Homework Set #5

S4.2.3

$$\psi'(x) = e^{i/\hbar p_0 x}\,\psi(x)$$

$$\langle \psi' | P | \psi' \rangle = \int dx\, \psi'(x)^* \frac{\hbar}{i}\frac{d}{dx}\,\psi'(x)$$

$$= \int dx\, \psi'(x)^* \frac{\hbar}{i}\frac{d}{dx}\, e^{i/\hbar p_0 x}\,\psi(x)$$

$$= \int dx\, \psi'(x)^*\, e^{i/\hbar p_0 x}\left(p_0 + \frac{\hbar}{i}\frac{d}{dx}\right)\psi(x)$$

$$= \int dx\, \psi(x)^*\left(p_0 + \frac{\hbar}{i}\frac{d}{dx}\right)\psi(x)$$

$$= p_0 + \langle \psi | P | \psi \rangle$$

S 5.1.1

$$1 = \int_{-\infty}^{+\infty} dp \; |p><p|$$

$$= \int_{-\infty}^{0} dp \; \left|E = \frac{p^2}{2m}, -\right\rangle\left\langle E = \frac{p^2}{2m}, -\right|$$

$$+ \int_{0}^{\infty} dp \; \left|E = \frac{p^2}{2m}, +\right\rangle\left\langle E = \frac{p^2}{2m}, +\right|$$

$$= \int_{0}^{\infty} dE \left|\frac{dp}{dE}\right| \sum_{\epsilon=\pm} |E,\epsilon><E,\epsilon|$$

$$= \sum_{\epsilon=\pm} \int_{0}^{\infty} dE \sqrt{\frac{m}{2E}} \; |E,\epsilon><E,\epsilon|$$

§5.1.2 $\langle x|H|E\rangle = E\langle x|E\rangle$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) = E\,\psi_E(x)$$

The most general solution is

$$\psi_E(x) = A_+ e^{+\frac{i}{\hbar}\sqrt{2mE}\,x} + A_- e^{-\frac{i}{\hbar}\sqrt{2mE}\,x}$$

$$\therefore \langle x|E\rangle = \beta\langle x|E,+\rangle + \gamma\langle x|E,-\rangle$$

$$\beta = A_+\sqrt{2\pi\hbar}$$

$$\gamma = A_-\sqrt{2\pi\hbar}$$

$$\therefore |E\rangle = \beta|E,+\rangle + \gamma|E,-\rangle$$

§5.1.3

$$U(t)\,\psi(x,0) = \exp\left(\frac{i}{\hbar}t\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\frac{1}{\pi^{1/4}}e^{-x^2/2}$$

$$= \frac{1}{\pi^{1/4}}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m!\,k!}\left(\frac{i}{\hbar}t\frac{\hbar^2}{2m}\right)^l\left(\frac{-1}{2}\right)^k\frac{d^{2l}}{dx^{2l}}x^{2k}$$

$$\frac{d^p}{dx^p}x^q = 0 \quad \text{if} \quad p > q$$

$$= q(q-1)\cdots(q-p+1)\,x^{q-p} \quad \text{if} \quad p \le q$$

$$= \frac{1}{\pi^{1/4}}\sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\frac{1}{l!\,k!}\left(\frac{itk}{2m}\right)^l\left(\frac{-1}{2}\right)^k\frac{(2k)!}{(2k-2l)!}x^{2(k-l)}$$

$$m = k-l \qquad k = l+m$$

$$= \frac{1}{\pi^{1/4}}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(2l+2m)!}{l!\,(l+m)!\,(2m)!}\left(\frac{-1}{2}\right)^m\left(\frac{-it\hbar}{4m}\right)^l x^{2m}$$

$$\frac{(2p)!}{p!} = \frac{(2p)(2p-1)(2p-2)\cdots 2\cdot 1}{p\cdot(p-1)\cdots 1}$$

$$= 2^p (2p-1)(2p-3)\cdots 1$$

$$= 2^{2p}\left(p-\tfrac{1}{2}\right)\left(p-\tfrac{3}{2}\right)\cdots \tfrac{3}{2}\cdot\tfrac{1}{2}$$

$$U(t)\psi(x,0) = \frac{1}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}\left(-\frac{1}{2}\right)^n \times$$

$$\times \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{-i\hbar t}{4m}\right)^l 2^{2(l+n)}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\left(l+n-\frac{1}{2}\right)$$

$$= \frac{1}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{1}{n!\,2^{2n}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n-\frac{1}{2}\right)}x^{2n}\left(\frac{-1}{2}\right)^n 2^{2n}$$

$$\times \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{i\hbar t}{m}\right)^l \frac{1}{2}\cdot\frac{3}{2}\cdots\left(l+n-\tfrac{1}{2}\right)$$

