# Homework Set # 3

S 1.9.1 $\Omega^\dagger = \Omega \quad \therefore \quad \exists S : S^\dagger S = \mathbb{1}$ and

$$S^\dagger \Omega S = \text{diag}(\omega_1, \ldots \omega_d) \equiv \Delta$$

$$S^\dagger f(\Omega) S = \sum_{n=0}^{\infty} S^\dagger \Omega^n S$$

$$= \sum_{n=0}^{\infty} (S^\dagger \Omega S)^n = \sum_{n=0}^{\infty} \Delta^n$$

$$= \sum_{n=0}^{\infty} \text{diag}(\omega_1^n, \ldots, \omega_d^n)$$

$$= \text{diag}\left(\frac{1}{1-\omega_1}, \ldots, \frac{1}{1-\omega_d}\right)$$

provided $|\omega_i| < 1$ all $i = 1 \ldots n$

(otherwise the series does not converge)

$$= \frac{1}{\mathbb{1} - \Delta} = \frac{1}{\mathbb{1} - S^\dagger \Omega S}$$

Thus $(\mathbb{1} - S^\dagger \Omega S)\, S^\dagger f(\Omega) S = \mathbb{1}$

$$S^\dagger (\mathbb{1} - \Omega) f(\Omega) S = \mathbb{1}$$

$$(\mathbb{1} - \Omega) f(\Omega) = \mathbb{1}$$

similarly $f(\Omega)(\mathbb{1} - \Omega) = \mathbb{1}$

§1.9.2 $U \equiv \sum_{n=0}^{\infty} \frac{1}{n!} i^n H^n \qquad H^\dagger = H.$

$$U^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n H^n = e^{-iH}$$

$$U^\dagger U = e^{-iH} e^{iH} = \mathbb{1}$$

§1.9.3 $\exists S : \; S^\dagger H S = \mathrm{diag}(\omega_1, \ldots \omega_d)$

$$\equiv \Delta$$

$$S^\dagger U S = \sum_{n=0}^{\infty} \frac{1}{n!} i^n (S^\dagger H S)^n$$

$$= \mathrm{diag}\left(e^{i\omega_1}, \ldots, e^{i\omega_d}\right)$$

$$\det(S^\dagger U S) = \det U$$

$$= e^{i(\omega_1 + \cdots + \omega_d)} = e^{i \mathrm{Tr} H}$$

S 1.10.1

$$\int_{-\infty}^{+\infty} f(x)\ \delta(ax)\,dx = \int_{-\infty}^{+\infty} f\left(\frac{y}{a}\right)\delta(y)\,\frac{dy}{a}$$

$$= \frac{1}{a} f(0) = \int_{-\infty}^{+\infty} f(x)\,\frac{1}{a}\,\delta(x)\,dx$$

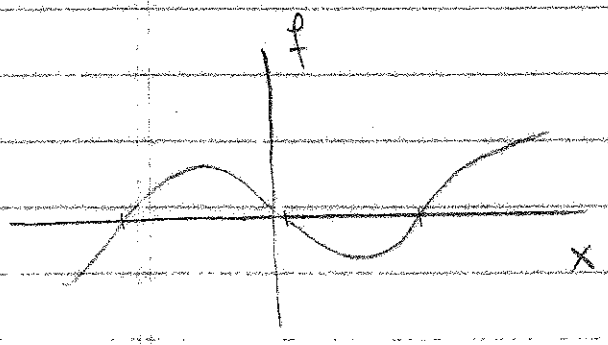
for all $f(x)$ ← not officially part of the homework

S 1.10.2

$$\int_{-\infty}^{+\infty} g(x)\ \delta[f(x)]\,dx$$

$$= \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\ \delta[f(x)]\,dx$$

where the $x_i$ are the zeros of $f(x)$




We assume that the $x_i$ are simple zeros.

