# Chapter 2

# Conformal Field Theory

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Following *"String Theory"* by J. Polchinski, Vol.I.
Notes written by students (work still in progress).
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## 2.1 Massless scalars in two dimensions

We will start by looking at the Polyakov action with one change. The metric has been replaced with a Euclidean metric $\delta_{a,b}$ with signature (+,+).

$$S = \frac{T}{2}\int d\tau d\sigma(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu) \tag{2.1}$$

where T, $\dot{X}^\mu$, $X'^\mu$ represent the string tension, $\partial_\tau X^\mu$, $\partial_\sigma X^\mu$ respectively.
We can derive the equation of motion by varying the action with respect to the coordinate X. We find:

$$\delta_X S = 0 \Rightarrow X''_\mu + \ddot{X}^\mu = \nabla^2 X^\mu = 0, \qquad \text{where} \qquad \nabla^2 = \partial^2_{\sigma_1} + \partial^2_{\sigma_2}$$

We can define $z$ and $\bar{z}$ as linear combinations of $\sigma_1$ and $\sigma_2$. These will represent the new worldsheet coordinates.

$$z = \sigma_1 + i\sigma_2, \qquad \bar{z} = \sigma_1 - i\sigma_2 \tag{2.2}$$

The bar denotes complex conjugate. We can also invert the coordinate transformation.

$$\sigma_1 = \frac{z+\bar{z}}{2}, \qquad \sigma_2 = \frac{z-\bar{z}}{2}$$

Define differentiation:

$$\partial_z = \partial = \frac{1}{2}(\partial_1 + i\partial_2) \qquad \text{and} \qquad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 - i\partial_2) \tag{2.3}$$

$\nabla^2$ can be written as $4\partial\bar{\partial}$, and the volume element is given by

$$d^2 z = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} d\sigma d\tau = 2 d\sigma d\tau. \tag{2.4}$$

Also define

$$\int d^2 z\, \delta^2(z,\bar{z}) = 1 \tag{2.5}$$

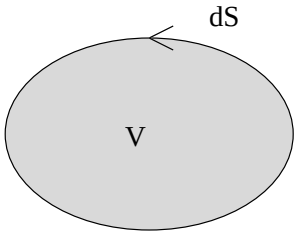
so that $\delta(\sigma^1)\delta(\sigma^2) = \delta^2(z,\bar{z})$. The action in complex coordinates is

$$S = T\int d^2z\partial X^\mu \bar{\partial} X_\mu \ . \tag{2.6}$$

After varying the action with respect to the coordinate $X_\mu$, we get the equation of motion.

$$\partial\bar{\partial}X^\mu = 0 \tag{2.7}$$

This implies $\bar{\partial}X$ is a holomorphic function, ie a function of $z$. Also, $\partial X$ is an antiholomorphic function, ie a function of $\bar{z}$.
Another useful result is the divergence theorem for complex coordinates. First, let's look at the divergence theorem for three dimensions ie, electrostatics. The divergence theorem states that for any well behaved vector field $\mathbf{E}(\mathbf{x})$ defined within a volume V surrounded by the closed surface S the relation




$$\int_V d^3x\ \boldsymbol{\nabla}\cdot\boldsymbol{E} = \oint_S \boldsymbol{E}\cdot\boldsymbol{n}\ da$$

holds between the volume integral of the divergence of $\mathbf{E}$ and the surface integral of the outwardly directly normal component of $\mathbf{E}$.
In 2D:

$$\int d^2z\ (\partial E^z + \bar{\partial}E^{\bar{z}}) = i\oint_{\partial R}(E^z dz - E^{\bar{z}}d\bar{z}) \qquad \text{where} \qquad \hat{n} = (-d\bar{z}, dz). \tag{2.8}$$

We can now write the mode expansions in terms of the complex coordinates.

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{z-\bar{z}}{2}\frac{p^\mu}{p^+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\{\alpha_n^\mu e^{\frac{2n\pi iz}{l}} + \tilde{\alpha}_n^\mu e^{\frac{-2n\pi i\bar{z}}{l}}\} \tag{2.9}$$

As one can see, this mode expansion can be broken into two pieces (left and right handed).

$$X^\mu(z,\bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z})$$

We see the left handed piece cooresponds to a holomorphic function and the right handed piece to a antiholomorphic function. We will never look at the mode expansion itself. Instead we will always look at various derivatives of the mode expansion.

$$\partial X_L^\mu = \frac{i}{2}\frac{p^\mu}{p^+} + -\frac{\pi}{l}\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu exp[\frac{in\pi z}{l}], \tag{2.10}$$

$$\bar{\partial}X_R^\mu = \frac{i}{2}\frac{p^\mu}{p^+} + -\frac{\pi}{l}\sqrt{2\alpha'}\sum_{n\neq 0}\tilde{\alpha}_n^\mu exp[\frac{in\pi\bar{z}}{l}] \tag{2.11}$$

We can absorb the $p^\mu$ by defining $\alpha_0^\mu$ a certain way.

$$\text{Let } \alpha_0^\mu = -i\frac{l}{2\pi\sqrt{2\alpha'}}\frac{p^\mu}{p^+}$$

Now the derivatives on the fields simplify.

$$\partial X_L^\mu = -\frac{\pi}{l}\sqrt{2\alpha'}\sum_{all\ n}\alpha_n^\mu exp[\frac{in\pi z}{l}], \tag{2.12}$$

$$\bar{\partial} X_R^\mu = -\frac{\pi}{l}\sqrt{2\alpha'}\sum_{all\ n}\tilde{\alpha}_n^\mu exp[\frac{in\pi \bar{z}}{l}] \tag{2.13}$$

## 2.2 Solution to the Boundary-Value Problem with Green function

The solution to the Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S can be obtained by means of Greens theorem. In general, we want to solve the equation,

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -\mathbf{4}\pi\delta(\mathbf{x} - \mathbf{x}') \tag{2.14}$$

where G(x,x') is the potential and the delta function is a point source. The solution for G is given as

$$G(\mathbf{x}, \mathbf{x}') = \frac{\mathbf{1}}{|\mathbf{r} - \mathbf{r_0}|} + \mathbf{F}(\mathbf{x}, \mathbf{x}') \tag{2.15}$$

with F satisfying the Laplace equation inside the volume V:

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = \mathbf{0}$$

In two dimensions the Poisson equation is given by

$$\begin{aligned} \nabla^2 G(z, 0; \bar{z}, 0) &= -2\pi\delta(z)\delta(\bar{z}) \qquad \text{where} \qquad \nabla^2 = \partial\bar{\partial} \\ &= -2\pi\delta^2(z, \bar{z}) \end{aligned} \tag{2.16}$$

where

$$G(z, \bar{z}) = \ln|z|^2 = \ln|z| + \ln|\bar{z}| \tag{2.17}$$

This is just the solution for the potential to a line charge in two dimensions. We can prove that G is a solution of the Poisson equation in problem 1.

$$\partial\bar{\partial} G = \partial\bar{\partial}\ln|z|^2 = \partial\frac{1}{\bar{z}} + \bar{\partial}\frac{1}{z} = 2\pi\delta^2(z, \bar{z}), \quad z = 0 \tag{2.18}$$

Now that we see G is directly related to the potential, we can take the gradient to get the electric field.

$$E_z = \partial G, \quad E_{\bar{z}} = \bar{\partial} G$$

$$\int_V d^3x\ \nabla \cdot E = \oint_S E \cdot d\hat{n} = 2\pi R(\frac{1}{R}) = 2\pi \tag{2.19}$$

## 2.3 Amplitudes

$$\langle 0|X^{\mu_1}(z_1)X^{\mu_2}(z_2)\cdots X^{\mu_n}(z_n)|0\rangle$$

Simplest amplitude: 2 point function.
$X^\mu = X^\mu_L = i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n \exp[\frac{-2\pi i n z}{l}]$.

$$\begin{aligned} A^{\mu\nu} &= \langle 0|X^\mu(z)X^\nu(z')|0\rangle \\ &= \frac{\alpha'}{2}\sum_{n,m\neq 0}\frac{1}{nm}exp[\frac{-2\pi i(n-m)z}{l}]\langle 0|\alpha^\mu_n\alpha^\nu_m|0\rangle \end{aligned} \tag{2.20}$$

We can write the expectation value as a commutator plus a term which kills the vacuum.