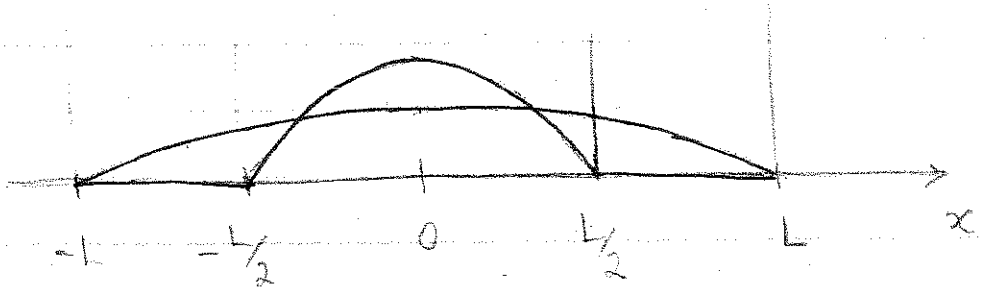
$$U(t)\psi(x,0) = \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \left(\frac{-1}{2}\right)^n \cdot$$

$$\left[1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left(\frac{-i\cdot t\hbar}{m}\right)^l \left(n+\tfrac{1}{2}\right)\left(n+\tfrac{3}{2}\right)\cdots\left(l+n-\tfrac{1}{2}\right)\right]$$

$$= \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \left(-\frac{1}{2}\right)^n \left(1+\frac{it\hbar}{m}\right)^{-n-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\pi}\left(1+\frac{it\hbar}{m}\right)^{1/2}} \exp\left[-\frac{1}{2}\cdot\frac{x^2}{\left(1+\frac{it\hbar}{m}\right)}\right]$$

S.B.2.1



$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right) \quad \text{for } |x| \leq \frac{L}{2}$$

$$= 0 \quad \text{for } \frac{L}{2} \leq |x| \leq L$$

$$\psi'(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{\pi}{2L}(x + L)\right)$$

$$P = |\langle \psi' | \psi \rangle|^2 = \left| \int_{-L}^{+L} dx \; \psi'(x)^* \, \psi(x) \right|^2$$

$$= \left(\sqrt{\frac{2}{L}} \int_{-L/2}^{L/2} dx \; \sin\left(\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right) \sin\left(\frac{\pi}{2L}(x + L)\right) \right)^2$$

$$= \left(\frac{\sqrt{2}}{L} \int_{0}^{L} dx' \; \sin\left(\frac{\pi x'}{L}\right) \sin \frac{\pi}{2L}\left(x' + \frac{L}{2}\right) \right)^2$$

$$\phi = \left(\frac{\sqrt{2}}{L} \int_0^{\pi} d\theta \; \frac{L}{\pi} \sin\theta \sin\left(\frac{1}{2}\theta + \frac{\pi}{4}\right) \right)^2$$

$$\theta = \frac{\pi x'}{L}$$

$$\phi = \left(\frac{\sqrt{2}}{\pi} \int_0^{\pi} d\theta \; \frac{1}{2} \left[\cos\left(\frac{1}{2}\theta - \frac{\pi}{4}\right) - \cos\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) \right] \right)^2$$

$$= \left(\frac{1}{\sqrt{2}\pi} \left[2\sin\left(\frac{1}{2}\theta - \frac{\pi}{4}\right) - \frac{2}{3}\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) \right]_0^{\pi} \right)$$

$$= \left[\frac{1}{\sqrt{2}\pi} \left(2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \frac{2}{3}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \right) \right]^2$$

$$= \left[\frac{2}{\pi} \left(1 + \frac{1}{3}\right) \right]^2 = \left(\frac{8}{3\pi}\right)^2$$

