PHY 6645 - Quantum Mechanics I - Fall 2011
Homework set # 8, due October 19

1. Let $H(\lambda)$ be the Hamiltonian of a system which depends explcitly upon a parameter λ. Consider the eigenvalues $E_n(\lambda)$ and eigenstates $|E_n(\lambda)>$ of $H(\lambda)$. Show that

$$< E_n(\lambda)|\frac{dH}{d\lambda}|E_n(\lambda)>=\frac{dE_n}{d\lambda} \quad , \tag{0.1}$$

for all n.

2. Show that the free particle one-dimensional Schrödinger equation is invariant with respect to Galilian transformations. Do this by showing that, when the transformation $x' = x - vt$, $t' = t$ is applied, the transformed wavefunction $\psi'(x',t') = f(x,t)\psi(x,t)$, with $f(x,t)$ a phase factor, is a solution with respect to the primed variables if $\psi(x,t)$ is a solution with respect to the unprimed ones. Find the phase factor $f(x,t)$, and show that the travelling wave solution $\psi(x,t) = Ae^{i(xt-\omega t)}$ transforms as expected.

3. Problems 7.3.1, 7.3.4 and 7.3.6 in Shankar's book.

1. $$H(\lambda)\,|E_n(\lambda)\rangle = E_n(\lambda)\,|E_n(\lambda)\rangle$$

$$\frac{d}{d\lambda}H(\lambda)\,|E_n(\lambda)\rangle + H(\lambda)\,\frac{d}{d\lambda}|E_n(\lambda)\rangle$$

$$= \frac{dE_n}{d\lambda}\,|E_n(\lambda)\rangle + E_n(\lambda)\,\frac{d}{d\lambda}\,|E_n(\lambda)\rangle$$

$$\langle E_n(\lambda)|\,\frac{dH(\lambda)}{d\lambda}\,|E_n(\lambda)\rangle + \langle E_n(\lambda)|H(\lambda)\frac{d}{d\lambda}\,|E_n(\lambda)\rangle$$

$$= \frac{dE_n}{d\lambda} + E_n(\lambda)\,\langle E_n(\lambda)|\,\frac{d}{d\lambda}\,|E_n(\lambda)\rangle$$

$$\langle E_n(\lambda)|\,H(\lambda) = \langle E_n(\lambda)|\,E_n(\lambda)$$

$$\therefore \quad \frac{dE_n}{d\lambda} = \langle E_n(\lambda)|\,\frac{dH}{d\lambda}\,|E_n(\lambda)\rangle$$

2.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) = +i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'(x',t')$$

$$= -\frac{\hbar^2}{2m}\left(\underbrace{\frac{\partial x}{\partial x'}}_{=1}\frac{\partial}{\partial x} + \underbrace{\frac{\partial t}{\partial x'}}_{=0}\frac{\partial}{\partial t}\right)^2 f(x,t)\,\psi(x,t)$$

since $x = x' + vt'$ $\quad t = t'$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} f(x,t)\,\psi(x,t)$$

$$i\hbar\frac{\partial}{\partial t'}\psi'(x',t')$$

$$= i\hbar\left(\frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\right) f(x,t)\,\psi(x,t)$$

$$= i\hbar\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) f(x,t)\,\psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'(x',t') = i\hbar\frac{\partial}{\partial t'}\psi'(x',t')$$

for any $\psi(x,t)$ which solves the Schrödinger equation, provided

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 f}{\partial x^2}\psi + 2\frac{\partial f}{\partial x}\frac{\partial \psi}{\partial x}\right)$$

$$= i\hbar\left[\psi\left(v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right) + vf\frac{\partial \psi}{\partial x}\right]$$

for any such ψ... We must have:

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial x^2} = i\hbar\left(v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right) & \text{and} \\ -\frac{\hbar^2}{m}\frac{\partial f}{\partial x} = i\hbar v f \end{cases}$$

$$f(x,t) = f(t)\, e^{-\frac{i}{\hbar}mvx}$$

$$\therefore \quad \frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \frac{i\hbar}{2m}\frac{\partial^2 f}{\partial x^2}$$

$$= \left[-v\left(\frac{-i}{\hbar}mv\right) + \frac{i\hbar}{2m}\left(\frac{-i}{\hbar}mv\right)^2\right] f$$

$$= \frac{i}{2\hbar}mv^2 f$$

$$f(x,t) = c\, e^{-\frac{i}{\hbar} m v x + \frac{i}{\hbar} \frac{1}{2} m v^2 t}$$