$$[\alpha^\mu_n, \alpha^\nu_m] = n\eta^{\mu\nu}\delta_{m+m,0}$$

$$A^{\mu\nu} = \frac{\alpha'}{2}\sum_{n\neq 0}\frac{1}{n}exp[-\frac{2\pi i n(z-z')}{l}] \tag{2.21}$$

This is a sum of the form $\sum\frac{\omega^n}{n}$, but only converges for $|\omega|<1$.

$$|exp[-\frac{2\pi i(z-z')}{l}]| = |exp[\frac{2\pi(z-z')}{l}]| < 1 \Rightarrow z_2 - z_2' < 0, z_2 < z_2'$$

So we see the Xs must be time ordered to give a correct result i.e.,

$$\langle 0|XX|0\rangle \Rightarrow \langle 0|T[XX]|0\rangle, \;\; \mathrm{T-time\ ordering.}$$

We define time ordering as

$$T[X^\mu(z)X^\nu(z')] = \begin{cases} X^\mu(z)X^\nu(z') & \text{for } z_2 < z_2' \\ X^\nu(z)X^\mu(z') & \text{for } z_2 > z_2' \end{cases}, \tag{2.22}$$

or in terms of the step function.

$$T[X^\mu(z)X^\nu(z')] = \theta(z_2'-z_2)X^\mu(z)X^\nu(z') + \theta(z_2-z_2')X^\nu(z)X^\mu(z')$$

Evaluating the sum:

$$A^{\mu\nu} = \frac{\alpha'}{2}\sum_n\frac{e^n}{n} = \frac{\alpha'}{2}ln|1-e^{-\beta}| \quad : \partial\bar\partial A^{\mu\nu} \sim \delta(z-z') \rightarrow A^{\mu\nu} \text{ is a Green fn.} \tag{2.23}$$

If we apply the D'Alembertian to A, we see A is a two-point Green function.

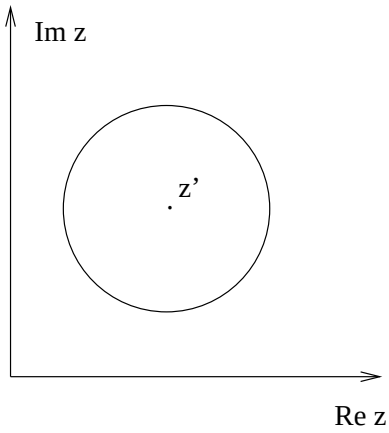
$$\begin{aligned} \partial\bar\partial T[X^\mu(z)X^\nu(z')] &= (\partial^2+\bar\partial^2)T[X^\mu(z)X^\nu(z')] \\ &= T[\partial_1^2X^\mu X^\nu] + T[\partial_1(-\delta(z_2-z_2')X^\mu X^\nu + \delta(z_2-z_2')X^\mu X^\nu)] \\ &= T[\partial_1^2X^\mu X^\nu] + T[\partial_2\delta(z_2-z_2')[X^\mu, X^\nu]] + T[\partial_2X^\mu X^\nu)] \\ &= T[\partial_1^2X^\mu X^\nu] + \delta(z_2'-z_2)[\partial_2X^\mu, X^\nu] + T[\partial_2^2X^\mu, X^\nu] \\ &= T[(\partial_1^2X^\mu + \partial_2^2X^\mu)X^\nu] + \delta(z_2'-z_2)[\partial_2X^\mu, X^\nu] \\ &= \delta(z_2'-z_2)[\partial_2X^\mu, X^\nu] \\ &= \pi\alpha'\delta(z-z', \bar z-\bar z')\eta^{\mu\nu} \end{aligned} \tag{2.24}$$

A must be of the form of a Green function, $A^{\mu\nu} = \eta^{\mu\nu} G(z, z')$.

For future reference X will imply only the holomorphic piece, unless specified otherwise.
Write the time-ordered product in terms of the normal ordered product minus the singularity.

$$T[X^\mu X^\nu] =: X^\mu X^\nu : -\frac{\alpha'}{2} ln|z - z'|\eta^{\mu\nu} \tag{2.25}$$

As z $\to z'$:




$$X^\mu(z)X^\nu(z') = -\frac{\alpha'}{2}\eta^{\mu\nu} \ln|z - z'| + \sum_{k>0}^{\infty} \frac{(z - z')^k}{k!} : X^\nu \partial_k X^\mu(z') : \tag{2.26}$$

$$\sum_k \frac{1}{k!}(-\frac{1}{2}\frac{\alpha'}{2} \int dzdz'\eta^{\mu\nu} \ln|z - z'| \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta X^\nu})^k = exp[-\frac{\alpha'}{4} \int dzdz'\eta^{\mu\nu} \ln|z - z'| \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta X^\nu}] \tag{2.27}$$

Define an operator $\hat{\mathcal{O}}$ such that $\hat{\mathcal{O}} = X^\mu X^\nu$.

$$: \hat{\mathcal{O}} := exp[-\frac{\alpha'}{4} \int dzdz'\eta^{\mu\nu} \ln|z - z'| \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta X^\nu}]\hat{\mathcal{O}} \tag{2.28}$$

This needs to be inverted.

$$\hat{\mathcal{O}} = exp[\frac{\alpha'}{4} \int dzdz'\eta^{\mu\nu} \ln|z - z'| \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta X^\nu}] : \hat{\mathcal{O}} : \tag{2.29}$$

Now $\hat{\mathcal{O}}$ should have no singularities. We have to define the product between two $\hat{\mathcal{O}}$s. This product will have singularities, unless the product of the two is normal ordered.
:$\hat{\mathcal{O}}_1$::$\hat{\mathcal{O}}_2$: $\leftarrow$ has singularities. :$\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2$: $\leftarrow$ has no singularities.

$$: \hat{\mathcal{O}}_1[X]\hat{\mathcal{O}}_2[Y] := exp[-\frac{\alpha'}{2} \int dzdz'\eta^{\mu\nu} \ln|z - z'| \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta X^\nu}] : \hat{\mathcal{O}}_1 :: \hat{\mathcal{O}}_2 : \tag{2.30}$$

invert

$$: \hat{\mathcal{O}}_1[X] :: \hat{\mathcal{O}}_2[Y] := exp[\frac{\alpha'}{2} \int dzdz'\eta^{\mu\nu} \ln|z - z'| \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta Y^\nu}] : \hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 : \tag{2.31}$$

Example:

Let $\hat{\mathcal{O}}_1 = \hat{\mathcal{O}}_2 = \partial X^\mu \partial X_\mu = T$

$$: T(z) :: T(z') :=: \partial X^\mu \partial X_\mu(z) :: \partial' X^\nu \partial' X_\nu :$$

There are four possible single contractions, and two possible double contractions.

$$\begin{aligned} &= : T(z)T(z') : -\frac{\alpha'}{2}\eta^{\mu\nu} \ln|z-z'| * 4 : \partial X_\mu \partial' X_\nu : + \frac{\alpha'^2}{2}\, \eta^{\mu\nu}\eta_{\mu\nu}(\partial\partial' \ln|z-z'|)^2 * 2 \\ : T(z) :: T(z') : &= \frac{\alpha'^2}{2}\frac{d}{(z-z')^4} - \frac{2\alpha'}{(z-z')^2} : T : -\frac{\alpha'}{z-z'} : \partial' T(z') : + : T(z)T(z') : \end{aligned} \tag{2.32}$$

Example:

Let $\hat{\mathcal{O}}_1 =: e^{ik_1 X(z)} :$ $\hat{\mathcal{O}}_2 =: e^{ik_2 X(z')} :$ $\frac{\delta}{\delta X^\mu}\hat{\mathcal{O}}_1 = ik_{1\mu}\hat{\mathcal{O}}_1$

$$\begin{aligned} : \hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 : &= exp[\frac{\alpha'}{2}\ln|z-z'|\eta^{\mu\nu}(ik_{1\mu})(ik_{2\nu})] : \hat{\mathcal{O}}_1 :: \hat{\mathcal{O}}_2 : \\ &= exp[-\frac{\alpha'}{2}\ln|z-z'|k_1\cdot k_2] : \hat{\mathcal{O}}_1 :: \hat{\mathcal{O}}_2 : \\ &= (z-z')^{-\frac{\alpha'}{2}k_1\cdot k_2} : \hat{\mathcal{O}}_1 :: \hat{\mathcal{O}}_2 : \end{aligned} \tag{2.33}$$

$$\begin{aligned} \Rightarrow : \hat{\mathcal{O}}_1 :: \hat{\mathcal{O}}_2 : &= (z-z')^{\frac{\alpha'}{2}k_1\cdot k_2} : \hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 : \\ &= (z-z')^{-\frac{\alpha'}{2}k_1\cdot k_2} : e^{i(k_1+k_2)\cdot X}(1+\hat{\mathcal{O}}(z-z')) : \end{aligned} \tag{2.34}$$