S5.2.2 a

$$H|E_n\rangle = E_n|E_n\rangle$$

E_0 is the lowest energy eigenvalue

$$\langle E_n|E_{n'}\rangle = \delta_{nn'}$$

$$\sum_{n=0}^{\infty} |E_n\rangle\langle E_{n'}| = 1$$

$$H = H\sum_{n=0}^{\infty} |E_n\rangle\langle E_n|$$

$$= \sum_{n=0}^{\infty} E_n |E_n\rangle\langle E_n|$$

$$\langle\psi|H|\psi\rangle = \sum_{n=0}^{\infty} E_n \langle\psi|E_n\rangle\langle E_n|\psi\rangle$$

$$= \sum_{n=0}^{\infty} E_n |\langle\psi|E_n\rangle|^2$$

$$= E_0 + \sum_{n=0}^{\infty} (E_n - E_0)|\langle\psi|E_n\rangle|^2$$

≥ 0 $\quad$ ≥ 0

since $\sum_{n=0}^{\infty} |\langle\psi|E_n\rangle|^2 = \langle\psi|\psi\rangle = 1$

Note: $\langle\psi|H|\psi\rangle = E_0$ only if $|\psi\rangle = |E_0\rangle$

b.

$$\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

$$\int_{-\infty}^{+\infty} dx \, |\psi_\alpha(x)|^2 = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} \, dx = 1$$

$$V(x) = -|V(x)| \quad \text{for all } x$$

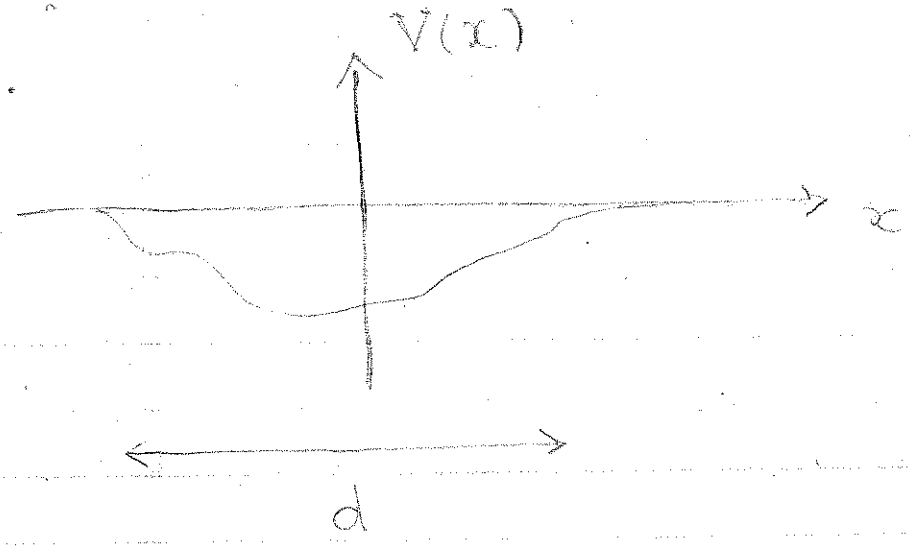
$$E_\alpha = \langle \psi_\alpha | H | \psi_\alpha \rangle$$

$$= \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2/2} \left(-\frac{k^2}{2m} \frac{d^2}{dx^2} - |V(x)| \right) e^{-\alpha x^2/2}$$

$$= \frac{1}{2m} \langle \psi_\alpha | P^2 | \psi_\alpha \rangle - \langle \psi_\alpha | |V(x)| | \psi_\alpha \rangle$$

$$\langle \psi_\alpha | P^2 | \psi_\alpha \rangle = \frac{k^2}{2\Delta^2} \Big|_{\Delta = \frac{1}{\sqrt{\alpha}}} = \frac{\hbar^2 \alpha}{2}$$

since $\psi_\alpha(x)$ is a Gaussian



$$\langle \psi_\alpha | V(x) | \psi_\alpha \rangle = \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\, V(x)$$

$$\to \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx\, V(x) \qquad \text{for} \quad \alpha^{-1} \gg d$$

$$\alpha \ll d^{-1}$$

$$= -\sqrt{\alpha}\, C \qquad \text{with}$$

$$C = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx\, |V(x)| \,, \qquad C < 0 \,,$$

$$E_\alpha \to \frac{\hbar^2 \alpha}{4m} - \sqrt{\alpha}\, C < 0 \qquad \text{for } \alpha \text{ sufficiently small}$$

$$E_0 \leq \langle \psi_\alpha | H | \psi_\alpha \rangle$$

$$\therefore \; E_0 < 0$$

$\therefore$ there is at least one bound state.