Over the small interval $[x_i-\epsilon, x_i+\epsilon]$ we can change variables $y_i = f(x)$

$$y_i = 0 + \frac{df}{dx}(x_i)\,(x - x_i) + \cdots$$

$$x = x_i + \frac{1}{\frac{df}{dx}(x_i)}\, y_i + \cdots$$

$$\int_{-\infty}^{+\infty} g(x)\, \delta[f(x)]\, dx$$

$$= \sum_i \int_{f(x_i-\epsilon)}^{f(x_i+\epsilon)} dy_i \; g\left(x_i + \frac{1}{\frac{df}{dx}(x_i)}\, y_i + \cdots\right) \times$$

$$\times\, \delta(y_i)\, \frac{1}{\frac{df}{dx}(x_i) + \cdots}$$

$$= \sum_i g(x_i)\, \frac{1}{\left|\frac{df}{dx}(x_i)\right|}$$

$$= \int_{-\infty}^{+\infty} dx\; g(x) \sum_i \delta(x_i)\, \frac{1}{\left|\frac{df}{dx}(x_i)\right|}$$

for all $g(x)$

S1.10.3

$$\int_{-\infty}^{x} \delta(x')\,dx' = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

definition of $\theta(x)$

$$\theta(x) = \int_{-\infty}^{x} dx'\,\delta(x')$$

this statement part of the framework

S1.10.4

$$\epsilon \frac{\partial^2 \psi}{\partial t^2} - \tau \frac{\partial^2 \psi}{\partial x^2} = 0$$

$\epsilon$ = mass per unit length of the string

$\tau$ = tension in the string

$$\psi(0) = \psi(L) = 0$$

$$\psi_n(x,t) = \sin\left(\frac{n\pi}{L}x\right) e^{-i\omega_n t}$$

with $\omega_n = \frac{n\pi}{L}\left(\frac{\tau}{\epsilon}\right)^{1/2}$

$\left(\frac{\tau}{\epsilon}\right)^{1/2}$ is the phase and group velocity of transverse oscillations of the string

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(A_n \cos\omega_n t + B_n \sin\omega_n t\right)$$

$$\dot{\psi}(x,0) = 0 \qquad \therefore \quad B_n = 0$$

$$\psi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right)\psi(x,0)$$

$$= \int_0^L dx \sum_{n'=1}^{\infty} A_{n'} \sin\left(\frac{n'\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{L}{\pi}\sum_{n'=1}^{\infty} A_{n'} \int_0^{\pi} d\theta \sin n'\theta \sin n\theta$$

$$\sin n'\theta \sin n\theta = \frac{1}{2}\left[\cos((n'-n)\theta) - \cos((n'+n)\theta)\right]$$

$$= \frac{L}{4\pi}\sum_{n'=1}^{\infty} A_{n'} \int_0^{2\pi} d\theta \left[\cos((n'-n)\theta) - \cos((n'+n)\theta)\right]$$

$$= \frac{L}{4\pi} \sum_{m'=1}^{\infty} A_{n'} \, 2\pi \, \delta_{nn'} = \frac{L}{2} A_n$$

$$= \int_0^{L/2} dx \, \frac{2xh}{L} \sin\left(\frac{n\pi x}{L}\right) + \int_{L/2}^{L} dx \, \frac{2h}{L}(L-x) \sin\left(\frac{n\pi x}{L}\right)$$

$x \to L - x$

$$= \int_0^{L/2} dx \, \frac{2xh}{L} \left[ \sin\left(\frac{n\pi x}{L}\right) + \sin\left(n\pi - \frac{n\pi x}{L}\right) \right]$$

$$-(-1)^n \sin\left(\frac{n\pi x}{L}\right)$$

$$\begin{cases} = 0 & \text{for } n \text{ even} \\ = \frac{4h}{L} \int_0^{L/2} dx \, x \sin\left(\frac{n\pi x}{L}\right) & \text{for } n \text{ odd} \end{cases}$$

$$= \frac{4h}{L} \left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi/2} d\vartheta \; \vartheta \sin\vartheta$$

$$= \frac{4hL}{n^2\pi^2} \left[ -\vartheta \cos\vartheta + \sin\vartheta \right]_0^{\frac{n\pi}{2}}$$

$$= \frac{4hL}{n^2\pi^2} \sin\frac{n\pi}{2}$$

$$A_n = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\psi(x,t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) \frac{1}{n^2} \sin\left(\frac{n\pi x}{L}\right) \cos\omega_n t$$