$\uparrow$

constant $c = 1$

since $f(x,t) = 1$ when $v = 0$

$$\therefore \quad \psi'(x',t') = e^{-\frac{i}{\hbar} m v x + \frac{i}{\hbar} \frac{1}{2} m v^2 t} \psi(x,t)$$

v = boost velocity

Consider $\psi(x,t) = A\, e^{i(kx - \omega t)}$

In this state, the particle has

momentum $p = \hbar k$

energy $E = \hbar \omega$

ψ satisfies the Schrodinger equation

provided $E = \frac{1}{2m} p^2$

The velocity of the particle is

$$v_p = \frac{1}{m} p \qquad E = \frac{1}{2m} v_p^2$$

$$\psi'(x',t') = A\, e^{+\frac{i}{\hbar}(m v_p - m v)x - \frac{i}{\hbar}\left(\frac{1}{2} m v_p^2 - \frac{1}{2} m v^2\right)t}$$

$$= A\, e^{\frac{i}{\hbar} m (v_p - v)(x' + v t') - \frac{i}{\hbar}\left(\frac{1}{2} m v_p^2 - \frac{1}{2} m v^2\right)t'}$$

$$= A\, e^{\frac{i}{\hbar} m (v_p - v) x' - \frac{i}{\hbar} \frac{m}{2} (v_p^2 + v^2 - v_p v) t'}$$

$$= A\, e^{\frac{i}{\hbar} m v_p' x' - \frac{i}{\hbar} \frac{m}{2} v_p'^2 t'}$$ ✓

with $v_p' = v_p - v$

S7.3.1

$$\psi'' + (2\epsilon - y^2)\psi = 0$$

$$\psi(y) = \sum_{n=0}^{\infty} A_n y^n$$

$$\sum_{n=0}^{\infty} A_n \left[n(n-1) y^{n-2} + 2\epsilon y^n - y^{n+2} \right] = 0$$

$$\sum_{n=0}^{\infty} y^n \left[A_{n+2}(n+2)(n+1) + 2\epsilon A_n - A_{n-2} \right] = 0$$

with $A_{-1} = A_{-2} = 0$

$$A_{n+2} = \frac{A_{n-2} - 2\epsilon A_n}{(n+2)(n+1)}$$

$$A_2 = \frac{-2\epsilon A_0}{2 \cdot 1}, \quad A_4 = \frac{A_0 - 2\epsilon A_2}{4 \cdot 3}$$

$$A_4 = A_0 \left(\frac{1}{4 \cdot 3} + \frac{4\epsilon^2}{4 \cdot 3 \cdot 2 \cdot 1} \right)$$

$$= A_0 \frac{1}{4 \cdot 3} \left(1 + \frac{(2\epsilon)^2}{2 \cdot 1} \right)$$

$$A_6 = \frac{1}{6.5}\left(A_2 - 2\epsilon A_4\right)$$

$$= \frac{A_0}{6.5}\left(\frac{-2\epsilon}{2.1} - 2\epsilon\left(\frac{1}{4.3} + \frac{4\epsilon^2}{4.3.2.1}\right)\right)$$

$$= \frac{A_0}{6.5}(-2\epsilon)\left(\frac{1}{2.1} + \frac{1}{4.3} + \frac{4\epsilon^2}{4.3.2.1}\right)$$

$$A_{2p} = \frac{1}{(2p)(2p-1)}\left(A_{2p-4} - 2\epsilon A_{2p-2}\right)$$

$$= \frac{1}{(2p)(2p-1)}\left[\frac{1}{(2p-4)(2p-5)}\left(A_{2p-8} - 2\epsilon A_{2p-6}\right)\right.$$

$$\left. - \frac{1}{(2p-2)(2p-3)}\left(A_{2p-6} - 2\epsilon A_{2p-4}\right)\right]$$

$$= \ldots$$

Indeed, the solution does not follow so readily.

