Homework Set #7

S5.3.1 $\quad V(\vec{r}) = V_r(\vec{r}) - i V_i$

$$H = \frac{1}{2m} p^2 + V_r(\vec{r}) - i V_i \neq H^\dagger$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + \left[V_r(\vec{r}) - i V_i \right] \psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + \left[V_r(\vec{r}) + i V_i \right] \psi^*$$

$$\frac{\partial}{\partial t} |\psi|^2 = \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi$$

$$= \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2m} \left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right) - 2i V_i |\psi|^2 \right]$$

$$= \frac{+i\hbar}{2m} \vec{\nabla} \cdot \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)$$

$$- \frac{2}{\hbar} V_i |\psi|^2$$

$$\frac{\partial}{\partial t}\left[|\psi|^2 e^{\frac{2}{\hbar} V_i t} \right] + \vec{\nabla} \cdot \vec{j} = 0$$

$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

[Note If $\psi'(\vec{r},t)$ is solution of

$$i\hbar\, \partial_t \psi' = -\frac{\hbar^2}{2m}\nabla^2\psi' + V_r(\vec{r})\,\psi'$$

then

$$\psi(\vec{r},t) = e^{-\frac{1}{\hbar} V_i t}\, \psi'(\vec{r},t) \;]$$

$$P = \int d^3r \; |\psi(\vec{r},t)|^2$$

$$\frac{dP}{dt} = -\frac{2}{\hbar} V_i P$$

$$P(t) = P(0)\; e^{-\frac{2}{\hbar} V_i t}$$

S5.3.2

$$\psi = \zeta \tilde{\psi} \qquad \psi' \text{ real}$$

ζ = complex number

$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

$$= \frac{\hbar}{2mi} |c|^2 \left(\tilde{\psi}^* \vec{\nabla}\tilde{\psi} - \tilde{\psi} \vec{\nabla}\tilde{\psi}^*\right) = 0$$

S 5.3.3

$$\psi_{\vec{p}} = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

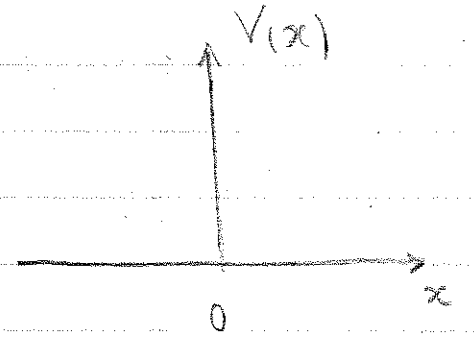
$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^*_{\vec{p}}\,\vec{\nabla}\,\psi_{\vec{p}} - \psi_{\vec{p}}\,\vec{\nabla}\,\psi^*_{\vec{p}}\right)$$

$$= \frac{\hbar}{2mi}\left(\frac{1}{\hbar}\vec{p}\right) 2\,\frac{1}{(2\pi\hbar)^3} = \frac{\vec{p}}{m}\,\frac{1}{(2\pi\hbar)^3}$$

$$\rho = \psi^*_{\vec{p}}\,\psi_{\vec{p}} = \frac{1}{(2\pi\hbar)^3}$$

$$\vec{j} = \vec{v}\,\rho \qquad \text{with} \qquad \vec{v} = \frac{\vec{p}}{m}$$

S 5.4.2 a



$$V(x) = a V_0 \delta(x)$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}(0_+) - \frac{d\psi}{dx}(0_-)\right]$$

$$= \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} dx \,(E - V(x))\,\psi(x)$$

$$= -a V_0 \psi(0)$$

$x < 0 \qquad \psi(x) = A_+ e^{ikx} + A_- e^{-ikx}$

$x > 0 \qquad \psi(x) = B e^{ikx}$

$$k = \frac{1}{\hbar}\sqrt{2mE}$$

$$\begin{cases} A_+ + A_- = B \\ +\frac{\hbar^2}{2m} ik \left[B - (A_+ - A_-)\right] = +a V_0 B \end{cases}$$

$$A_+ + A_- = B$$

$$A_+ - A_- = B\left(1 - \frac{2m a V_0}{ik\hbar^2}\right)$$

$$A_+ = B_-\left(1 - \frac{maV_0}{i\hbar k^2}\right)$$

$$A_- = B\,\frac{maV_0}{i\hbar k^2}$$

$$R = \frac{|A_-|^2}{|A_+|^2} = \frac{\left(\frac{maV_0}{\hbar k^2}\right)^2}{1+\left(\frac{maV_0}{\hbar k^2}\right)^2}$$

$$T = \frac{|B|^2}{|A_+|^2} = \frac{1}{1+\left(\frac{maV_0}{\hbar k^2}\right)^2}$$

