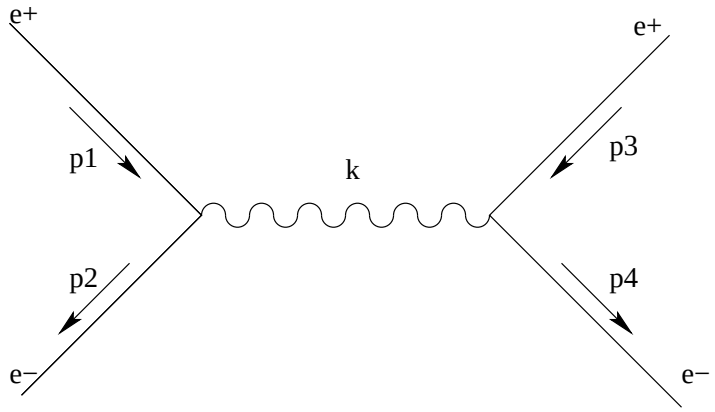
Chapter 6

Tree-level Amplitudes

Following *"String Theory"* by J. Polchinski, Vol.I.
Notes written by students (work still in progress).
For more information contact George Siopsis
gsiopsis@utk.edu

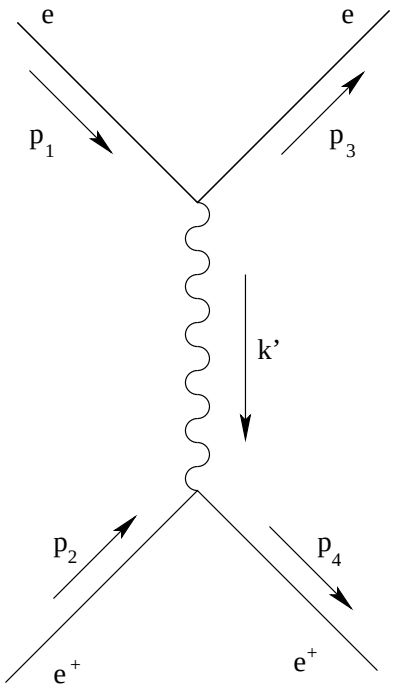
Feynman diagrams for Scattering:

$$Amplitude \sim \frac{1}{k^2} = \frac{1}{(p_1+p_2)^2} \tag{6.1}$$



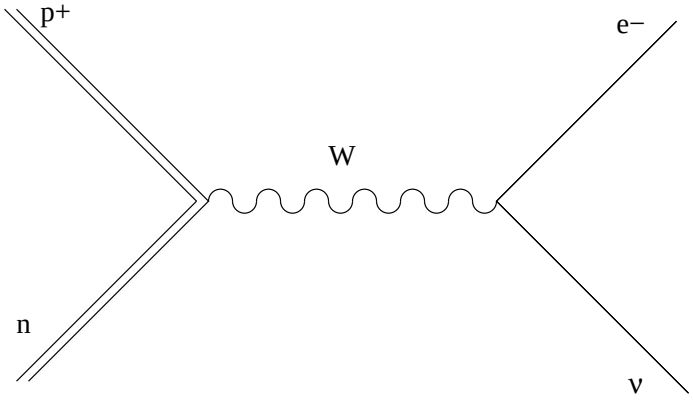
$$comes\ from \quad \Box\phi = 0;\ \Box^{-1} = \frac{1}{k^2} in\ fourier\ modes \tag{6.2}$$

$$Amplitude \sim \frac{1}{k'^2} = \frac{1}{(p_1-p_4)^2} \tag{6.3}$$



β decay

$$\sim \frac{1}{k'^2 - m_w^2} \qquad (\Box - m_w^2)\phi = 0 \tag{6.4}$$



(6.5)

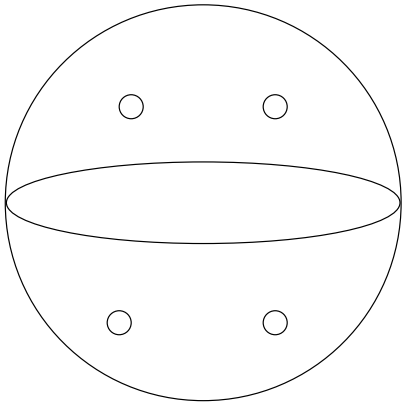
Strings

Two colliding closed strings

$$pole\ at\ k^2 = m_w^2 \tag{6.6}$$

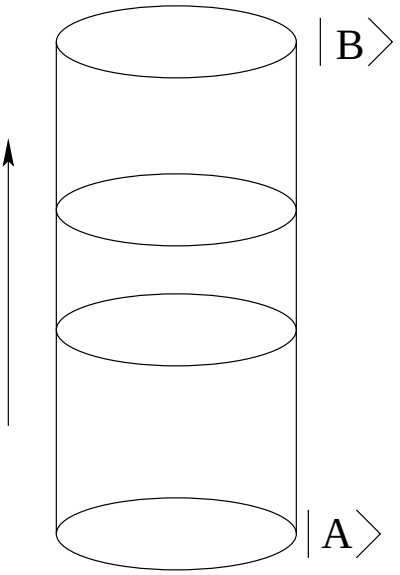
Each leg is semi infinite, where the strings join is up to you. → no need for renormalization

Now we'll map this to a sphere with 4 punctures

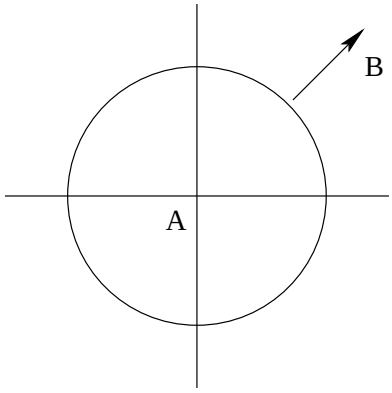


T-channel looks identical to s-channel: they are topologically equivalent.∴ only need 1 diagram