## 2.4 Noether's Theorem

For every symmetry in a theory, there must be some conserved current, ie

$$\partial_\mu j^\mu = 0 \Rightarrow \text{Symmetry(S)}. \tag{2.35}$$

We can integrate over the zeroth component of the conserved current to get the charge.

$$Q = \int d^3x \partial_0 j^0 \rightarrow \frac{dQ}{dt} = \int_R d^3x j^0 = \int d^3x \nabla\cdot\vec{j} = \int d\vec{s}\cdot\vec{j} = 0 \tag{2.36}$$

Q generates transformations.

$$\delta A = i\epsilon[Q, A] = i\epsilon(QA - AQ) \qquad \epsilon << 1$$

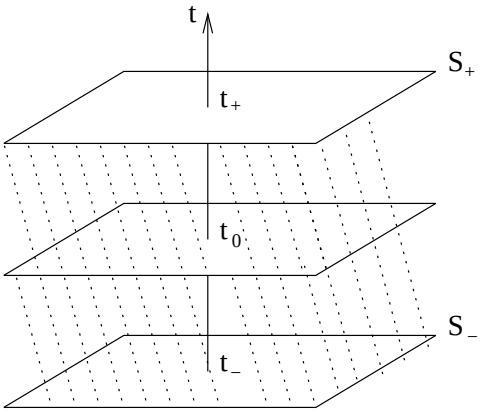
Example: Let Q=H

$$\delta A = i\epsilon[H, A] = \dot{A}$$

or Q=$\vec{p}$

$$\vec{\nabla} A = i[\vec{p}, A]$$

Look at $A(t_0)$:




Region R is bounded by $[S_+, S_-]$.

$$\begin{aligned}
\delta A &= i\epsilon(Q(t_+)A(t_0) - A(t_0)Q(t_-)) \\
&= i\epsilon\{\int_{S_+} d^3x\, j^0 A(t_0) - \int_{S_-} d^3x\, j^0 A(t_0)\} \\
&= i\epsilon \int_{\partial R} dS_\mu j^\mu A(t_0) \\
&= i\epsilon \int_{\partial R} dS_\mu T[j^\mu A(t_0)] \\
\delta A &= i\epsilon \int_R d^4x\, \partial_\mu T[j^\mu A(t_0)] \quad \text{"Ward Identity"}
\end{aligned} \tag{2.37}$$

Show the Ward Identity explicitly in two dimensions:

$$\begin{aligned}
\delta A &= \frac{i\epsilon}{2\pi} \int_R d^2z\, \partial_a T[j^a A(z_0)], \quad j = (j_z, j_{\bar{z}}) \\
&= \frac{i\epsilon}{2\pi} \oint_{\partial R} (dz j_z - d\bar{z} j_{\bar{z}}) A(z_0)
\end{aligned} \tag{2.38}$$

$\partial_a j^a = \partial_{\bar{z}} j_z + \partial_z j_{\bar{z}} = 0$ for special case $j_z$ is holomorphic, and $j_{\bar{z}}$ is antiholomorphic.

$$\oint \frac{dz}{2\pi} j_z A(z_0) = \lambda(z_0): \quad j_z A(z_0) = \ldots + \frac{\lambda(z_0)}{z - z_0} + \ldots \tag{2.39}$$

$$\delta A = \epsilon\lambda + \epsilon\bar{\lambda} = -\epsilon(\lambda - \bar{\lambda}) \tag{2.40}$$

Example:

Define the transformation on X and the current density.

$$X^\mu \to x^\mu + \epsilon a^\mu: \quad j_a^\mu = \frac{i}{\alpha'} \partial_a A^\mu \quad j_z^\mu = \frac{i}{\alpha'} \partial X^\mu$$

$$\bar{\partial} j_z^\mu = 0 \quad \text{remember X is holomorphic}$$

Let $A(z_0) =: e^{ik \cdot X}:$

$$\begin{aligned} \delta A &= :e^{ik\cdot(X+\epsilon a)}: \\ &= (1+i\epsilon k\cdot a)A(z_0) \end{aligned} \tag{2.41}$$

$$\begin{aligned} j_z^\mu A(z_0) &= \frac{i}{\alpha'}\frac{\alpha'}{2}\eta^{\mu\nu}\partial\ln|z-z_0|(ik_\nu):A(z_0):+\text{regular terms} \\ &= -\frac{1}{2}\frac{k^\mu}{z-z_0}A(z_0) \\ \Rightarrow\quad \lambda &= \frac{1}{2}k^\mu A(z_0) = -\bar{\lambda}: \quad \text{Residue} \end{aligned} \tag{2.42}$$

$$\epsilon(\lambda+\bar{\lambda}) = \epsilon(k^\mu A(z_0))$$

$$\epsilon a^\mu(\lambda-\bar{\lambda}) = \epsilon a^\mu k_\mu A(z_0)$$

Example:

$$\begin{aligned} \delta z = -\epsilon, \delta X^\mu &= X^\mu(z-\epsilon) - X^\mu(z) \\ &= -\epsilon\partial X_\mu \end{aligned} \tag{2.43}$$

Noether Current

$$S = -\frac{1}{2\pi\alpha'}\int \partial X^\mu\bar{\partial}X_\mu$$

$$\begin{aligned} \delta z = -\epsilon(z,\bar{z}) \qquad \delta S &= \frac{1}{2\pi\alpha'}\int \partial(\epsilon\partial X^\mu)\bar{\partial}X_\mu + \partial X^\mu\bar{\partial}(\epsilon\partial X_\mu) \\ &= \frac{1}{\pi\alpha'}\int \bar{\partial}\epsilon(\partial X^\mu\partial X_\mu) \end{aligned} \tag{2.44}$$

## 2.5 Conformal invariance

$$T = \frac{1}{\pi\alpha'}:\partial X^\mu\partial X_\mu: \qquad \bar{\partial}T = 0 \quad \text{conserved, T is holomorphic} \tag{2.45}$$

$$\bar{T} = \frac{1}{\pi\alpha'}:\bar{\partial}X^\mu\bar{\partial}X_\mu: \qquad \partial\bar{T} = 0 \quad \bar{\text{T}} \text{ is antiholomorphic} \tag{2.46}$$

$$T_{\tau\tau} = -\frac{1}{2\alpha'}:\dot{X}^2 + x'^2 := T_{\sigma\sigma}$$

$$T_{\tau\sigma} = T_{\sigma\tau} = -\frac{1}{\alpha'}\dot{X}\cdot X': \qquad \text{Traceless} \quad T^a_a = 0.$$

For an arbitrary function v(z):

$$j(z) = iv(z)T(z) \qquad \bar{\partial} j = 0 \Rightarrow \text{infinite number of conservation laws.}$$

From $j = iv(z)T(z)$, $\partial^a j_a = 0$

$$\bar{\partial} j = 0 \text{ or } \partial \bar{\mathrm{j}} = 0$$
$$\bar{\partial}(v(z)T(z)) = 0$$

v(z)T(z) is a conserved current, where

$$vTx^\mu \sim \frac{v\partial x^\mu}{z - z'}.$$

Let z transform as $\delta z = z + \varepsilon V$, then

$$x^\mu \longrightarrow x^\mu - \varepsilon v \partial x^\mu$$

and $T(z)A(z')$ can be expanded as Laurent series

$$T(z)A(z') = \frac{a_{-1}}{z - z'} + \frac{a_{-2}}{(z - z')^2} + ... + \text{regular terms}, \tag{2.47}$$

then $\lambda$ is

$$\begin{aligned}
\lambda &= \oint \frac{dz}{2\pi} iv(z)T(z)A(z') \\
&= \oint \frac{dz}{2\pi} i[\frac{a_{-1}v(z)}{z - z'} + \frac{a_{-2}v(z)}{(z - z')^2} + ..] \\
&= ia_{-1}v(z') + ia_{-2}\partial v(z') + \frac{i}{2!}a_{-3}\partial^2 v(z') + \cdots . \quad \text{ie.} \qquad (2.48) \\
\delta A &= -\varepsilon a_{-1}v - \varepsilon a_{-2}\partial v - \frac{\varepsilon}{2!}a_{-3}\partial^2 v - \cdots , \\
&= -\varepsilon \sum_{n=0}^{\infty} \frac{1}{n!} a_{-n-1}\partial^n v. \qquad (2.49)
\end{aligned}$$