S.7.3.4

$$\Delta = \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

$$\langle n'|X|n\rangle = \int_{-\infty}^{+\infty} dx\; H_{n'}\left(\frac{x}{\Delta}\right) H_n\left(\frac{x}{\Delta}\right) \frac{x\, e^{-\left(\frac{x}{\Delta}\right)^2}}{\sqrt{\pi}\,\Delta\left(2^{n+n'} n!\, n'!\right)^{1/2}}$$

$$= \frac{\Delta}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\; H_{n'}(y)\, H_n(y)\, y\, e^{-y^2}$$

$$= \frac{\Delta}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\; \frac{1}{2} e^{y^2} H_{n'}(y) \left[H_{n+1}(y) + 2n\, H_{n-1}(y)\right]$$

$$= \frac{1}{2}\Delta\left[\delta_{n',n+1}\sqrt{2(n+1)} + 2n\,\delta_{n',n-1}\,\frac{1}{\sqrt{2n}}\right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\delta_{n',n+1}\sqrt{n+1} + \delta_{n',n-1}\sqrt{n}\right]$$

$$\langle n'|P|n\rangle = \int_{-\infty}^{+\infty} dx\ \psi_{n'}(x)\ \frac{\hbar}{i}\frac{d}{dx}\ \psi_n(x)$$

$$= \frac{\hbar}{i}\frac{\Delta^{-1}}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\ H_{n'}(y)\, e^{-\frac{1}{2}y^2}\ \frac{d}{dy}\ H_n(y)\, e^{-\frac{1}{2}y^2}$$

$$= \frac{\hbar}{i}\frac{\Delta^{-1}}{\sqrt{2^{n+n'}\,\pi\, n!\, n'!}} \int_{-\infty}^{+\infty} dy\ H_{n'}(y)\, e^{-y^2} \underbrace{\left[H'_n(y) - y\, H_n(y)\right]}$$

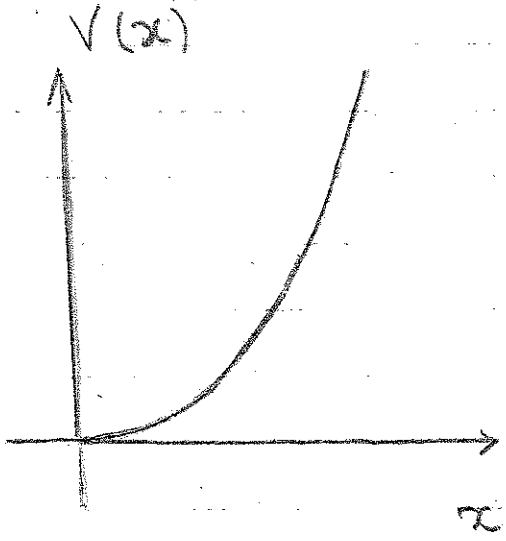
$$2n H_{n-1}(y) - \left(\frac{1}{2}H_{n+1}(y) + n H_{n-1}(y)\right)$$

$$= n H_{n-1}(y) - \frac{1}{2}H_{n+1}(y)$$

$$= \frac{\hbar}{i}\left[\ \delta_{n',n-1}\ n\ \frac{1}{\sqrt{2n}} - \frac{1}{2}\delta_{n',n+1}\sqrt{2(n+1)}\ \right]\Delta^{-1}$$

$$= \left(\frac{m\omega k}{2}\right)^{1/2}\frac{1}{i}\left[\ \delta_{n',n-1}\sqrt{n} - \delta_{n',n+1}\sqrt{n+1}\ \right]$$

S7.3.6



$$V(x) = \frac{1}{2} m\omega^2 x^2 \quad \text{for } x > 0$$

$$= \infty \quad \text{for } x \leq 0$$

$$\psi(x=0) = 0$$

$$\left[-\frac{k^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right] \psi_n(x) = E_n \psi_n(x)$$

Clearly

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\,\Delta\, 2^n n!}} H_n\left(\frac{x}{\Delta}\right) e^{-\frac{1}{2}\frac{x^2}{\Delta^2}} \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x < 0$$

for n odd , are solutions.

There are no other solutions.

Indeed let us assume there is another eigenfunction $\psi'(x)$ of eigenvalue E'.

$$\psi'(x) = \psi_{E'}(x) \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x < 0$$

Consider the wavefunction

$$\psi''(x) = \psi_{E'}(x)\frac{1}{\sqrt{2}} \quad \text{for } x > 0$$

$$= -\psi_{E'}(-x)\frac{1}{\sqrt{2}} \quad \text{for } x < 0$$

$\psi''(x)$ is a eigenfunction of the harmonic oscillator hamiltonian with eigenvalue E'.

But then $\psi'(x)$ is not a new solution.