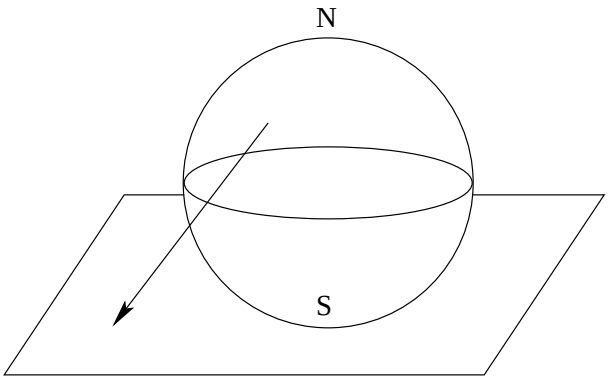
i.e. Map cylinder → complex plane



Amplitude is $< B|A >$
Scattering one string to itself
Now, map to complex plane

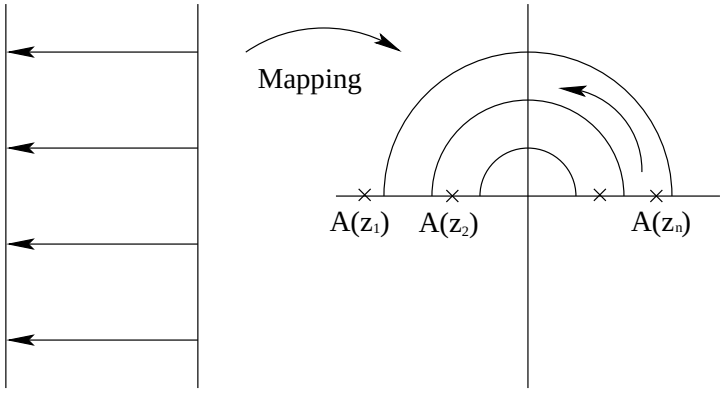


i.e. Cylinder $\rightarrow$ C $\cup \{\infty\} = S^2 \leftarrow$ sphere ; looks like stereographic projection



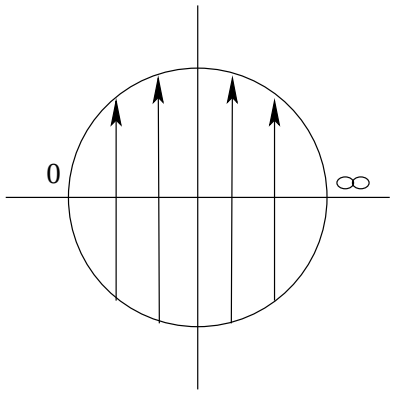
z $\to \frac{1}{z}$, Now zero is special

Open Strings Cut cylinder along axis $\to$ 2 strips



unit disk: $z \to \frac{z-i}{z+i}$

$z \in R \quad \left|\frac{z-i}{z+i}\right| = 1$



Scattering of N strings Scattering of N strings corresponds to N punctures on S^2

A reminder about primary (vertex) operators:

$$A(z) \sim : \partial^{m_1} X \partial^{m_2} X ... e^{ik\dot{x}} : \tag{6.7}$$

$$h = m_1 + m_2 + ... + \frac{\alpha' k^2}{4} = 1 \tag{6.8}$$

i.e. tachyon: A(z) = $: e^{ik\dot{x}} :$ $\quad k^2 = \frac{1}{\alpha'}$

i.e. vector: A(z)= $: \partial X e^{ik\dot{x}} :$ $\quad \frac{\alpha' k^2}{4} + 1 = 1 \quad \therefore \quad k^2 = 0$ (photon)

$$z \to z + \epsilon v(z) \qquad \delta z = \epsilon v(z) \tag{6.9}$$

$$v(z) : analytic \ \ at \ \ \infty \tag{6.10}$$

$$z \to \frac{1}{z} = z' \tag{6.11}$$

$$\delta z' = \frac{dz'}{dz} \delta z = -\frac{1}{z^2} \epsilon v(z) = -z'^2 \epsilon v(\frac{1}{z'}) \tag{6.12}$$

$$v(z) = a_o + a_1 z + ... if \ v(z) \ is \ analytic \tag{6.13}$$

$$v(\frac{1}{z'}) = a_o + a_1 \frac{1}{z'} + ... \tag{6.14}$$

$$\delta z' = -\epsilon (a_o z'^2 + a_1 z' + a_2 + \frac{a_3}{z'} + ...) \ \ to \ be \ analytic \tag{6.15}$$

$$v(z) = a_o + a_1 z + a_2 z^2 \tag{6.16}$$

a_o is generated L_{-1}, a_1 is generated L_o, a_2 is generated L_1

$v(z) = z^{m+1}$ generated by L_m
6 parameter family of generators (a_o, a_1, a_2)
all isomometries of S^2
includes rotations SO(3)

$$[L_m, A] = z^{m+1}\partial A + h(m+1)z^m A \tag{6.17}$$

For m=-1:

$$[L_{-1}, A] = \partial A \qquad L_{-1}\ generates\ translations:\ z \to z + a \tag{6.18}$$

$$A(z) \to e^{aL_{-1}} A(z) e^{-aL_{-1}} = A(z+a) \tag{6.19}$$

If a is small:

$$A(z) \to A(z) + b\,[L_o, A] = A(z) + bA + bz\partial A \tag{6.20}$$

For m=0:

$$[L_o, A] = z\partial A + A \qquad L_o\ is\ the\ Hamiltonian \tag{6.21}$$

$$A(z) \to e^{bL_o} A(z) e^{-bL_o} = A(e^b z) \tag{6.22}$$

If b is small:

$$A(z) \to A(z) + b\,[L_o, A] = A(z) + bA + bz\partial A \tag{6.23}$$

For m=1:

$$[L_1, A] = z^2\partial A + 2zA \tag{6.24}$$

$$A(z) \to e^{cL_1} A(z) e^{-cL_1} = A\left(\frac{z}{1-cz}\right) \tag{6.25}$$

$$Combination:\ z \to \frac{az+b}{cz+d} \quad ad - bc = 1 \tag{6.26}$$

$$SL(2,C) = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1\right] \tag{6.27}$$

