Chapter 4

The String Spectrum

Following *"String Theory"* by J. Polchinski, Vol.I.
Notes written by students (work still in progress).
For more information contact George Siopsis
gsiopsis@utk.edu

I. BRST quantization

We have a number of different types of quantization schemes which have been proposed over the decades, each with their own merits and drawbacks.

1) The canonical quantization method:

At the level of QED, the canonical quantization method is not too difficult, but the canonical quantization of more complicated theories, such as non-Abelian theories, is often prohibitively tedious.

2) The Gupta-Bleuler or covariant quantization method:

Contrary to canonical quantization, it maintains full Lorentz symmetry, which is a great advantage. The disadvantage of this approach is that ghosts or unphysical states of negative norm are allowed to propagete in the theory.

3) The path integral method:

This is perhaps the most elegant and powerful of all quantization programs. One advantage is that one can easily go back and forth between many of the other quantization programs to see the relationships between them. The disadvantage of the path integral approach is that functional integration is a mathematically delicate operation that may not even exist in Minkowski space.

4) The Becchi-Rouet-Stora-Tyupin(BRST) quantization method:

This is one of the most convenient and practical covariant approaches used for gauge theories. Like the covariant quantization program, negative norm states or ghosts are allowed to propagate and are eliminated by applying the BRST condition onto the state vectors. All the information is contained in a single operator, making this a very attractive formalism.

We will especially discuss the BRST quantization method in this chapter.

II. BRST quantization of the point particle

The action for a point particle of rest mass m is

$$S = \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\dot{X}^\mu \dot{X}_\mu - \eta m^2\right). \tag{4.1}$$

Using the equations of motion

$$P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{1}{\eta}\dot{X}_\mu, \quad P_\eta = \frac{\partial L}{\partial \dot{\eta}} = 0 \tag{4.2}$$

where η is not dynamic variable, the action becomes

$$\begin{aligned} S &= \frac{1}{2}\int d\tau \left(P_\mu \dot{X}^\mu - \eta m^2\right) \\ &= \int d\tau \left(P_\mu \dot{X}^\mu - \frac{1}{2}P_\mu \dot{X}^\mu - \frac{1}{2}\eta m^2\right) \\ &= \int d\tau \left[P_\mu \dot{X}^\mu - \frac{\eta}{2}(P_\mu P^\mu + m^2)\right] \\ &= \int d\tau (P_\mu \dot{X}^\mu - \eta\chi) \end{aligned} \tag{4.3}$$

where $\chi = \frac{1}{2}(P_\mu P^\mu + m^2)$.
Now, we consider the following transformation

$$X \longrightarrow X + \delta X, \quad P \longrightarrow P + \delta P.$$

From the quantization condition $[X^\mu, P^\nu] = i\eta^{\mu\nu}$, we obtain

$$\begin{aligned} \delta X^\mu &= i\epsilon[X^\mu, \chi] \\ &= \frac{i\epsilon}{2}[X^\mu, P_\nu P^\nu + m^2] \\ &= \frac{i\epsilon}{2}\left(P_\nu [X^\mu, P^\nu] + [X^\mu, P_\nu]P^\nu\right) \\ &= \frac{i\epsilon}{2}(iP_\nu \eta^{\mu\nu} + i\eta^\mu_\nu P^\nu) \\ &= -\epsilon P^\mu \end{aligned} \tag{4.4}$$

$$\begin{aligned} \delta P_\mu &= i\epsilon[P_\mu, \chi] \\ &= \frac{i}{2}\epsilon[P_\mu, P^\nu P_\nu] \\ &= 0 \end{aligned} \tag{4.5}$$