An example for v(z)=z, z $\rightarrow z + \delta z = (1 + \varepsilon)z$, then $\delta A = -h\varepsilon A$, ie.

$$\begin{aligned}
A' &= (1 + \varepsilon)^{-h} A \\
&= (\zeta)^{-h} A. \qquad (2.50)
\end{aligned}$$

or $A(z, \tilde{z})$ transforms as

$$A(z, \bar{z}) \longrightarrow \zeta^{-h} \bar{\zeta}^{-\tilde{h}} A.$$

For $\zeta = re^{i\theta}$,

$$A \longrightarrow r^{-(h+\tilde{h})} e^{-(h-\tilde{h})\theta} A$$

$h + \tilde{h}$ represents the magnitude and $h - \tilde{h}$ represents the spin of A. If A is order h, then $\partial A$ is order h+1, ie.

$$\begin{aligned}
\frac{\partial A}{\partial z} &= \frac{\partial z'}{\partial z}\frac{\partial A}{\partial z'} \\
&\rightarrow (1 - \varepsilon z)(1 - h\varepsilon z)\partial A \\
&\rightarrow (1 - (h + 1)\varepsilon z)\partial A.
\end{aligned}$$

or

$$\bar{\partial}A, \quad \tilde{h} \to \tilde{h}+1$$

Compare the coefficient of $\partial v$ with the equations for $\delta A$. This implies $a_{-2}$ will be $hA$. For simplicity, let $v(z)=1$. Do a translation transformation on z, $z \to z+\varepsilon$. Then $\delta A = A(z-\varepsilon) - A(z) = \varepsilon \partial A \longrightarrow a_{-1} = \partial A$. For an arbitary $v(z)$, $z \to z+\varepsilon v(z)$, $\delta A$ is given by

$$\begin{aligned} \delta A &= -h\varepsilon \partial v A \\ A' &= (\tfrac{\partial \zeta}{\partial z})^{-h} (\tfrac{\partial \bar{\zeta}}{\partial \bar{z}})^{-\tilde{h}} A \\ A' &= (1+\varepsilon \partial v)^{-h} A. \end{aligned}$$

If $\delta A$ has only two singularity terms in the form below, call $A$ a primary field, ie.

$$T(z)A(z') = \frac{\partial A}{z-z'} + \frac{h\partial A}{(z-z')^2}. \tag{2.51}$$

Some examples for $T(z)A(z')$

$$\begin{aligned} T(z)X^\mu(z') &= -\frac{1}{\alpha'} : \partial X^\nu \partial X_\nu : X^\mu(z') = \frac{1}{2}\eta^{\mu\nu}\partial \ln(z-z')\partial X_\nu \\ &= \frac{1}{z-z'}\partial X^\mu, \text{ ie. h} = \tilde{\text{h}} = 0 \qquad (2.52) \\ T(z)\partial^2 X^\mu &= \frac{1}{2}\eta^{\mu\nu}\partial\partial^2 \ln(z-z')\partial X_\nu \\ &= \frac{2}{(z-z')^3}\partial X^\mu(z) \\ &= \frac{2}{(z-z')^3}[\partial X^\mu(z') + (z-z')\partial^2 X(z') + \frac{1}{2!}(z-z')^2\partial^3 X^\mu(z') + ... \\ &= \frac{2}{(z-z')^3}\partial X^\mu + \frac{2}{(z-z')^2}\partial^2 X^\mu(z') + \frac{1}{z-z'}\partial\partial^2 X^\mu + ... \qquad (2.53) \\ T(z) : e^{ikX} : &= \frac{1}{\alpha'}(\frac{\alpha'}{2}\eta^{\mu\nu}k_\nu \partial \ln(z-z'))^2 : e^{ik\cdot X} : + \eta^{\mu\nu}k_\nu \partial \ln(z-z') : \partial X_\mu e^{ik\cdot X} : \\ &= \frac{\alpha'}{4}\frac{k^2}{(z-z')^2} : e^{ik\cdot X} : + \frac{1}{z-z'} : \partial e^{ik\cdot X} \qquad (2.54) \end{aligned}$$

For $A = e^{ik\cdot X}$, $T(z)A(z')$ implies $h = \frac{\alpha' k^2}{4}$ and $\bar{T}(\bar{z})A(\bar{z}')$ implies $\tilde{h} = \frac{\alpha' k^2}{4}$. Therefore A has weight $(h, \tilde{h}) = (\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$. If a translation is applied to z ($z \to z+\varepsilon v$), then $e^{ik\cdot X(z)} \to e^{ik\cdot X(z-\varepsilon v)}$ and $\delta(e^{ik\cdot X(z)}) = -\varepsilon v \partial e^{ik\cdot X(z)}$, which means that h=0. We have just shown that h is not equal to zero, therefore this result must come from a quantum effect, where

$$\begin{aligned} \partial X e^{ik\cdot X} &\longrightarrow h = 1 + \tfrac{\alpha' k^2}{4} \\ \partial^2 X e^{ik\cdot X} &\longrightarrow h = 2 + \tfrac{\alpha' k^2}{4} \\ \partial^m X e^{ik\cdot X} &\longrightarrow h = m + \tfrac{\alpha' k^2}{4} \\ \partial^{m_n} X^{\mu_n} ... \partial^{m_2} X^{\mu_2} \partial^{m_1} X^{\mu_1} e^{ik\cdot X} &\longrightarrow h = m_n + ... + m_2 + m_1 + \tfrac{\alpha' k^2}{4} \end{aligned}$$

| | | | |
|---|---|---|---|
| $X^\mu$ | $h=0$ | $\tilde{h}=0$ | $(0,0)$ |
| $\partial X^\mu$ | $h=1$ | $\tilde{h}=0$ | $(1,0)$ |
| $\bar{\partial} X^\mu$ | $h=0$ | $\tilde{h}=1$ | $(0,1)$ |
| $\partial^2 X^\mu$ | $h=2$ | $\tilde{h}=0$ | $(2,0)$ |
| $e^{ik\cdot X}$ | $h=\frac{\alpha' k^2}{4}$ | $\tilde{h}=\frac{\alpha' k^2}{4}$ | $(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$ |
| $\partial X e^{ik\cdot X}$ | $h=1+\frac{\alpha' k^2}{4}$ | $\tilde{h}=1+\frac{\alpha' k^2}{4}$ | $(1+\frac{\alpha' k^2}{4}, 1+\frac{\alpha' k^2}{4})$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

## 2.6 Free CFTs

### 2.6.1 Linear dilaton

The TT OPE is given by:

$$T(z)T(z') = \frac{D}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{2}{\alpha'(z-z')} : \partial^2 X_\mu \partial X^\mu :$$

where $\partial^2 X_\mu \partial X^\mu$ can be written as $\frac{1}{2}(\partial(\partial X_\mu \partial X^\mu))$

If $T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : + V_\mu \partial^2 X^\mu$, the TT OPE is given by

$$\begin{aligned} T(z)T(z') &\sim \frac{D}{2(z-z')^4} + V_\mu V^\mu \frac{\alpha'}{2}\partial^2\partial'^2 \ln(z-z') + ... \\ &= \frac{D}{2(z-z')^4} + \frac{6V_\mu V^\mu \alpha'}{2(z-z')^4} + ... \\ &= \frac{c}{2(z-z')^4} + ... \,, \end{aligned} \tag{2.55}$$

where $c = D + 6\alpha' V_\mu V^\mu$.