Mapping a vector to a matrix:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix} \tag{6.28}$$

Open Strings

$$a, b, c, d \in R \qquad z = x + iy \tag{6.29}$$

$$x \to \frac{ax+b}{cx+d} \in R \quad upper\ half\ plane\ mapped\ into\ itself \tag{6.30}$$

$$A = \int dz_1 dz_2 ... dz_n \langle 0|A_1(z_1)A_2(z_2)...A_n(z_n)|0\rangle \quad \propto Vol(SL(2,R)) = \infty \tag{6.31}$$

We are free to fix three positions of three operators in the z-plane. By convention, and for convenience, we'll choose $(0, 1, \infty)$
Now

$$A = \int dz_4 ... dz_n \langle 0|A_1(\infty)A_2(1)...A_n(0)|0\rangle \tag{6.32}$$

Example 1: Scattering Amplitude for 3 Tachyons

$$A_i =: e^{ik_i \cdot X} : \qquad i = 1, ...3 \tag{6.33}$$

$$A \sim \langle 0|A_1(\infty)A_2(1)A_3(0)|0\rangle \tag{6.34}$$

$$X^\mu = x^\mu - i\alpha' p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \frac{1}{m}\alpha_m^\mu z^{-m} \tag{6.35}$$

Now taking the dot product with k, as is found in the exponential:

$$k \cdot X = k_\mu x^\mu - i\frac{\alpha'}{2}k_\mu p^\mu ln(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \frac{1}{m}k_\mu \alpha_m^\mu z^{-m} \tag{6.36}$$

$$\therefore A_3(0)|0\rangle = e^{ik_3 \cdot X}(0)|0\rangle = |0; k_3\rangle \tag{6.37}$$

$$\langle 0|A_1(\infty) = \langle 0; -k_1| \tag{6.38}$$

$$\therefore Amp \sim \langle 0; -k_1|A_2(1)|0; k_3\rangle \tag{6.39}$$

$$k_2 \cdot X(1) = k_{2\mu}x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \frac{1}{m}k_{2\mu}\alpha_m^\mu \qquad \ln(1) = 0 \tag{6.40}$$

Using the relation:

$$e^{A+B} = e^A e^B e^{[A,B]} \tag{6.41}$$

$$e^{k\cdot X(1)} = e^{k_\mu x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_\mu\alpha_m^\mu} = e^{k_\mu x^\mu} e^{i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_\mu\alpha_m^\mu} \tag{6.42}$$

$$\therefore A \sim \langle 0; -k_1|e^{k_\mu x^\mu} e^{i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_\mu\alpha_m^\mu}|0; k_3\rangle = \langle 0; -(k_1+k_2)|e^{i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_\mu\alpha_m^\mu}|0; k_3\rangle \tag{6.43}$$

$$A \sim \langle 0; -(k_1+k_2)||0; k_3\rangle = \delta^D(k_1+k_2+k_3) \tag{6.44}$$

Example 2: Scattering Amplitude for 2 Tachyons and 1 Vector

$$A_i(z) =: e^{ik_i \cdot X} : \quad i = 1, 3 \tag{6.45}$$

$$A_2(z) = \epsilon_\mu : \partial X^\mu e^{ik_2 \cdot X}(z) : \tag{6.46}$$

$$A \sim \langle 0|A_1(\infty)A_2(1)A_3(0)|0\rangle = \langle 0| : e^{ik_1 \cdot X}(\infty) :: \partial X^\mu e^{ik_2 \cdot X}(1) :: e^{ik_3 \cdot X}(0) : |0\rangle \tag{6.47}$$

$$X^\mu = x^\mu - i\alpha' p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \frac{1}{m}\alpha_m^\mu z^{-m} \tag{6.48}$$

$$\partial X^\mu = -i\alpha'\frac{p^\mu}{z} - i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \alpha_m^\mu z^{-m-1} \tag{6.49}$$

$$\epsilon_\mu \partial X^\mu = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \epsilon_\mu \alpha_m^\mu z^{-m-1} \qquad \epsilon_\mu p^\mu = 0 \tag{6.50}$$