By giving dynamics to anticommuting variable ϵ like

$$\epsilon \longrightarrow -\epsilon c(\tau)$$

we can obtain the BRST transformation;

$$\begin{aligned} \delta_B X^\mu &= \epsilon c P^\mu \\ \delta_B P^\mu &= 0 \\ \delta_B b &= -\epsilon\chi \\ \delta_B c &= 0 \end{aligned} \tag{4.6}$$

where new coordinate c and new momentum b are anticommuting fields.
Under this BRST transformation (4.6), the action can be invariant by adding the additional term associated with new fields b and c to the old action. That is, the new action can be written as

$$S' = S + S_{b,c} \tag{4.7}$$

where $S_{b,c} = \int d\tau b\dot{c}$. Then, the variation of action is

$$\delta_B S' = \delta(S + S_{b,c}) = \delta \int d\tau (P_\mu \dot{X}^\mu + b\dot{c})$$

$$
\begin{aligned}
&= \int d\tau (P_\mu \delta \dot{X}^\mu + \delta b \dot{c} + b \delta \dot{c}) \\
&= \int d\tau \left[P_\mu \epsilon \frac{d}{d\tau}(cP^\mu) - \epsilon \chi \dot{c} \right] \\
&= \int d\tau \left[\frac{\epsilon}{2} \dot{c}(P_\mu P^\mu - m^2) + \epsilon c P_\mu \dot{P}^\mu \right] \\
&= \int d\tau \left[\epsilon \dot{c} \left(\frac{1}{2} P_\mu P^\mu - \frac{1}{2} m^2 \right) + \epsilon c \frac{d}{d\tau} \left(\frac{1}{2} P_\mu P^\mu - \frac{1}{2} m^2 \right) \right] \\
&= \int d\tau \epsilon \frac{d}{d\tau} \left[c \left(\frac{1}{2} P_\mu P^\mu - \frac{1}{2} m^2 \right) \right] \\
&= 0. \qquad (4.8)
\end{aligned}
$$

Define the BRST charge as $Q_B \equiv \frac{c}{2} \left(P_\mu P^\mu + m^2 \right) = c\chi$ where c is anticommuting. Using $\{c, c\} = 0$, we obtain

$$Q_B^2 = c^2 \chi^2 = 0. \qquad (4.9)$$

That is, the BRST charge is nilpotent. The physical state condition is given by

$$Q_B \left| \Psi \right\rangle = 0. \qquad (4.10)$$

The states that satisfy this condition are the physical states of the system and physical states must be BRST invariant. We now have a compact and elegant statement of the physical state condition.

III. BRST quantization of the string

The action for the string is

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu \qquad (4.11)$$

and the classical equation of motion is

$$\partial \bar{\partial} X^\mu (z, \bar{z}) = 0.$$

Now, we can consider the new action by adding an additional term $S_{b,c}$ and it becomes

$$
\begin{aligned}
S &= S_{old} + S_{bc} \\
&= \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu + \frac{1}{2\pi} \int d^2 z b \bar{\partial} c \qquad (4.12)
\end{aligned}
$$

When a transformation is given by

$$z \to z + \epsilon v(z) \quad \Longrightarrow \quad \delta X^\mu = -\epsilon v \partial X^\mu$$

then, the BRST transformation is

$$
\begin{aligned}
\delta_B X^\mu &= i\epsilon c \partial X^\mu \\
\delta_B b &= i\epsilon T + i\epsilon T^{bc} \\
\delta_B c &= i\epsilon c \partial c \qquad (4.13)
\end{aligned}
$$

where $T = \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu :$ and $T^{bc} =: (\partial b) c : - \lambda \partial (: bc :)$. Here, T^{bc} is added for invariance of S. Under the BRST transformation, the resulting change in S is

$$\delta_B S = \int d^2 \epsilon \bar{\partial} j_B \qquad (4.14)$$

where the BRST current is

$$\begin{aligned} j_B &= cT + \frac{1}{2} : cT^{bc} : + \frac{3}{2}\partial^2 c \\ &= cT + : bc\partial c : + \frac{3}{2}\partial^2 c. \end{aligned}$$

Here, $\lambda = 2$ was chosen because we will have a holomorphic bc theory with $\lambda = 2$ in string theory. The BRST operator is given by

$$\begin{aligned} Q_B &= \oint \frac{dz}{2\pi i} j_B \\ &= \oint \frac{dz}{2\pi i} \left(cT + \frac{1}{2} : cT^{bc} : + \frac{3}{2}\partial^2 c \right). \end{aligned} \tag{4.15}$$