§ 5.3.4

$$\psi = A e^{\frac{i}{\hbar} p x} + B e^{-\frac{i}{\hbar} p x}$$

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right)$$

$$= \frac{\hbar}{2mi}\left[\left(A^* e^{-\frac{i}{\hbar} p x} + B^* e^{+\frac{i}{\hbar} p x}\right) \times \right.$$

$$\left. \times \frac{ip}{\hbar}\left(A e^{\frac{i}{\hbar} p x} - B e^{-\frac{i}{\hbar} p x}\right) - \text{h.c.}\right]$$

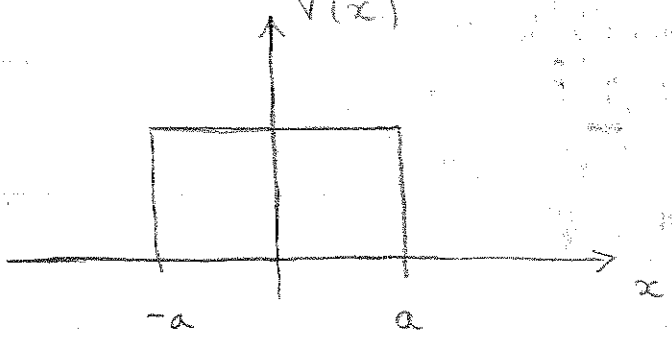
$$= \frac{\hbar}{2mi}\left\{\left[|A|^2 - |B^*|^2 + \underbrace{\left(B^* A e^{\frac{2i}{\hbar} p x} - \text{h.c.}\right)}_{\text{pure imaginary}}\right]\frac{ip}{\hbar}\right.$$

$$\left. - \text{h.c.}\right\}$$

$$= \frac{\hbar}{2mi}\frac{ip}{\hbar} 2\left(|A|^2 - |B|^2\right)$$

$$= \frac{p}{m}\left(|A|^2 - |B|^2\right)$$

b.



$V(x) = 0$ for $|x| > a$

$= V_0$ for $|x| < a$

$E < V_0$

$x > a \quad \psi_E(x) = C e^{ikx}$

$|x| < a \quad \psi_E(x) = B_+ e^{+\kappa x} + B_- e^{-\kappa x}$

$x < -a \quad \psi_E(x) = A_+ e^{ikx} + A_- e^{-ikx}$

$$k = \frac{1}{\hbar}\sqrt{2mE} \qquad \kappa = \frac{1}{\hbar}\sqrt{2m(V_0 - E)}$$

$$C e^{ika} = B_+ e^{\kappa a} + B_- e^{-\kappa a}$$

$$ik C e^{ika} = \kappa (B_+ e^{\kappa a} - B_- e^{-\kappa a})$$

$$A_+ e^{-ika} + A_- e^{+ika} = B_+ e^{-\kappa a} + B_- e^{\kappa a}$$

$$ik(A_+ e^{-ika} - A_- e^{ika}) = \kappa(B_+ e^{-\kappa a} - B_- e^{\kappa a})$$

$$B_{\pm} = \frac{e^{\mp ka}}{2}\left(1 \pm \frac{ik}{\kappa}\right) e^{ika} C$$

$$A_{\pm} = \frac{e^{ika}}{2}\left[\left(1 \mp \frac{ik}{\kappa}\right) B_{+} e^{-\kappa a} + \left(1 \pm \frac{ik}{\kappa}\right) B_{-} e^{\kappa a}\right]$$

$$A_{\pm} = \frac{e^{2ika}}{4}\left[\left(1 \mp \frac{ik}{\kappa}\right)\left(1 + \frac{ik}{\kappa}\right) e^{-2\kappa a}\right.$$

$$\left. + \left(1 \pm \frac{ik}{\kappa}\right)\left(1 - \frac{ik}{\kappa}\right) e^{2\kappa a}\right] C$$

$$A_{\pm} = \frac{e^{2ika}}{4}\left[\left(2 + i\left(\frac{k}{\kappa} \mp \frac{\kappa}{k}\right)\right) e^{-2\kappa a}\right.$$

$$\left. + \left(2 - i\left(\frac{k}{\kappa} \mp \frac{\kappa}{k}\right)\right) e^{2\kappa a}\right] C$$

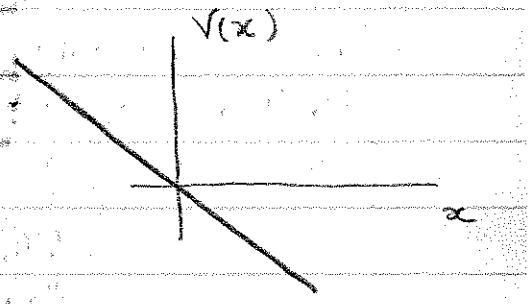
$$T = \frac{|C|^2}{|A_{+}|^2} = \frac{4}{\left[1 + \frac{i}{2}\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right)\right] e^{-2\kappa a} + \left[1 - \frac{i}{2}\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right)\right] e^{2\kappa a}}$$

$$= \frac{1}{\cosh 2\kappa a + \frac{1}{4}\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right)^2 \sinh 2\kappa a}$$