Therefore the central charge for the Linear Dilaton theory can be any number. When the number of dimensions is one or two the theory is exactly solvable.

$$\begin{aligned} T(z)X^\mu(z') &\sim \partial \ln(z-z')\partial X^\mu + V^\mu \frac{\alpha'}{2}\partial^2 \ln(z-z') + ... \\ &= \frac{1}{z-z'}\partial X^\mu + \frac{V^\mu \alpha'}{2(z-z')^2}, \quad h = 0 \text{ and X}^\mu \text{ is not a primary field,} \end{aligned} \tag{2.56}$$

$$v(z)T(z)X^\mu(z') \sim \frac{V^\mu \alpha'}{2(z-z')^2}[v(z') + (z-z')\partial v + ...] + \frac{v}{z-z'}\partial X^\mu + .... \tag{2.57}$$

For $\delta X^\mu = -\varepsilon,\ \lambda = \frac{1}{2}V^\mu \alpha' \partial v + v\partial X^\mu$.

### 2.6.2 bc theory

Let b and c be anticommuting fields, ie spinors. The action can be writen as

$$S = \frac{1}{2\pi}\int d^2 z b\bar{\partial} c \tag{2.58}$$

The equations of motion are given by: $\bar{\partial} c = 0$, $\bar{\partial} b = 0$, where b and c are holomorphic. If we let $l = 2\pi$, then we can write b and c as

$$b(z) = i\sum b_n e^{inz},\ c(z) = i\sum c_n e^{inz}, \tag{2.59}$$

where

$$\{b(z), c(z')\} = \delta(\sigma - \sigma')_{\text{equaltime}},\ z_2 = z_2' \text{ and } 0 < \sigma < 2\pi \text{ or} \tag{2.60}$$

$$\{b_m, c_n\} = \delta_{m+n,0} \tag{2.61}$$

$$< 0|b(z)c(z')|0 > = \frac{1}{1 - e^{i(z-z')}}$$

$$\sim \frac{1}{z-z"} + \text{regular terms, then} \tag{2.62}$$

$$: b(z)c(z') : = b(z)c(z') - \frac{1}{z-z'} \tag{2.63}$$

If b has weight $h_b = \lambda$, then $h_c = 1 - \lambda$. This is known since the action has weight 0 and the volume element has weight (-1,-1). From the transformation $\delta z = \varepsilon(z)$, b will change to

$$\begin{aligned} b' &= (\frac{\partial z'}{\partial z})^{\lambda} b(z-\varepsilon) \\ &= (1-\lambda\partial\varepsilon)(b-\varepsilon\partial b), \text{ then} && (2.64) \\ \delta b &= -\lambda\partial b - \varepsilon\partial b, \text{ and} && (2.65) \\ \delta c &= -(1-\lambda)\partial c - \varepsilon\partial c && (2.66) \\ \delta S &= \int \bar{\partial}\varepsilon((\partial b)c - \lambda\partial(bc)), && (2.67) \end{aligned}$$

where $(\partial b)c - \lambda\partial(bc)$ is the Noether current in this case, then T is

$$\begin{aligned} T &= : (\partial b)c : -\lambda\partial(: bc :), \text{ and} && (2.68) \\ c(z)b(z') &= \frac{1}{z-z'} + ... && (2.69) \\ b(z)c(z') &= \frac{1}{z-z'} + ... && (2.70) \\ T(z)T(z') &= -(\frac{1}{z-z'})^2 - 2\lambda\partial(\frac{1}{z-z'}\partial\frac{1}{z-z'}) + \lambda^2\partial\partial'\frac{1}{(z-z')^2} \\ &= \frac{-1+6\lambda-6\lambda^2}{2(z-z')^4} + ... \\ &= \frac{c}{2(z-z')^4} && (2.71) \\ c &= -2+12\lambda-12\lambda^2 = 1-3(2\lambda-1)^2, \text{ central charge.} && (2.72) \end{aligned}$$

From the Linear Dilaton theory $c = D + 6\alpha V^2$. Let the charges from the two theories be equal, ie. $D + 6\alpha V^2 = 1 - 3(2\lambda-1)^2$ and solve for V. For the case when D=1, one will obtain

$$V = \frac{1}{\sqrt{2\pi}}(2\lambda - 1). \tag{2.73}$$

In the special case ($\lambda = \frac{1}{2}$), V=0, c=1, b and c can be writen as a linear combination of scalar fields $\Psi_1$ and $\Psi_2$

$$b(z) = \frac{1}{\sqrt{2}}(\Psi_1 + i\Psi_2)$$

$$c(z) = \frac{1}{\sqrt{2}}(\Psi_1 - i\Psi_2)$$

$$S = \frac{1}{4\pi}\int d^2z(\Psi_1\bar{\partial}\Psi_1 + \Psi_2\bar{\partial}\Psi_2)$$

$$T = -\frac{1}{2}\Psi_1\partial\Psi_1$$

Note that if $\lambda = 2$ and V=0, then c is -26 from $c = 1 - 3(2\lambda-1)^2$. This is the result obtained in chapter 1.

### 2.6.3 $\beta\gamma$ theory

The next example, the bosonic case, let $\beta$ and $\gamma$ be commuting scalar fields. The action is given by:

$$S = \frac{1}{\sqrt{2\pi}} \int d^2z \beta \bar{\partial} \gamma, \tag{2.74}$$

where

$$\bar{\partial}\beta = \bar{\partial}\gamma = 0 \tag{2.75}$$

The procedure is the same as for the spinor case.

$$\begin{aligned} \beta(z)\gamma(z') &= \frac{1}{z-z'} + ... && (2.76)\\ \gamma(z)\beta(z') &= -\frac{1}{z-z'} + ... && (2.77)\\ c &= -1 + 3(2\lambda - 1)^2 && (2.78) \end{aligned}$$

Note that if $\lambda = \frac{3}{2}$, then c is 11.

$$\left.\begin{array}{cccc} X^\mu & D & (b,c) & -26 \\ \Psi^\mu & \frac{d}{2} & (\beta,\gamma) & 11 \end{array}\right\} D + \frac{d}{2} - 26 + 11 = 0 \Longrightarrow D = 10.$$

From

$$X_L^\mu = x^\mu - \frac{\alpha'}{2} p^\mu z = i\sqrt{\frac{\alpha'}{2}}\, \Sigma \frac{1}{n} \alpha_n^\mu e^{inz}$$

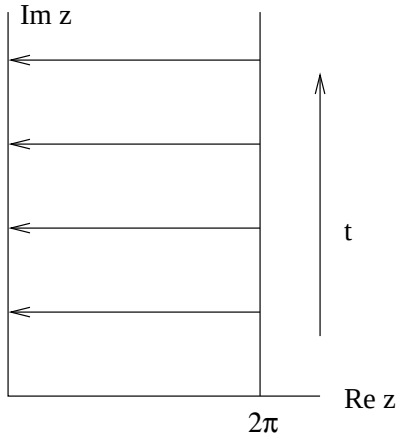
$$\begin{aligned} < X_L(z), X_L(z') > &= \ln(1 - e^{i(z-z')}) \\ &\sim \ln(z-z') + ... && (2.79) \end{aligned}$$

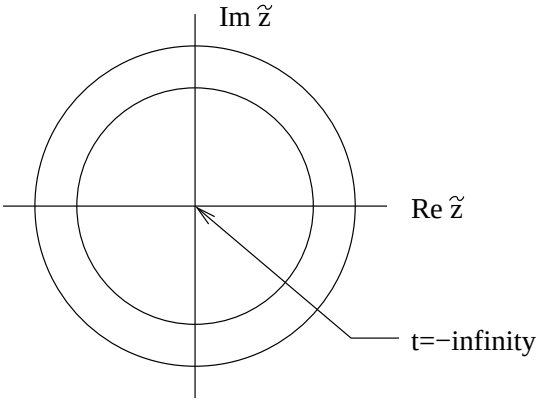
## 2.7 The Virasoro algebra

The worldsheet of a free closed string moving through space-time looks like a cylinder whose radius may fluctuate depending on the excitation mode. We can map this cylinder to the complex plane. This map would be equivalent to sqeezing one end of the string so it looks like a cone. Then smash the cone into a plane. Now the worldsheet coordinates can be expressed as complex coordinates, where r will represent tau and the phase term will represent the position on the string, sigma.

$z \to \tilde{z}$ closed strings $l = 2\pi, 0 < \sigma < 2\pi$

$$\tilde{z} = e^{-iz}$$




Instead of expanding in Fourier modes, do a Laurent Expansion.

$$X^\mu \sim \sum \frac{1}{n} \alpha_n^\mu \tilde{z}^{-n}$$

What happens to T(z):

$$\begin{aligned} T(z) &= \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \\ &= \sum T_m e^{imz} \\ &\stackrel{?}{=} \sum T_m \tilde{z}^{-m} \end{aligned}$$

T doesn't transform as easily as guessed.