We will be using the Glauber Formula:

$$e^{A+B} = e^a e^b e^{[A,B]} \tag{6.51}$$

$$\therefore e^{ik_3 \cdot X}(0)|0\rangle = e^{ik_3 \cdot x + \alpha' k_{3\mu} p^\mu \ln z - \sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_3 \cdot \alpha_m^\mu z^{-m}}(0)|0\rangle \tag{6.52}$$

$$= e^{ik_3 \cdot x + \alpha' k_{3\mu} p^\mu \ln z} e^{\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}k_3 \cdot \alpha_{-m}^\mu z^m}(0)|0\rangle \tag{6.53}$$

$$= e^{ik_3\cdot x+\alpha' k_{3\mu}p^\mu \ln z}(0)|0\rangle \tag{6.54}$$

Now the commutation in the Glauber formula yields:

$$[ik_{3\mu}x^\mu, \alpha' k_3\cdot p ln(z)] = i\alpha' k_{3\mu}^2 ln|z|\,[x^\mu, p^\mu] = -ln|z| \tag{6.55}$$

$$e^{ik_3\cdot X}(0)|0\rangle = e^{ik_3\cdot x}e^{+\alpha' k_{3\mu}p^\mu \ln z}e^{-ln(z)}(0)|0\rangle = e^{ik_3\cdot x}e^{-ln(z)}e^{+\alpha' k_{3\mu}p^\mu \ln z}(0)|0\rangle \tag{6.56}$$

$$= e^{ik_3\cdot x}z^{\alpha' k_{3\mu}p^\mu -1}(0)|0\rangle = z^{\alpha' k_{3\mu}p^\mu +1}e^{ik_3\cdot x}(0)|0\rangle\ [\ according\ to\ GSW\ 7.1.6\ ] \tag{6.57}$$

$$z^{\alpha' k_{3\mu}p^\mu +1}(0)|0:k_3\rangle = z^{\alpha' k_{3\mu}p^\mu +1}(0)|0:k_3\rangle \tag{6.58}$$

$$= |0:k_3\rangle \quad since\ \ k_{3\mu}^2 = -\frac{1}{\alpha'} \tag{6.59}$$

$$\langle 0|e^{ik_1\cdot X}(\infty) = \langle 0|e^{ik_{1\mu}x^\mu+\alpha' k_{1\mu}p^\mu \ln z-\sqrt{\frac{\alpha'}{2}}k_{1\mu}\sum_{m=-\infty}^{+\infty}\frac{1}{m}\alpha_m^\mu z^{-m}}(\infty) \tag{6.60}$$

$$= \langle 0|e^{ik_{1\mu}x^\mu+\alpha' k_{1\mu}p^\mu \ln z}e^{-\sqrt{\frac{\alpha'}{2}}k_{1\mu}\sum_{m=-\infty}^{+\infty}\frac{1}{m}\alpha_m^\mu z^{-m}}(\infty) \tag{6.61}$$

$$= \langle 0|e^{-\sqrt{\frac{\alpha'}{2}}k_{1\mu}\sum_{m>0}\frac{1}{m}\alpha_m^\mu z^{-m}}e^{ik_{1\mu}x^\mu+\alpha' k_{1\mu}p^\mu \ln z}(\infty) \tag{6.62}$$

$$= \langle 0|e^{ik_{1\mu}x^\mu+\alpha' k_{1\mu}p^\mu \ln z}(\infty) \tag{6.63}$$

$$= \langle 0|e^{ik_{1\mu}x^\mu}e^{\alpha' k_{1\mu}p^\mu \ln z}e^{-\alpha' k_{1\mu}^2 ln|z|}(\infty) \tag{6.64}$$

$$= \langle 0;-k_1|e^{\alpha' k_{1\mu}p^\mu \ln z}e^{-\alpha' k_{1\mu}^2 ln|z|}(\infty) \tag{6.65}$$

$$= \langle 0;-k_1|e^{\alpha' k_{1\mu}^2 ln|z|}e^{-\alpha' k_{1\mu}^2 ln|z|}(\infty) \tag{6.66}$$

$$= \langle 0;-k_1| \tag{6.67}$$

$$\therefore A\sim \epsilon_\mu\langle 0;-k_1|\partial X^\mu e^{ik_2\cdot x}(1)|0;k_3\rangle \tag{6.68}$$

$$= -i\epsilon_\mu\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\langle 0;-k_1|\alpha'^\mu_m z^{-m-1}e^{ik_2\cdot X}(1)|0,k_3\rangle \tag{6.69}$$

$$= -i\epsilon_\mu\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\langle 0;-k_1|\alpha'^\mu_m z^{-m-1}e^{ik_2\cdot X}(1)|0,k_3\rangle \tag{6.70}$$

$$e^{ik_2\cdot X}(1)|0,k_3\rangle = e^{ik_{2\mu}x^\mu+\alpha' k_{2\mu}p^\mu \ln z-\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_{2\mu}\alpha_m^\mu z^{-m}}(1)|0;k_3\rangle \tag{6.71}$$

$$= e^{ik_{2\mu}x^\mu+\alpha' k_{2\mu}p^\mu \ln z}e^{-\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{1}{m}k_{2\mu}\alpha_m^\mu z^{-m}}(1)|0;k_3\rangle \tag{6.72}$$

$$= e^{ik_{2\mu}x^\mu+\alpha' k_{2\mu}p^\mu \ln z}e^{+\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}k_{2\mu}\alpha_{-m}^\mu}(1)|0;k_3\rangle \tag{6.73}$$

$$= e^{+\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}k_{2\mu}\alpha_{-m}^\mu}e^{ik_{2\mu}}e^{\alpha' k_{2\mu}p^\mu \ln z}e^{-ln|z|}(1)|0;k_3\rangle \tag{6.74}$$

$$= e^{+\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}k_{2\mu}\alpha_{-m}^\mu}e^{ik_{2\mu}}(1)|0;k_3\rangle \tag{6.75}$$

$$= e^{+\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}k_{2\mu}\alpha_{-m}^\mu}|0;k_2+k_3\rangle \tag{6.76}$$

$$\therefore A\sim -i\epsilon_\mu\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\langle 0;-k_1|\alpha'^\mu_m e^{\sqrt{\frac{\alpha'}{2}}k_2\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}^\mu}|0,k_2+k_3\rangle \tag{6.77}$$