Using mode expansion $c(z) = \sum_n c_n z^{-n+1}$ and $T(z) = \sum_n L_n z^{-n-2}$, the BRST charge in string can be written as

$$Q_B = \sum_{n=-\infty}^{\infty} : c_n \left(L_{-n} + \frac{1}{2} L^{bc}_{-n} - \delta_{n0} \right) : \tag{4.16}$$

where $L^{bc}_{-n} = \sum_m (m-n) c_m b_{-m-n}$. In the previous discussion(BRST quantization of the point-particle), we found that $Q_B^2 = 0$, but we now wish to ask whether this is true in case of the string. Using the Eq.(4.16),

$$\begin{aligned} Q_B^2 &= \frac{1}{2}\{Q_B, Q_B\} \\ &= \frac{1}{2} \sum_{n,m} c_n c_m \left([L^{Tot}_{-n}, L^{Tot}_{-m}] - (m-n) L^{Tot}_{-n-m} \right) \\ &= \frac{1}{2} \sum_{n,m} c_n c_m \left(\frac{D(m^3 - m)}{12} \delta_{m+n,0} + \frac{c^{bc}(m^3 - m)}{12} \delta_{m+n,0} \right) \end{aligned} \tag{4.17}$$

where $L^{Tot}_n = L_n + L^{bc}_n$ and $c^{bc} = 1 - 3(2\lambda - 1)^2 \xrightarrow{\lambda=2} -26$. Therefore, $Q_B^2 = 0$ for $D = 26$.
Like a point particle, we can find the BRST invariant physical state

$$Q_B \left| phys \right\rangle = 0. \tag{4.18}$$

When we define a new state such as

$$\left| phys \right\rangle_{new} = \left| phys \right\rangle + Q_B \left| A \right\rangle, \tag{4.19}$$

then we can find

$$\begin{aligned} {}_{new}\left\langle phys_1 \middle| phys_2 \right\rangle_{new} &= \Big(\langle phys_1| + \langle A_1| Q_B \Big) \cdot \Big(|phys_2\rangle + Q_B |A_2\rangle \Big) \\ &= \langle phys_1 | phys_2 \rangle + \langle A_1 | Q_B | phys_2 \rangle + \langle phys_1 | Q_B | A_2 \rangle + \langle A_1 | Q_B Q_B | A_2 \rangle \\ &= \langle phys_1 | phys_2 \rangle . \end{aligned} \tag{4.20}$$

Therefore, $|phys\rangle + Q_B |A\rangle$ is a same physical state as $|phys\rangle$:

$$Q_B \Big(|phys\rangle + Q_B |A\rangle \Big) = 0. \tag{4.21}$$

These two physical states, $|phys\rangle_{new}$ and $|phys\rangle$, have the same inner products with all physical states and are therefore physically equivalent. So we identify the true physical space with a set of equivalence classes. This is a natural construction for a nilpotent operator, and is known as the cohomology of Q_B.

equivalence classes

$$\begin{aligned} \mathcal{H}_{phys} &= \{\, |phys\rangle : Q_B\,|phys\rangle = 0\,\} \\ \mathcal{H}_{exact} &= \{\, |A\rangle : |A\rangle = Q_B\,|\varphi\rangle\,\} \\ \Rightarrow &\quad \mathcal{H}_{exact} \subseteq \mathcal{H}_{phys} \\ \Longrightarrow &\quad \mathcal{H}_{BRST} = \frac{\mathcal{H}_{phys}}{\mathcal{H}_{exact}} \end{aligned} \tag{4.22}$$

The complete Virasoro generators are given by

$$L_m^{Tot} = L_m + L_m^{bc} = \{Q_B, b_m\}. \tag{4.23}$$

Then, from $Q_B^2 = 0$ we obtain

$$[L_m^{Tot}, Q_B] = [\{Q_B, b_m\}, Q_B] = 0. \tag{4.24}$$

In a similar manner, we can see that the algebra of the complete Virasoro generators L_m closes using Jacobi identity:

$$\begin{aligned} [L_m^{Tot}, L_n^{Tot}] &= [L_m^{Tot}, \{Q_B, b_n\}] = \{Q_B, [L_m^{Tot}, b_n]\} \\ &= (m-n)\{Q_B, b_{m+n}\} = (m-n)L_{m+n}^{Tot} \end{aligned} \tag{4.25}$$

General Algebra

$$\begin{aligned} [L_i, L_j] &= i f_{ij}^k L_k \\ \{c^i, b_j\} &= \delta_j^i, \quad \{c^i, c^j\} = \{b_i, b_j\} = 0 \end{aligned} \tag{4.26}$$