Subtract term with central charge to make it a primary field (tensor). Then transform each coordinate.

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2} T(z') + \frac{1}{z-z'} \partial T(z') + \dots \tag{2.80}$$

multiply both sides by v(z)

$$v(z)T(z)T(z') = v(z)[\dots]$$

$$\delta T = -\epsilon\lambda = \epsilon \frac{c}{2} \frac{1}{3!} \partial^3 v - 2\epsilon \partial v T - \epsilon v \partial T \tag{2.81}$$

$$z \to z + \epsilon v(z) \;\; z \to \tilde{z} = e^{-iz}$$

$$T(z) = (\frac{\partial \tilde{z}}{\partial z})^2 T(\tilde{z}) + \frac{c}{12} \{\tilde{z}, z\} \tag{2.82}$$

where $\{\tilde{z}, z\}$ is known as a Schwarzian derivative, defined below:

$$\{\tilde{z}, z\} = \frac{2\tilde{z}'''\tilde{z}' - 3(\tilde{z}'')^2}{2(\tilde{z}')^2} \tag{2.83}$$

=1/2 for our example.

$$T = -\tilde{z}^2\tilde{T} + \frac{c}{24}$$

We can invert this and solve for $\tilde{z}$,

$$\begin{aligned} \tilde{T} &= -\tilde{z}^{-2}\tilde{T} + \frac{c}{24}\tilde{z}^{-2} \\ &= \sum L_m z^{-m-2}, \qquad L_m = -T_m + \frac{c}{24}\delta_{m,0} \end{aligned} \tag{2.84}$$

Invert the equation and solve for L.

$$L_m = \oint \frac{d\tilde{z}}{2\pi i}\tilde{z}^{m+1}\tilde{T} \tag{2.85}$$

The Hamiltonian $\int_0^{2\pi}\frac{d\sigma}{2\pi}T$ can now be written in terms of L.

$$H = L_0 - \frac{c}{24} \tag{2.86}$$

We can see that $\bar{\partial}L_m = 0$ since $\bar{\partial}[\tilde{z}^{m+1}\tilde{T}] = 0$. This implies all $L_m$s generate symmetries.
The Ward identity for any operator A:

$$\delta A = i\epsilon[Q, A]$$

implies Q can be written as an integral of the current, j. $Q = \oint \frac{dz}{2\pi i}j(z)$

The OPE of T with some primary field of weight h is given by:

$$T(z)A(z') = \frac{h}{(z-z')^2}A(z') + \frac{\partial A(z')}{z-z'} + \dots \tag{2.87}$$

Look at the variation of A:

$$\delta A = -\epsilon h \partial v A - \epsilon v \partial A$$

choose $v = \tilde{z}^{m+1} \quad j = \tilde{z}^{m+1}\tilde{T} \quad Q = \oint j \sim L_m$

$$\Rightarrow \delta A = i\epsilon[L_m, A]$$

$$[Q, A] = h(m+1)\tilde{z}^m + \tilde{Z}^{m+1}\partial A = [L_m, A]$$

expand A in a Laurent expansion:

$$\begin{aligned} A &= (\frac{\partial\tilde{z}}{\partial z})^{-h}\sum A_m\tilde{z}^{-m} \\ &= \sum A_m\tilde{z}^{-m-h} \end{aligned} \tag{2.88}$$

This is the expansion for a primary field in the $\tilde{z}$ coordinates. Look at the commutator of L with A in these cordinates.

$$\begin{aligned}[L_m, A_n] &= h(m+1)A_{n+m} - (n+m+h)A_{n+m} \\ &= [(h-1)m - n]A_{m+n} \end{aligned} \tag{2.89}$$

We know the algebra, but what are the representations of the algebra.

$$\delta T = \text{expected} + \frac{c}{12}\partial^3 v$$

$$[L_m, T] = \text{expected} + \frac{c}{12}(m+1)m(m-1)\tilde{z}^{m-2}$$

$$\begin{aligned}[L_m, L_n] &= \text{expected} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \text{ "VirasoroAlgebra"} \end{aligned} \tag{2.90}$$

Look at the case m=0:

$$[L_0, A_n] = -nA_n, \quad [L_0, L_n] = -nL_n \tag{2.91}$$

$$L_0|\psi\rangle = E|\psi\rangle \quad |\psi'\rangle = L_n|\psi\rangle$$

$$\begin{aligned} L_0|\psi'\rangle &= [L_0, L_n]|\psi\rangle + L_n L_0|\psi\rangle \\ &= -nL_n|\psi\rangle + EL_n|\psi\rangle \\ &= (E-n)L_n|\psi\rangle \\ &= (E-n)|\psi'\rangle \end{aligned} \tag{2.92}$$

For $n < 0 \Rightarrow E' > E$ : E'=E-n $\rightarrow$ E is increasing.
For $n > 0 \Rightarrow E' < E$: Hence E is decreasing forever and there is no ground state energy. If H isn't bounded from below, the theory has a major problem!
Look at n=-1,0,1. Is this analogous to the raising and lowering operators for angular momentum.

$$[L_0, L_1] = -L_1$$

$$[L_0, L_{-1}] = L_{-1}$$

$$[L_1, L_{-1}] = 2L_0$$

We notice that this closed algebra is independent of c, therefore for every CFT we should get this subalgebra. This is a Lie algebra SL(2,R) and is not compact.
Look at Quantum Mechanics:

$$[L_0, L_+] = L_+$$

$$[L_0, L_-] = -L_-$$

$$[L_+, L_-] = 2L_0$$

This is close to the algebra above, but not the same. The difference is actually very important!

## 2.8 More expansions

### 2.8.1 Free scalars

Now that we know how to transform to these $\tilde{z}$ coordinates, we can look at the same calculations and look for similarities. The mode expansion is given by,

$$X_L^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \ln\tilde{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}\alpha_m^\mu \tilde{z}^{-m}. \tag{2.93}$$

The two-point function which has to be radially ordered is given by,

$$\begin{aligned}\langle|X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}')|\rangle &= \frac{\alpha'}{2}\sum\frac{1}{m}(\frac{\tilde{z}'}{\tilde{z}})\eta^{\mu\nu}|\tilde{z}'| < |\tilde{z}| \ \rightarrow \text{ time ordering...i.e. radial ordering} &(2.94)\\ &= \frac{\alpha'}{2}\ln|1-\frac{\tilde{z}'}{\tilde{z}}|\eta^{\mu\nu}. &(2.95)\end{aligned}$$

The normal ordered product is given by,

$$:XX: = XX = \frac{\alpha'}{2}\ln|z-z'| \text{ in z picture.} \tag{2.96}$$

Now we can compare our definition for :: to switching $a$ and $a^\dagger$ around.

$$\begin{aligned}:X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}'): &= :(X_L^+ + X_L^-)^\mu(X_L^+ + X_L^-)^\nu:\\ &= X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}') + [X^{(-)}(z'), X^{(+)}(z)]\\ &= X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}') - \frac{\alpha'}{2}\sum_n\frac{1}{n}\eta^{\mu\nu}\frac{\tilde{z}'}{\tilde{z}}^n \end{aligned}\tag{2.97}$$

$$[x^\mu, -i\frac{\alpha'}{2}p^\mu\ln\tilde{z}] = XX + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|\tilde{z}-\tilde{z}'|. \tag{2.98}$$

From normal ordering

$$X^\mu(z)X^\nu(z') = \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z') + :X^\mu(z)X^\nu(z'):,$$

the operator product can be writen as

$$\text{product} = \text{singularity} + \text{normal ordering product}$$

From

$$L^m = \oint\frac{dz}{2\pi i}z^{m+1}T(z), \quad \text{charge}$$

where $z^{m+1}T(z)$ is a conserved current and

$$\begin{aligned}\partial X^\mu &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\alpha_m^\mu z^{-m-1}\\ P^\mu &= \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu\\ X^\mu &= x^\mu + p^\mu\ln z + i\sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu z^{-m}\end{aligned}$$

$$\begin{aligned} T(z) &= \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \\ &= \frac{1}{2} \sum_{n_1,n_2} z^{-n_1-1} z^{-n_2-1} : \alpha^\mu_{n_1} \alpha_{n_2 \mu} : \\ L_m &= \frac{1}{2} \sum_n : \alpha^\mu_{m-n} \alpha_{n\mu} : \end{aligned} \tag{2.99}$$

