lim_{r \to 0} U_2 \frac{dU_1}{dr} - U_1 \frac{dU_2}{dr} = c_1 c_2 r^{\alpha_1 + \alpha_2 - 1} (\alpha_1 - \alpha_2)$$

For $l \geq 1$, $\alpha_1 = \alpha_2 = l + 1$

(cf. Shankar, p 352). Hence $c = 0$.

For $l = 0$, if V is less singular than r^{-2} as $r \to 0$,

then $U \to r^{\alpha}$ with $\alpha = 0$ or 1,

by the same argument as on page 352

of Shankar's book. $\alpha = 0$ is clearly

not allowed (cf page 351 of Shankar).

Thus $\alpha_1 = \alpha_2 = 1$. Hence $c = 0$.

In all cases that I am aware off, the boundary conditions at $r = 0$ require $\zeta = 0$. Hence

$$U_2 \frac{dU_1}{dr} - U_1 \frac{dU_2}{dr} = 0$$

$$\therefore \quad \frac{d}{dr}\left(\frac{U_1}{U_2}\right) = 0$$

$$\therefore \quad U_1 = \text{constant} \times U_2$$

Hence U_1 and U_2 describe the same state.

Hence for each set of values for E, l and m, there is at most one state (whether $E > 0$ or $E < 0$).

§12.6.8

$$\psi_{E\ell m}(r,\theta,\varphi) = R_{E\ell}(r)\, Y_\ell^m(\theta,\varphi)$$

$$\frac{-\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right] R_{E\ell}(r) = E R_{E\ell}(r)$$

$$\therefore\ R_{E\ell}(r) \sim r^\ell \quad \text{as } r \to 0$$

[For $\ell = 0$, $R_{E\ell}(r) \sim \frac{1}{r}$ as

not allowed because

$$\nabla^2 \frac{1}{r} = -4\pi\,\delta^3(\vec{x}) \neq 0 .$$

For $\ell \neq 0$, see discussion on p. 352]

$$R_{E\ell}(r) = N\, j_\ell(kr) \qquad E = \frac{\hbar^2 k^2}{2\mu}$$

$$= N(-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{\sin\rho}{\rho}\Bigg|_{\rho = kr}$$

The energy eigenvalues $E_{ln} = \frac{\hbar^2 k_{ln}^2}{2\mu}$ are determined by the boundary condition

$$\psi_{Elm}(r_0, \theta, \varphi) = 0$$

$\therefore$ k_{ln} is such that $j_l(k_{ln} r_0) = 0$

For $l = 0$

$$j_0(k_n r_0) = \frac{\sin(k_n r_0)}{k_n r_0} = 0$$

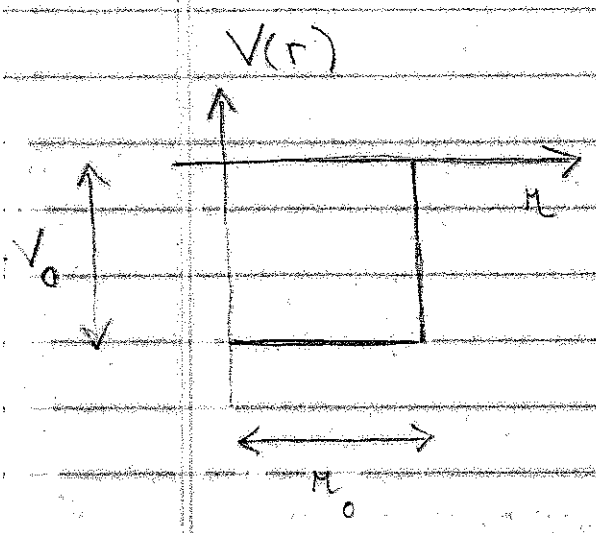
$$k_n r_0 = n\pi \qquad n = 1, 2, 3, \ldots$$

$$E_n = \frac{\hbar^2 \pi^2}{2\mu} \frac{n^2}{r_0^2}$$

$$\psi_{n00} = N_n \frac{1}{r} \sin\left(\frac{n\pi}{r_0} r\right)$$

12.6.9

$$\psi_{Elm}(r,\theta,\varphi) = R_{El}(r)\, Y_l^m(\theta,\varphi)$$



$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)+V(r)\right]R_{El}(r) = E\,R_{El}(r)$$

$(V_0<0)$ ——— $R_{El}(r) = \frac{1}{r}U_{El}(r)$, $E<0$

$$\left[-\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right]U_{El}(r) = 0 \quad \text{for } r<r_0$$

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-\kappa^2\right]U_{El}(r) = 0 \quad \text{for } r>r_0$$

with $E-V_0 = \frac{\hbar^2k^2}{2\mu}$ and $-E = \frac{\hbar^2\kappa^2}{2\mu}$

$U_{El}(r)$ and $\frac{dU_{El}}{dr}$ must be continuous at $r=r_0$

For $\ell = 0$

$$U(r) = A \sin kr \quad \text{for } r < r_0$$

$$= B e^{-\kappa r} \quad \text{for } r > r_0$$

($U(r) \sim \cos kr$ for $r < r_0$ does not solve the Schrödinger equation because $\nabla^2 \frac{1}{r} = -4\pi \delta^3(\vec{x}) \neq 0$)

$$\begin{cases} A \sin kr_0 = B e^{-\kappa r_0} \\ A k \cos kr_0 = -B\kappa e^{-\kappa r_0} \end{cases}$$