The general form of the BRST operator is

$$\begin{aligned} Q_B &= c^i L_i - \frac{i}{2} f_{ij}^k c^i c^j b_k \\ &= c^i \left(L_i + \frac{1}{2} L_i^{bc} \right). \end{aligned} \tag{4.27}$$

where $L_i^{bc} = -i f_{ij}^k c^j b_k$. Using the Eqs.(4.26), (4.27) and Jacobi identity, we obtain

$$\begin{aligned} Q_B^2 &= \frac{1}{2}\{Q_B, Q_B\} \\ &= \frac{1}{2}\{c^i L_i, c^j L_j\} - i f_{lm}^k c^l c^m \{c^i, b_k\} L_i - \frac{1}{2} f_{ij}^k f_{kl}^m c^i c^j c^l b_m \\ &= i c^i c^j f_{ij}^k L_k - i f_{lm}^i c^l c^m L_i - \frac{1}{2} f_{ij}^k f_{kl}^m c^i c^j c^l b_m \\ &= 0. \end{aligned} \tag{4.28}$$

BRST cohomology of the string

Let us now focus on the open string. Physical state satisfies the additional condition

$$b_0\,|\varphi\rangle = 0. \tag{4.29}$$

Using $|\chi\rangle = c_0\,|\varphi\rangle$, we can see that

$$\langle\varphi|\varphi\rangle = \langle\varphi|\varphi\rangle = 0, \quad \langle\chi|\varphi\rangle \neq 0, \quad \langle\varphi|\, c_0\,|\varphi\rangle \neq 0. \tag{4.30}$$

The Eq.(4.29) implies

$$L_0^{Tot}\,|\varphi\rangle = \{Q_B, b_0\}\,|\varphi\rangle = 0 \tag{4.31}$$

since Q_B and b_0 both annihilate $|\varphi\rangle$. We denote by $\hat{\mathcal{H}}$ the space of states satisfying the conditions Eqs.(4.29) and (4.31);

$$\hat{\mathcal{H}} = \{\,|\varphi\rangle : \quad b_0\,|\varphi\rangle = 0, \quad L_0^{Tot}\,|\varphi\rangle = 0\,\}. \tag{4.32}$$

From the commutators $\{Q_B, b_0\} = L_0$ and $[Q_B, L_0^{Tot}] = 0$, it follows that Q_B takes $\hat{\mathcal{H}}$ into itself. If we consider a new state such as $|Z\rangle = Q_B\,|\varphi\rangle$, then

$$\begin{aligned} b_0\,|Z\rangle &= \{b_0, Q_B\}\,|\varphi\rangle = L_0^{Tot}\,|\varphi\rangle = 0 \\ L_0\,|Z\rangle &= [L_0^{Tot}, Q_B]\,|\varphi\rangle = 0. \end{aligned} \tag{4.33}$$

Therefore, we can see that

$$|Z\rangle = Q_B\,|\varphi\rangle \quad \in \quad \hat{\mathcal{H}}. \tag{4.34}$$

Example 1.
Let us now work out the first levels of the $D = 26$ flat spacetime string. If we write the state at the lowest level as $\left|\varphi; \vec{k}\right\rangle$, then we have

$$L_0\left|\varphi; \vec{k}\right\rangle = (\alpha' k^2 - 1)\left|\varphi; \vec{k}\right\rangle = 0 \tag{4.35}$$

using $L_0\left|\varphi; \vec{k}\right\rangle = 0$. Therefore, we obtain

$$m^2 = -k^2 = -\frac{1}{\alpha'}. \tag{4.36}$$

This state is invariant,

$$Q_B\left|\varphi; \vec{k}\right\rangle = 0, \tag{4.37}$$

because every term in Q_B contains either lowering operators or L_0^{Tot}. This also shows that there are no exact states at this level, so each invariant state corresponds to a cohomology class. The Eq.(4.36) means that these are just the states of the tachyon.