§5.4.3

$$U(p,t;p',0) = \langle p | e^{-\frac{i}{\hbar} H t} | p' \rangle$$

$$H = \frac{P^2}{2m} - fx$$



$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - fx \right) \psi_E(x) = E\,\psi_E(x)$$

$$\left(+\frac{p^2}{2m} + \frac{\hbar}{i} f \frac{d}{dp} \right) \tilde{\psi}_E(p) = E\,\tilde{\psi}_E(p)$$

$$\frac{1}{\tilde{\psi}_E(p)} \frac{\hbar}{i} f \frac{d}{dp} \tilde{\psi}_E(p) = \frac{\hbar f}{i} \frac{d}{dp} \ln \tilde{\psi}_E(p)$$

$$= \left(E - \frac{p^2}{2m} \right) = \frac{d}{dp}\left(Ep - \frac{p^3}{6m} \right)$$

$$\tilde{\psi}_E(p) = N \exp\left[\frac{i}{\hbar f} \left(Ep - \frac{p^3}{6m} \right) \right]$$

$$= \langle p | E \rangle$$

$$\langle E' | E \rangle = |N|^2 \int_{-\infty}^{+\infty} dp \, e^{\frac{i}{\hbar f}(E - E')p}$$

$$= \delta(E - E') \quad \text{provided} \quad |N|^2 = \frac{1}{2\pi\hbar f}$$

With this normalization

$$\int_{-\infty}^{+\infty} dE \; |E\rangle\langle E| = 1$$

$$U(p,t;p',0) = \int_{-\infty}^{+\infty} dE \, \langle p|E\rangle\langle E|p'\rangle \, e^{-\frac{i}{\hbar}Et}$$

$$= |N|^2 \int_{-\infty}^{+\infty} dE \; e^{\frac{i}{\hbar f}E(p - p' - ft)} \; e^{-\frac{i}{\hbar f}\frac{p^3 - p'^3}{6m}}$$

$$= |N|^2 \, 2\pi\hbar f \, \delta(p - p' - ft) \; e^{\frac{i}{6m\hbar f}(p'^3 - p^3)}$$

$$= \delta(p - p' - ft) \; e^{\frac{i}{6m\hbar f}(p'^3 - p^3)}$$

$$U(x,t;x',0) = \langle x | e^{-\frac{i}{\hbar}Ht} | x' \rangle$$

$$= \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dp' \langle x|p\rangle \langle p | e^{-\frac{i}{\hbar}Ht} |p'\rangle \langle p'|x'\rangle$$

$$= \int_{-\infty}^{+\infty} dp \; \frac{1}{\sqrt{2\pi\hbar}} e^{+\frac{i}{\hbar}px} \; e^{\frac{i}{6m\hbar f}\left[(p-ft)^3 - p^3\right]} \cdot$$

$$\cdot \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}(p-ft)x'}$$

$$= \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \; e^{\frac{1}{6m\hbar f}\left[-3ftp^2 + 3(ft)^2 p - (ft)^3\right]} \cdot$$

$$\cdot e^{\frac{i}{\hbar}(x-x')p} \; e^{\frac{i}{\hbar}ftx'}$$

$$\frac{1}{2m}\left[-p^2 t + ft^2 p\right] + (x-x')p$$

$$= \frac{-t}{2m}\left[p^2 - \left(ft + \frac{2m}{t}(x-x')\right)p\right]$$

$$= \frac{-t}{2m}\left[p - \frac{1}{2}\left(ft + \frac{2m}{t}(x-x')\right)\right]^2 + \frac{t}{8m}\left(ft - \frac{2m}{t}(x-x')\right)^2$$

$$U(x,t;x',0)$$

$$= \frac{1}{2\pi\hbar}\left[\pi\,\frac{2m\hbar}{it}\right]^{1/2}\exp\left\{\frac{-i}{6m\hbar}f^2t^3\right.$$

$$\left.+\frac{i}{\hbar}\,\frac{t}{8m}\left(ft+\frac{2m}{t}(x-x')\right)^2+\frac{i}{\hbar}ftx'\right\}$$

$$= \sqrt{\frac{m}{2\pi\hbar it}}\exp\left\{\frac{+i}{\hbar}\left[ft\,\frac{1}{2}(x+x')+\frac{m}{2t}(x-x')^2\right.\right.$$

$$\left.\left.-\frac{1}{24m}f^2t^3\right]\right\}$$