$\tan kr_0 = -k/\kappa$ determines the discrete spectrum of bound state energies

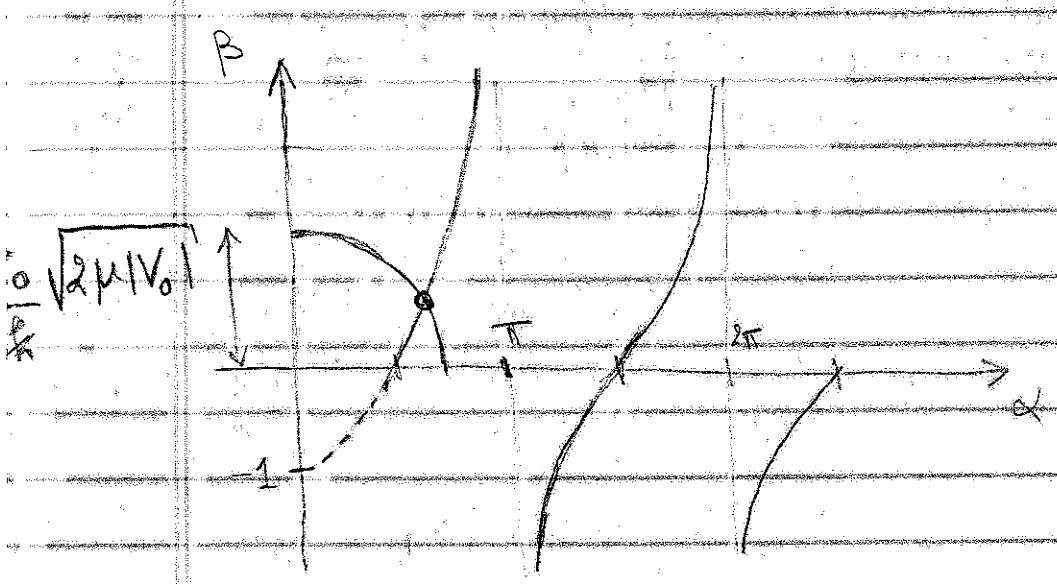
$$\kappa^2 = 2\mu\left(\frac{-E}{\hbar^2}\right) \qquad k^2 = 2\mu\left(\frac{E - V_0}{\hbar^2}\right)$$

$$\kappa^2 + k^2 = \frac{2\mu}{\hbar^2}(-V_0)$$

Let $Kr_0 \equiv \beta$ and $kr_0 \equiv \alpha$

$$\alpha^2 + \beta^2 = \frac{2\mu}{\hbar^2} r_0^2 |V_0|$$

$$\beta = -\frac{\alpha}{\tan\alpha}$$



There are no bound states for

$$\frac{r_0}{\hbar}\sqrt{2\mu|V_0|} < \frac{\pi}{2}$$

or

$$|V_0| < \frac{\pi^2\hbar^2}{8\mu r_0^2}$$

12.6.10

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} c_l P_l(\cos\theta) j_l(kr)$$

$$\int_{-1}^{+1} d(\cos\theta)\, P_l(\cos\theta)\, e^{ikr\cos\theta}\, d\cos\theta$$

$$= c_l \frac{2}{2l+1} j_l(kr)$$

$$= \int_{-1}^{+1} dx \frac{r}{2^l l!} \frac{d^l (x^2-1)^l}{dx^l} e^{ikrx}$$

$$= \frac{1}{2^l l!} (-ikr)^l \int_{-1}^{+1} dx\, (x^2-1)^l\, e^{ikrx}$$

by integrating by parts l times

$$= \frac{2}{2^l l!} (ikr)^l \int_0^1 dx\, (1-x^2)^l\, e^{ikrx}$$

Take the limit $kr \to 0$ on both sides

$$j_\ell(kr) \to \frac{(kr)^\ell}{(2\ell+1)!!} \quad \text{as} \quad kr \to 0$$

$$e^{ikrx} \to 1 \quad \text{as} \quad kr \to 0$$

$$C_\ell \quad \frac{2}{2\ell+1} \quad \frac{1}{(2\ell+1)!!}$$

$$= \frac{2}{2^\ell \ell!} i^\ell \int_0^1 dx \; (1-x^2)^\ell$$

$$= \frac{2}{2^\ell \ell!} i^\ell \frac{(2\ell)!!}{(2\ell+1)!!}$$

Hence $$C_\ell = (2\ell+1) i^\ell \frac{(2\ell)!!}{2^\ell \ell!}$$

$$= (2\ell+1) i^\ell \quad \text{since}$$

$$(2\ell)!! = (2\ell)(2\ell-2) \cdots 4 \cdot 2$$

$$= \ell(\ell-1) \cdots 2 \cdot 1 \; 2^\ell$$

$$= \ell! \, 2^\ell$$