Example 2.
At the next level, there are $26 + 2$ states,

$$\left|Z, \vec{k}\right\rangle = \left(A_\mu(\vec{k})\alpha_{-1}^\mu + \beta(\vec{k})b_{-1} + \gamma(\vec{k})c_{-1}\right)\left|\varphi; \vec{k}\right\rangle, \quad -k^2 = 0, \tag{4.38}$$

depending on a 26-vector $A_\mu(\vec{k})$ and two constants, β and γ. The norm of this state is

$$\begin{aligned} \left\langle Z, \vec{k}\middle| Z, \vec{k'}\right\rangle &= \left\langle \varphi; \vec{k}\right| \left(A_\mu^*(\vec{k})\alpha_1^\mu + \beta^*(\vec{k})b_1 + \gamma^*(\vec{k})c_1\right)\left(A^\mu(\vec{k})\alpha_{\mu,-1} + \beta(\vec{k})b_{-1} + \gamma(\vec{k})c_{-1}\right)\left|\varphi; \vec{k'}\right\rangle \\ &= \left(A_\mu^* A^\mu + \beta^*\gamma + \beta\gamma^*\right)\left\langle \varphi; \vec{k}\middle|\varphi; \vec{k'}\right\rangle \end{aligned} \tag{4.39}$$

using $A_\mu A_\nu[\alpha_1^\mu, \alpha_{-1}^\nu] = A_\mu A_\nu \eta^{\mu\nu} = A_\mu A_\nu$, $\{b_m, c_n\} = \delta_{m,-n}$ and $\{b_m, b_n\} = \{c_m, c_n\} = 0$. From the eq.(4.39), we can see that there are 26 positive norm states and 2 negative norm states because $|A_\mu|^2 > 0$ for $\mu = 1, 2, \cdots 25$, $-|A_\mu|^2 < 0$ for $\mu = 0$, $|\beta|^2 > 0$ for $\beta = \gamma$ and $-|\beta|^2 < 0$ for $\beta = -\gamma$. The BRST condition $Q_B\,|Z\rangle = 0$ gives

$$\begin{aligned} Q_B\,|Z\rangle &= \left(c_{-1}k_\mu\alpha_1^\mu + c_1 k_\mu \alpha_{-1}^\mu\right)|Z\rangle \\ &= \left(c_{-1}k_\mu\alpha_1^\mu + c_1 k_\mu \alpha_{-1}^\mu\right)\left(A_\mu(\vec{k})\alpha_{-1}^\mu + \beta(\vec{k})b_{-1} + \gamma(\vec{k})c_{-1}\right)\left|\varphi; \vec{k}\right\rangle \\ &= \left(c_{-1}k_\mu A^\mu + \beta k_\mu \alpha_{-1}^\mu\right)\left|\varphi; \vec{k}\right\rangle \\ &= 0 \end{aligned} \tag{4.40}$$

An invariant state therefore satisfies

$$k_\mu A^\mu = 0, \quad \beta = 0 \tag{4.41}$$

which get rid of negative norm states:

$$k_0 A^0 = 0, \quad k_0 \neq 0 \quad \Longrightarrow \quad A^0 = 0$$

$$\beta = 0 \quad \Longrightarrow \quad \beta = \gamma = 0.$$

Therefore, there are 24 linearly independent states remaining, which have positive norm, being orthogonal to all physical states including themselves. In the Eq.(4.40), we already see that

$$Q_B A_\mu \alpha^\mu_{-1} \left|\varphi; \vec{k}\right\rangle = c_{-1} k_\mu A^\mu \left|\varphi; \vec{k}\right\rangle$$

$$\Longrightarrow \quad c_{-1} \left|\varphi; \vec{k}\right\rangle = \frac{1}{k \cdot A} Q_B A_\mu \alpha^\mu_{-1} \left|\varphi; \vec{k}\right\rangle = exact.$$

Thus the ghost state $c_{-1} \left|\varphi; \vec{k}\right\rangle$ is BRST-exact.

IV. The no-ghost theorem

Define the transverse Hilbert space $\mathcal{H}^\perp$ to consist of states in $\hat{\mathcal{H}}$ that have no longitudinal (X^0, X^1, b, or c) excitations:

$$\mathcal{H}^\perp = \{\, |\varphi\rangle : no\ longitudinal\ (X^0, X^1, b, c)\ mode \,\} \tag{4.42}$$

Light-cone gauge-fixing elimilates the longitudinal oscillators directly - the light-cone Hilbert space is isomorphic to $\mathcal{H}^\perp$. Generally the BRST cohomology is isomorphic to $\mathcal{H}^\perp$. That is, it has the same number of states at each mass level, and has a positive inner product. This is the no-ghost theorem.