The Hamiltonian ($L_0$) is given by:

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \alpha^\mu_{-n} \alpha_{n\mu} \tag{2.100}$$

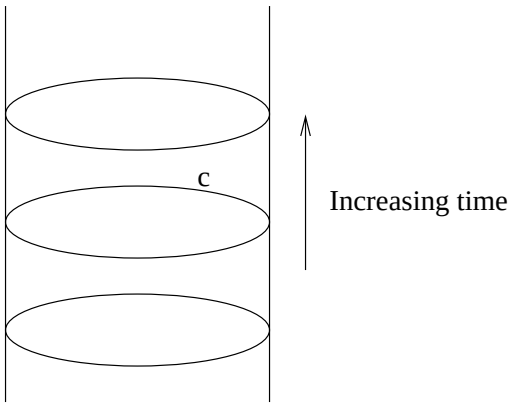
The commutation relations are:

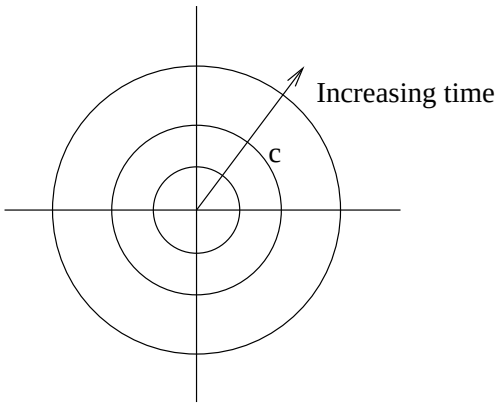
$$[X^\mu(\sigma), \Pi^\nu(\sigma')] = 2\pi i \delta(\sigma - \sigma') \tag{2.101}$$

$$[X^\mu(\sigma), X^\nu(\sigma')] = 0, \text{ where } \mathrm{X} = \mathrm{X_L(z)} + \mathrm{X_R}(\bar{\mathrm{z}}) \tag{2.102}$$

$$\begin{aligned} [X^\mu_L(z), X^\nu_L(z')] &= X^\mu_L(z) X^\nu_L(z') - X^\nu_L(z') X^\mu_L(z) \\ &= \frac{\alpha'}{2} \eta^{\mu\nu} \ln(z - z') - \frac{\alpha'}{2} \eta^{\mu\nu} \ln(z - z') \\ &= \frac{\pi i}{2} \alpha' \eta^{\mu\nu} \frac{d}{dz} (\text{step function}) \\ &= \frac{\pi i}{2} \alpha' \eta^{\mu\nu} \delta(z - z'), \end{aligned} \tag{2.103}$$

where differentiating a step function, a delta function is obtained.




An example for $: e^{ikX} :$ is

$$\begin{aligned} e^A e^B &= e^{[A,B]} e^B e^A, \text{ then} \\ : e^{ik_1 X_1} :: e^{ik_2 X_2} : &= e^{\pm i\pi \frac{\alpha'}{2} k_1 \cdot k_2} : e^{ik_2 X_2} :: e^{ik_1 X_1} :, \end{aligned}$$

For the special case D=1, $X^\mu = \frac{\alpha'}{2}\Psi$, $k_1^\mu = \pm\sqrt{\frac{2}{\alpha'}} = k_2^\mu$ and let $O_1 = e^{\pm i\psi}$ and $O_2 = e^{\pm i\psi}$, then

$$O_1 O_2 = e^{\pm i\pi} O_2 O_1 = -O_2 O_1$$

or

$$\begin{aligned}
\{O_1O_2, O_2O_1\} &= 0 && (2.104)\\
\psi(z)\psi(z') &\sim \ln(z-z'), \text{ then} && (2.105)\\
:e^{i\psi}(z)::e^{-i\psi}(z'): &\sim e^{-\ln(z-z')}:e^{i\psi}(z)e^{-i\psi}(z'): \\
&\sim \frac{1}{z-z'}:e^{i\psi}(z)e^{-i\psi}(z'): && (2.106)\\
:e^{i\psi}(z)::e^{i\psi}(z'): &\sim e^{ln(z-z')}:e^{i\psi}(z)e^{i\psi}(z'): \\
&\sim (z-z'):e^{i\psi}(z)e^{i\psi}(z'):. && (2.107)
\end{aligned}$$

### 2.8.2 bc CFT

We can write b and c in terms of $\psi$: $b =: e^{i\psi} :$ and $c =: e^{i\psi} :$,. The stress tensor is given as:

$$T_\psi =: \partial\psi\partial\psi : +V\partial^2\psi \tag{2.108}$$

and from $b(z) = \sum b_m z^{-m-\lambda}$ and $c(z) = \sum c_m z^{-m-1+\lambda}$ and if the weight $\lambda$ is an integer, $b(z)c(z') \sim \frac{1}{z-z'}$, then the anticommutator between b and c is

$$\{b_m, c_n\} = \delta_{m+n,0}. \tag{2.109}$$

$b_n$ and $c_n$ are annihilation operators for $n > 0$. For zero mode, n = 0

$$\{b_0, c_0\} = 1$$

If let $|\Psi\rangle$ be a null state of b, ie. $b_0|\psi\rangle = 0$ and $c_0|\psi\rangle = |\chi\rangle$,then

$$\begin{aligned}
c_0|\chi\rangle &= c_0c_0|\psi\rangle = 0,\ \{b_m, b_n\} = \{c_m, c_n\} = 0 && (2.110)\\
b_0|\chi\rangle &= b_0c_0|\psi\rangle = \{b_0, c_0\}|\psi\rangle = |\psi\rangle, \text{ then} && (2.111)\\
\langle\psi||\psi\rangle &= \langle\chi|b_0b_0|\chi\rangle = 0 && (2.112)\\
\langle\chi||\chi\rangle &= \langle\psi|b_0b_0|\psi\rangle = 0, \text{ and} && (2.113)\\
\langle\psi||\chi\rangle &\neq 0. && (2.114)
\end{aligned}$$

ie. defining the inner product $\langle\chi|b_0|\chi\rangle \neq 0$ if $|\chi\rangle \neq 0$

## 2.9 Vertex operators

Vertex operators are one-to-one corresponding to their states, $I \simeq |0>$ for an operator $\partial X^\mu$ and momentum operator $p^\mu$

$$\begin{aligned}
\partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha^\mu_m z^{-m-1} \\
p^\mu|0\rangle &= 0, \\
p^\mu|0;k\rangle &= k^\mu|0;k\rangle, \text{ then} \\
\partial X^\mu(z)|0\rangle &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=1}^{\infty}\alpha^\mu_{-m}z^{-m-1}|0\rangle \\
&= -i\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_{-1} + \alpha^\mu_{-2}z + \alpha^\mu_{-3}z^2 + ...)|0\rangle \text{ letting z} \longrightarrow 0 \\
\partial X^\mu(0) &= -i\sqrt{\frac{\alpha'}{2}}\alpha^\mu_{-1}|0\rangle && (2.115)
\end{aligned}$$

From $|A\rangle \simeq A(z)$ or $|A\rangle = A(0)|0\rangle$, $\partial X^\mu$ operating at a bra is

$$\begin{aligned}\langle 0|\partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\langle 0|\sum_{m=1}^{\infty}\alpha^\mu_{-m}z^{-m-1}, \text{ letting z} \longrightarrow \infty \\ \langle 0|\partial X^\mu(\infty) &= -i\sqrt{\frac{\alpha'}{2}}\langle 0|\alpha^\mu_{-1}, \end{aligned} \tag{2.116}$$