$$\sim i\epsilon_\mu\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\langle 0;-k_1|\alpha'^\mu_m\left(1+\sqrt{\frac{\alpha'}{2}}k_2\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}^\mu+...\right)|0,k_2+k_3\rangle \tag{6.78}$$

$$\sim i\epsilon_\mu k_2^\mu\frac{\alpha'}{2}\sum_{m>0}\sum_{n>0}\frac{1}{n}\langle 0;-k_1|\alpha'^\mu_m\alpha'^\mu_{-n}|0,k_2+k_3\rangle \tag{6.79}$$

$$\sim i\epsilon_\mu k_2^\mu\frac{\alpha'}{2}\sum_{m>0}\sum_{n>0}\frac{1}{n}\left(\langle 0;-k_1|\left[\alpha'^\mu_m,\alpha'^\mu_{-n}\right]|0,k_2+k_3\rangle+\langle 0;-k_1|\alpha'^\mu_{-n}\alpha'^\mu_m|0,k_2+k_3\rangle\right) \tag{6.80}$$

$$\sim \epsilon\cdot k_2\delta^D\left(k_1+k_2+k_3\right) \tag{6.81}$$

Example 3: Scattering Amplitude for 4 Tachyons In this case we have 4 individual tachyon amplitudes:

$$V_i =: e^{ik_i \cdot X} : \qquad i = 1, ...4 \tag{6.82}$$

In the previous examples we were dealing with 3 particles, and we were free to fix three positions in the z plane. Now, we have a fourth particle whose z co-ordinate cannot be similarly fixed, and when we work out the amplitude we must integrate over it.
The 3 fixed values of z are chosen to be at 0, 1, and ∞ as before. The free z co-ordinate can be placed in 3 regions: $z < 0$, $0 < z < 1$, and $z > 1$. We will examine the case for $0 < z < 1$, and generalize to the rest of the region later on.
So the Amplitude for our position of z is

$$\begin{aligned} A &= \int_0^1 dz \langle 0|V_1(\infty)V_2(1)V_3(z)V_4(0)|0\rangle & (6.83)\\ &= \int_0^1 dz \langle 0; -k_1| : e^{ik_2 \cdot X}(1) :: e^{ik_3 \cdot X}(z) : |0; k_4\rangle & (6.84) \end{aligned}$$

In the case of open strings,

$$X^\mu = x^\mu + i\alpha' p^\mu \ln z + i\sqrt{2\alpha'} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \alpha_m^\mu z^{-m} \tag{6.85}$$

So,

$$e^{ik_2 \cdot X}(1) = e^{ik_2 \cdot x - \sqrt{2\alpha'} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \alpha_m \cdot k_2} \tag{6.86}$$

$$e^{ik_3 \cdot X}(z) = e^{ik_3 \cdot x - \alpha' p \cdot k_3 \ln z - \sqrt{2\alpha'} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \alpha_m \cdot k_3 z^{-m}} \tag{6.87}$$

Now when $e^{ik_2 \cdot X}(1)$ acts left on the state $\langle 0; -k_1|$, elements of the sum over m for $m < 0$ annihilates it, and when $e^{ik_3 \cdot X}(z)$ acts right on the state $|0; k_4\rangle$, the sum over m for m¿0. Changing $m \to -m$ in the sum in $e^{ik_3 \cdot X}(z)$, and we have:

$$\begin{aligned} A &= \int_0^1 dz \langle 0; -k_1| e^{ik_2 \cdot x - \sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_m \cdot k_2} e^{ik_3 \cdot x - \alpha' p \cdot k_3 \ln z + \sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_{-m} \cdot k_3 z^m} |0; k_4\rangle & (6.88)\\ &= \int_0^1 dz \langle 0; -k_1 - k_2| e^{-\sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_m \cdot k_2} e^{\sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_{-m} \cdot k_3 z^m} e^{-\alpha' k_4 \cdot k_3 \ln z} |0; k_3 + k_4\rangle & (6.89) \end{aligned}$$

Where we have commuted the first two terms in $e^{ik_3 \cdot X}(z)$ through the sum, as x and p commute with the α_m's. Now we swap the order of the first two exponents using the commutation relation between the α_m's:

$$e^{-\sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_m \cdot k_2} e^{\sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_{-m} \cdot k_3 z^m} \tag{6.90}$$

$$= e^{\sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_{-m} \cdot k_3 z^m} e^{-\sqrt{2\alpha'} \sum_{m>0} \frac{1}{m} \alpha_m \cdot k_2} e^{-2\alpha' sum_{n,m} \frac{1}{m^2} z^m [\alpha_{-m}, \alpha m] k_2 \cdot k_3} \tag{6.91}$$

All terms in the first exponential apart from the '1' annihilate left on the state $\langle 0; -k_1 - k_2|$, and all terms in the second exponential except '1' annihilate the ket state. Using $[\alpha_{-m}, \alpha m] = m\delta_{n+m,0}$, we have

$$\begin{aligned} A &= \int_0^1 dz \langle 0; -k_1 - k_2| e^{-\alpha' k_4 \cdot k_3 \ln z} e^{-2\alpha' sum_m \frac{1}{m} z^m k_2 \cdot k_3} |0; k_3 + k_4\rangle & (6.92)\\ &= \int_0^1 dz z^{-2\alpha' k_4 \cdot k_3} (1 - z)^{2\alpha' k_3 \cdot k_4} \delta^D(k_1 + k_2 + k_3 + k_4) & (6.93) \end{aligned}$$