Proof

Before we prove the no-ghost theorem, Let us find the new operators Q_1 and S. Define the light-cone oscillators

$$\alpha^\pm_m = \frac{1}{\sqrt{2}}(\alpha^0_m \pm \alpha^1_m), \quad m, n \in \mathcal{Z} - \{0\} \tag{4.43}$$

which satisfy

$$[\alpha^+_m, \alpha^-_m] = -m\delta_{m,-n}, \quad [\alpha^+_m, \alpha^+_m] = [\alpha^-_m, \alpha^-_m] = 0. \tag{4.44}$$

Using the quantum number

$$N^{lc} = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \alpha^+_{-m} \alpha^-_m,$$

decompose the BRST generator:

$$Q_B = Q_1 + Q_0 + Q_{-1}, \tag{4.45}$$

where Q_j changes N^{lc} by j units, $[N^{lc}, Q_j] = jQ_j$. From $Q_B^2 = 0$, we obtain $Q_1^2 = 0$, that is, Q_1 itself is nilpotent and has a cohomology. Explicitly

$$Q_1 = -\sqrt{2\alpha'} k^+ \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \alpha^-_{-m} c_m. \tag{4.46}$$

Define $S \equiv \{Q_1, R\}$ where

$$R = \frac{1}{\sqrt{2\alpha'} k^+} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \alpha^+_{-m} b_m, \tag{4.47}$$

then S can be written as

$$S = \sum_{n=1}^{\infty} \left(n b_{-n} c_n + n c_{-n} b_n - \alpha^+_{-n} \alpha^-_n - \alpha^-_{-n} \alpha^+_n \right). \tag{4.48}$$

We can also see that

$$\begin{aligned} [Q_1, S] &= [Q_1, \{Q_1, R\}] \\ &= Q_1\{Q_1, R\} - \{Q_1, R\}Q_1 = Q_1 R Q_1 - Q_1 R Q_1 \\ &= 0. \end{aligned} \tag{4.49}$$

Now, we prove the no-ghost theorem. The proof has two parts of which the first step is to find the cohomology of a BRST operator Q_1 and the second step is to show that the cohomology of the full Q_B is identical to that of Q_1.

The 1st step :

Theorem : If $|\varphi\rangle$ is Q_1-invariant with $S|\varphi\rangle = s|\varphi\rangle$, then $|\varphi\rangle$ is Q_1-exact for nonzero s.

Proof

$$\begin{aligned} |\varphi\rangle &= S|Z\rangle = \{Q_1, R\}|Z\rangle \\ &= Q_1 R|Z\rangle + R Q_1|Z\rangle = Q_1 R|Z\rangle. \end{aligned} \tag{4.50}$$

Therefore the Q_1 cohomology can be nonzero only at $s = 0$.
$S|\varphi\rangle = 0$ iff $|\varphi\rangle$ has no $\alpha^{\pm}_{-m}$, b_{-m}, c_{-m}, $m > 0 \quad \Rightarrow \quad |\varphi\rangle \in \mathcal{H}^{\perp}$

$$\Longrightarrow \quad \textit{Cohomology of } Q_1 \equiv \mathcal{H}^{\perp}$$

The 2nd step :

$|\varphi\rangle \in$ the kernel of S.

Define

$$\{Q_B, R\} = \{Q_1, R\} + \{Q_B - Q_1, R\} = S + U \tag{4.51}$$

where $U \equiv \{Q_B - Q_1, R\}$.
$U|\varphi\rangle$ has no $\alpha^{\pm}_{-m}$ modes. $\quad \Rightarrow \quad U|\varphi\rangle \notin$ the kernel of S.
If $|\varphi\rangle$ is annihilated by S, then

$$|Z\rangle = \left(1 - S^{-1}U + S^{-1}US^{-1}U - \cdots\right)|\varphi\rangle$$

is annihilated by $(S + U)$:
$(S + U)|Z\rangle = S|\varphi\rangle = 0$.

Cohomology of Q_B
$\cong$ the kernel of $(S + U) \sim$ the kernel of S
$\cong$ Cohomology of Q_1.