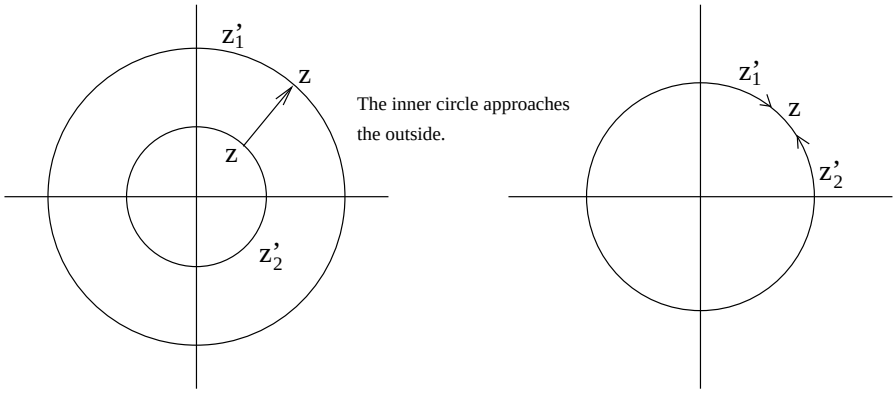
then the product can be consider as

$$\underbrace{\langle 0|}_{\text{time at } +\infty} \quad \text{time order stuffed} \quad \underbrace{|0\rangle}_{\text{time at } -\infty}$$

From the equation of $\partial X^\mu$, $\alpha^\mu_1$ is

$$\alpha^\mu_{-1} = \oint \frac{dz}{2\pi i}\ \underbrace{z^{-1}\partial X^\mu(z)}_{\text{conservedcurrent}}\ , \tag{2.117}$$

$$z \to e^{-iz}$$




$\alpha^\mu_{-2}|0\rangle$ also can be obtained from

$$\partial^2 X^\mu(z)|0\rangle = -i\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_{-2} + 2\alpha^\mu_{-3}z + ...)|0\rangle, \text{ for any } \alpha^\mu_{-\mathrm{m}} \tag{2.118}$$

$$\partial^m X^\mu(0) = -i\sqrt{\frac{\alpha'}{2}}((m-1)!\alpha^\mu_{-m} + O(z))|0\rangle, \text{letting z} \longrightarrow 0 \tag{2.119}$$

$$\alpha^\mu_{-m}(0) \simeq i\sqrt{\frac{2}{\alpha'}}\frac{1}{(m-1)!}\partial^m X^\mu(z). \tag{2.120}$$

$\alpha^\mu_{-m}\alpha^\nu_{-n}|0\rangle$ can be obtained from

$$:\partial^m X^\mu \partial^\nu X^\nu :|0\rangle = -\frac{\alpha'}{2}(m-1)!(n-1)!\alpha^\mu_{-m}\alpha^\nu_{-n} + \underbrace{...}_{\text{go to 0 in the infinite pass}}|0\rangle \tag{2.121}$$

Some examples of vertex operators are

$$\begin{aligned} e^{ik\cdot X}|0\rangle &= |0;k\rangle \\ \lim_{z\longrightarrow 0} :e^{ik\cdot X(z)}:|0\rangle &= e^{ik\cdot X}|0\rangle, \text{ then} \\ :\partial X^{m_1}\partial X^{m_2}...e^{ik\cdot X}: &\simeq \alpha_{-m_1}\alpha_{-m_2}....|0;k\rangle \end{aligned}$$

## 2.10 Primary fields

A primary field A, $A \Rightarrow |A\rangle, A(0)|0\rangle = |A\rangle$, with a state $|m_1 + m_2 + ..., k >$ can be writen as

$$\begin{aligned} |A\rangle &= \alpha - {m_1}^{\mu_1} \alpha^{\mu_2}_{-m_2} ...|0; k\rangle \text{where} && (2.122) \\ A(z) &= \frac{1}{(n_1-1)!}\frac{1}{(n_2-1)!} ... : \partial^{n_1} X^{\mu_1} \partial^{n_2} X^{\mu_2} ... e^{ik\cdot X} : . && (2.123) \end{aligned}$$

If we let $\lambda = 2$ for the bc theory:

$$\begin{aligned} b_{-m}|\psi\rangle &\longrightarrow \frac{1}{(m-2)!}\partial^{m-2} b,\ b_0|\psi\rangle = 0 && (2.124) \\ c_{-m}|\psi\rangle &\longrightarrow \frac{1}{(m+1)!}\partial^{m+1} c && (2.125) \\ T_{bc}(z) &= : (\partial b)c : -\lambda \partial : (bc) :, && (2.126) \end{aligned}$$

where

$$\begin{aligned} b(z) &= \sum b_m z^{-m-\lambda} \\ c(z) &= \sum c_m z^{-m+\lambda} \end{aligned}$$

The Virasoro operator in this case is

$$\begin{aligned} L^{bc}_m &= \oint \frac{dz}{2\pi i} z^{m+1} T_{bc} \\ &= \sum_n -(n+\lambda) b_n c_{m-n} + \lambda(m+1) b_n c_{n-m} + a\delta_{n,0} \\ &= \sum_n (m\lambda - n) : b_n c_{n-m} : + a\delta_{m,0} && (2.127) \\ T_{bc}(z)T_{bc}(z') &\sim \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2} T_{bc} + \frac{1}{z-z'}\partial T_{bc} && (2.128) \end{aligned}$$

where $c = 1 - 3(2\lambda - 1)^2$, and the commutation relations for L are given by:

$$[L^{bc}_m, L^{bc}_n] = (n-m)L^{bc}_{n+m} + \frac{c}{12}(m^3 - n)\delta_{n+m,0} \tag{2.129}$$

For m=1 and n=-1, the commutator is

$$\begin{aligned} [L^{bc}_1, L^{bc}_{-1}]|\Psi\rangle &= 2L^{bc}_0|\Psi\rangle \\ L_1 L_{-1}|\Psi\rangle - L_{-1}L_1|\Psi\rangle &= \lambda b_0 c_1 (1-\lambda) b_{-1} c_0 |\Psi\rangle \\ &= \lambda(1-\lambda) c_1 b_{-1}|\Psi\rangle \\ &= \lambda(1-\lambda)\{c_1, b_{-1}\}|\Psi\rangle \\ &= \lambda(1-\lambda)|\Psi\rangle && (2.130) \end{aligned}$$

## 2.11 Operator product expansion

Consider the commutator between $L_m$and A

$$\begin{aligned} [L_m, A] &= z^{m+1}\partial A + h(m+1)z^m A, \text{ where} && (2.131)\\ A(z) &= \sum A_n z^{-n-h} && (2.132) \end{aligned}$$

For $m = 0$,

$$\begin{aligned} [L_0, A] &= z\partial A + hA && (2.133)\\ [L_0, A_n] &= -nA_n && (2.134)\\ \text{as } z \to 0,\ [L_0, A(0)] &= hA(0) && (2.135)\\ L_0|A\rangle &= [L_0, A(0)]|0\rangle + A(0)L_0|0\rangle,\ L_0|0\rangle = 0, && \\ &= hA(0)|0\rangle = h|A\rangle && (2.136)\\ \text{as } z \to 0,\ [L_m, A] &= 0, \text{ for m } > 0, \text{ then} && (2.137)\\ L_m|A\rangle &= 0, m > 0, && (2.138) \end{aligned}$$

where $L_0$ is bounded from below.

## 2.12 Unitary CFTs

To correct the inner product with a negative norm, $\langle ...||...\rangle \geq 0$ , consider a Virasoro operator $L_m^+ = L_{-m}$ or

$$\langle \psi | L_m | \chi \rangle = \langle L_m^+ \psi || \chi \rangle \tag{2.139}$$

and also $X^\mu$ with $\langle 0; k||0; k'\rangle = 2\pi\delta(k - k')$. From $[\alpha_m^\mu, \alpha_n^\nu] = \eta^{\mu\nu}\delta_{m+n,0}$, for $\mu = \nu = 0$, we know that $\| \alpha_{-1}^\mu|0; k\rangle \|^2 < 0$. This can be corrected by letting $\Phi \to X'$, the action will be

$$S = \frac{1}{2\pi\alpha'}\int d^2z\partial\phi\bar{\partial}\phi \tag{2.140}$$

for $h \geq 0$, then

$$\begin{aligned} \langle A|[L_1, L_{-1}]|A\rangle &= \langle A|2L_0|A\rangle && (2.141)\\ \| L_{-1}|A\rangle \|^2 &= 2h \geq 0,\ L_0|vac\rangle = 0,\ c > 0, \text{ then} && (2.142)\\ \langle 0|[L_m, L{-}m] &= 2mL_0 + \frac{c}{12}(m^3 - m)|0\rangle && (2.143)\\ \| L_{-m}|0\rangle \|^2 &= \frac{c}{12}(m^3 - m) > 0 && (2.144) \end{aligned}$$

And we have gotten what we wanted; a positive norm for the highest weight state.