Where we have used

$$\langle 0; -k_1 - k_2|0; k_3 + k_4\rangle = \delta^D(k_1 + k_2 + k_3 + k_4) \tag{6.94}$$

and

$$\sum_m \frac{1}{m} z^m = \ln(1-z) \tag{6.95}$$

We now introduce Mandelstown variables:

$$s = -(k_1+k_2)^2 = -(k_3+k_4)^2 = -{k_3}^2 - {k_4}^2 - 2k_3 \cdot k_4 = -2k_3 \cdot k_4 - \frac{2}{\alpha'} \tag{6.96}$$

$$t = -(k_2+k_3)^2 = 2k_2 \cdot 2_3 - \frac{2}{\alpha'} \tag{6.97}$$

$$u = -(k_2+k_4)^2 \tag{6.98}$$

Using the mass relation $k_i^2 = -m^2 = \frac{2}{\alpha'}$. From these relations it can be seen that

$$s+t+u = 4m^2 = -\frac{4}{\alpha'} \tag{6.99}$$

Using these variables the Amplitude is written

$$A = \int_0^1 dz z^{-\alpha' s-2}(1-z)^{\alpha' t-2} \tag{6.100}$$

This integrates to a Beta function, which can be expressed in terms of Gamma functions:

$$A = B(-\alpha' s-1, -\alpha' t-1) = \frac{\Gamma(-\alpha' s-1)\Gamma(-\alpha' t-1)}{\Gamma(-\alpha' s-\alpha' t-2)} \tag{6.101}$$

This is the Veneziano Amplitude

Let us now examine the singularities in this function. Using the formula

$$x\Gamma(x) = \Gamma(x+1) \rightarrow \Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{1}{x} + analytic functions..... \tag{6.102}$$

we can see the poles are at $-\alpha' s - 1 = 0$ and $-\alpha' t - 1 = 0$ i.e. when $s = -1/\alpha'$ or

$$m^2 = -\frac{1}{\alpha'} \tag{6.103}$$

First Pole

$$For\ \alpha' s + 1 \cong 0: \qquad Amp \sim -\frac{1}{\alpha' s + 1} + ... \tag{6.104}$$

this gives a tachyon of mass $\frac{1}{\alpha'}$

$$\int d^D k \delta^D(-k + k_1 + k_2)\delta^D(-k + k_3 + k_4) = \delta^D(k_1 + k_2 + k_3 + k_4) \tag{6.105}$$

Second Pole

$$For\ \alpha' s \cong 0: \qquad \Gamma(-\alpha' s - 1) = -\frac{\Gamma(\alpha' s)}{\alpha' s + 1} = \frac{\Gamma(-\alpha' s + 1)}{(\alpha' s + 1)(\alpha' s)} = \frac{1}{\alpha' s} + ... \tag{6.106}$$

$$Amp \sim \frac{1}{\alpha' s}\frac{\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' t - 2)} = -\frac{\alpha' t - 2}{\alpha' s} + ... \tag{6.107}$$

$$= \frac{-t + 2/\alpha'}{s} + ... = \frac{u - t}{2s} + ... \tag{6.108}$$

Now, from before:

$$\sum_{polarizations} \sqrt{\frac{\alpha'}{2}}(k_1 - k_2) \cdot \epsilon \sqrt{\frac{\alpha'}{2}} \epsilon \cdot (k_3 - k_4) \tag{6.109}$$

$$\sum_{pol} \epsilon^\mu \epsilon^\nu = \eta^{\mu\nu} \tag{6.110}$$

$$= \frac{\alpha'}{2}(k_1 - k_2) \cdot (k_3 - k_4) = \frac{\alpha'}{2}(u - t) \tag{6.111}$$

$$\epsilon^\mu \not\propto k^\mu \quad because\ polarization\ is\ in\ transverse\ direction \tag{6.112}$$

if $\epsilon^\mu \propto k^\mu$:

$$(k_1 - k_2) \cdot \vec{k} = (k_1 - k_2) \cdot (k_1 + k_2) = k_1^2 - k_2^2 = 0 \quad gauge\ invariant \tag{6.113}$$

$$\epsilon^\mu \to \epsilon^\mu + \lambda k^\mu \tag{6.114}$$

$$A^\mu \to A^\mu + \partial^\mu \lambda \tag{6.115}$$

Same exact poles in t, only need one Feynman diagram in string theory unlike the point particles in QFT

Alternative derivation

$$Amp \sim \int_0^1 dz z^{-\alpha' s - 2}(1 - z)^{-\alpha' t - 2} = \int_0^1 dz z^{-\alpha' s - 2}\left(1 + (\alpha' t - 2)z + ...\right) \tag{6.116}$$

poles at $\alpha' s = -1, \alpha' s = 0, \alpha' s = 1$ from each term in expansion

$$as\ z \to 0: \qquad Amp \sim \int_0^1 dz z^{-\alpha' s - 2} = \left.\frac{z^{-\alpha' s - 1}}{-\alpha' s - 1}\right| + ... = \frac{1}{-\alpha' s - 1} + ... \tag{6.117}$$

This is the same as before, a tachyon with mass $\frac{1}{\alpha'}$

In limit $z \to 0$, intermediate particle propagates forever (classical particle)
In limit $z \to 1$, $k_2 \to k_3$
The infinite past is arbitrary, let k_2 go to infinite past

We did calculations for $0, z, 1$
For $t < 0$:

$$\int_{-\infty}^{0} = \int_{0}^{1} k_1 \leftrightarrow k_2 \; ; \; t \leftrightarrow u \tag{6.118}$$

For $t > 1$:

$$\int_{1}^{\infty} = \int_{0}^{1} k_2 \leftrightarrow k_3 \; ; \; s \leftrightarrow u \tag{6.119}$$

$$I(s,t) = \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' s - \alpha' t - 2)} \delta^D(k_1 + k_2 + k_3 + k_4) \tag{6.120}$$

full amplitude:

$$\sim \int_{0}^{1} + \int_{-\infty}^{0} + \int_{1}^{\infty} = I(s,t) + I(s,u) + I(t,u) \tag{6.121}$$

BRST Invariance

$$V(z) \; with \; weight \; h = 1 \tag{6.122}$$

$$V = \int dz V(z) \; with \; weight \; h = 0 \tag{6.123}$$

I claim that:

$$[Q_B, V] = 0 \tag{6.124}$$

From before:

$$Q_B = \sum : c_n \left(L_{-n} + \frac{1}{2} L^{bc}_{-n} - \delta_m \right) : \tag{6.125}$$

$$[Q_B, V] = \sum c_{-n} \left[L_n, V(z) \right] = \sum c_{-n} \left(z^{n+1} \partial V + (n+1) z^n V \right) = c(z)\partial V + (\partial c) V = \partial (cV) \tag{6.126}$$

$$\therefore \left[Q_B, \int V(z) \right] = \int \partial (cV) = 0 \tag{6.127}$$

$$A \sim \langle 0 | \int V_1 \int V_2 \int V_3 ... \, | 0 \rangle \tag{6.128}$$

Since $c(z)$ has weight $h = -1$, we'll stick a $c(z)$ on each operator to have weight 0

$$V_1(\infty)V_2(1)V_3(0) \longrightarrow c(\infty)V_1(\infty)c(1)V_2(1)c(0)V_3(0) \tag{6.129}$$

Now

$$\{Q_B, cV\} = \{Q_B, c\}V = c\,[Q_B, V] = c(\partial c)V - c\partial(cV) = 0 \tag{6.130}$$

$$|0\rangle\ c(\infty)c(1)c(0)\ |0\rangle \tag{6.131}$$

From before

$$b_0|\psi\rangle = 0 \qquad |\chi\rangle = c_0|\psi\rangle \tag{6.132}$$

$SL(2,\mathbb{R})$ invariance allowed us to fix position of operators

$$\left(L_1^{bc}, L_0^{bc}, L_{-1}^{bc}\right)|0\rangle = 0 \tag{6.133}$$

$$L_m^{bc} = \sum (2m-n) : b_n c_{m-n} : -\delta_{m,0} \tag{6.134}$$

6.1 A Short Course In Scattering

Here we briefly outline the theory of scattering particles. In any scattering process, we define a wavefunction representing the incidents particle(s) at the infinite past, $|in\rangle$, and a wavefunction representingg what we detect, in the infinite future $|out\rangle$. Then in between the scattering process in represented by the operator S:

$$|out\rangle = S\,|in\rangle \tag{6.135}$$

S is the limit of a time evolution operator as time goes to infinity:

$$S = \lim_{t\to\infty} \text{"}\ \ e^{iHt/\hbar}\ \text{"} \tag{6.136}$$

Conservation of probability demands that:

$$S^+ S = I \tag{6.137}$$

so that $\langle out|out\rangle = \langle in|in\rangle$.
Now, if we have an intial state $|i\rangle$ and a final state $|f\rangle$, the transition probability between the two states is given by

$$|\langle f_{+\infty}|i_{-\infty}\rangle|^2 \tag{6.138}$$

But we can write $\langle f_{+\infty}| = \langle f_{-\infty}|\,S$, so

$$|\langle f_{-\infty}|\,S\,|i_{-\infty}\rangle|^2 \tag{6.139}$$

i.e. we've mapped the states to the infinite past.
S can be written

$$S = I + iT \tag{6.140}$$

I is the identity, and represents scattering directly forward, in the direction of the initial particle beam, which can never be observed, so we can ignore that part. Then the transition probability becomes

$$|\langle f_{-\infty}|\,T\,|i_{-\infty}\rangle|^2 \tag{6.141}$$

Now equation (25) is

$$(I - iT^+)(I + iT) = I \tag{6.142}$$

$$iT - iT^+ + T^+T = 0 \tag{6.143}$$

$$i\langle f|\,T\,|i\rangle - i\langle f|\,T^+|i\rangle + \langle f|\,T^+T\,|i\rangle = 0 \tag{6.144}$$

And using $-i\langle f|\,T^+|i\rangle = +i\langle i|\,T\,|f\rangle^*$, we finally obtain

$$T_{fi} - T^*_{if} = \langle f|\,T^+T\,|i\rangle \tag{6.145}$$

Insering a complete set of states $I = \sum_n |n\rangle\langle n|$, where n are physical states, we get

$$T_{fi} - T^*_{if} = i\sum_n T^*_{nf}T_{ni} \tag{6.146}$$

Example - 2 tachyons scattering Let us consider the case of 2 tachyons incoming ($|i\rangle$), with momenta k_1, k_2, scattering somehow, and two tachyons leaving ($|f\rangle$) with momenta k_3, k_4. As we have seen, the scattering amplitude has a pole of the form

$$\frac{1}{s+m^2} \tag{6.147}$$

where $s = (k_1 + k_2)^2$. Physically, this means that there is an intermediate particle in the scattering (mass $m = -\frac{1}{\alpha'}$, so it is also a tachyon), with momentum $k = k_1 + k_2$:

Now, the formalism just discuused, this means that T_{fi} has a pole. To modulate the singularity, we add an infinitesimal imaginary component to the pole, making T complex:

$$T_{fi} \sim \frac{1}{s+m^2+i\epsilon} \tag{6.148}$$

and

$$T^*_{fi} \sim \frac{1}{s+m^2-i\epsilon} \tag{6.149}$$

So

$$T_{fi} - T^*_{if} = \frac{2i\epsilon}{(s+m^2)^2+\epsilon^2} \tag{6.150}$$

Now on the limit $\epsilon \to 0$,, this term is 0 everywhere except at the pole, where is is infinite, so the term is a delta function $\delta(s+m^2)$. To prove this, examine the integral:

$$\int_{-\infty}^{+\infty} \frac{2i\epsilon}{x^2+\epsilon^2}dx = 2\pi i \tag{6.151}$$

We want the integral to be one, which we can do by adding a factor:

$$T_{fi} - T^*_{if} = 2\pi i\delta(s+m^2) \tag{6.152}$$

Thus the amplitude consists of the amplitude for the k_1, k_2 vertex, the amplitude for the k_3, k_4 vertex, and an amplitude for an intermediate tachyon state. This must be formed by summing over all possible tachyon states that satisfy $s = -m^2$ - then the momentum is said to be "On Mass Shell".
This is another amazing thing about the theory we are constructing: there is no S matrix in conformal field theory, yet it comes out when one tries to write an amplitude for a physical state.

N-Point Amplitude(for closed strings)

$$Amp \sim \langle 0| : e^{ik_1\cdot X}(z_1) : \ ... \ : e^{ik_n\cdot X}(z_n) : |0\rangle = A\left(z_1, ..., z_n\right) \tag{6.153}$$

$$\partial_{z_1} Amp = \langle 0|\partial_{z_1} : e^{ik_1\cdot X}(z_1) : \ ... \ : e^{ik_n\cdot X}(z_n) : |0\rangle \tag{6.154}$$

The naive approach:

$$\partial_{z_1} Amp = \langle 0|ik_1\cdot\partial X(z) : e^{ik_1\cdot X}(z_1) : \ ... \ : e^{ik_n\cdot X}(z_n) : |0\rangle \tag{6.155}$$

Let's examine f^μ

$$f^\mu(z) = \langle 0|\partial X^\mu(z) : e^{ik_1\cdot X}(z_1) : \ ... \ : e^{ik_n\cdot X}(z_n) : |0\rangle \tag{6.156}$$

$$\partial X(z) : e^{ik_1\cdot X}(z_1) := -\frac{i\alpha'}{2}\eta^{\mu\nu}\partial ln(z-z_1)k_\nu : e^{ik\cdot X}(z_1) : + ... \tag{6.157}$$

$$= -i\alpha' k^\mu \frac{: e^{ik\cdot X}(z_1) :}{z-z'} + ... \tag{6.158}$$

$$f^\mu(z) = -\frac{i\alpha'}{2}k^\mu\frac{1}{z-z_1}(Amp) \longrightarrow -\frac{i\alpha'}{2}A(z_1, ..., z_n)\sum_{i=1}^{n} k_i\frac{1}{z-z'} + \ holomorphic\, terms \tag{6.159}$$

let $z' = \frac{1}{z}$

$$\frac{\partial X^\mu}{\partial z} = \frac{\partial z'}{\partial z}\frac{\partial X^\mu}{\partial z'} = -\frac{1}{z^2}\frac{\partial X^\mu}{\partial z'} \tag{6.160}$$

$$\therefore f^\mu(z) = -\frac{1}{z^2}\langle 0|\partial' X^\mu(z) : e^{ik_1\cdot X}(z_1) : \ ... \ : e^{ik_n\cdot X}(z_n) : |0\rangle \tag{6.161}$$

As $z \to \infty, z' \to 0$ Therefore, our function is analytic at $z' = 0$

$$\lim_{z\to\infty} f^\mu(z) = 0 \tag{6.162}$$

$$0 = \oint \frac{dz}{2\pi i} f^\mu(z) = \frac{i\alpha'}{2}A\sum_{i=1}^{n} k_i \tag{6.163}$$

$$\therefore \sum k_i^\mu = 0 \Longleftarrow \ conservation\ of\ momentum \tag{6.164}$$

To find Amplitude

$$ik_1\cdot\partial X^\mu(z) : e^{ik\cdot X}(z_1) := \frac{\alpha'}{2}\frac{1}{z-z_1'}k_1^2 : e^{ik_1\cdot X}(z_1) + \partial_1 : e^{ik_1\cdot X}(z_1) : \tag{6.165}$$

using

$$\partial_{z_1} lnA(z_1,...,z_n) = \frac{\alpha'}{2} A(z_1,...,z_n) \sum_{i\neq 1} \frac{k_1 \cdot k_i}{z - z_i} \tag{6.166}$$

$$ik_1 \cdot f(z) = \frac{\alpha'}{2} \frac{k^2}{z - z_1} A(z_1,...,z_n) + \frac{\alpha'}{2} A(z_1,...,z_n) \sum_{i\neq 1} \frac{k_1 \cdot k_i}{z - z_i} \tag{6.167}$$

$$lnA = \frac{\alpha'}{2} \sum_{i\neq 1} k_1 \cdot k_i \; ln\,|z_1 - z_i| + constant \tag{6.168}$$

$$lnA = \frac{\alpha'}{2} \sum_{i\neq j} k_j \cdot k_i \; ln\,(z_j - z_i) + constant \tag{6.169}$$

$$A(z_1,...z_n) = \prod_{i<j} (z_j - z_i)^{\frac{\alpha'}{2} k_i \cdot k_j} \;\; * \;\; anti - holomorphic\, term \tag{6.170}$$

For 4 point case:
$z_1 = \infty$
$z_2 = 1$
$z_3 = z$
$z_4 = 0$
For open strings:

$$A \propto \prod_{i<j} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \qquad z_i \in \mathbb{R} \tag{6.171}$$

For open strings, check $SL(2,\mathbb{R})$ invariance:

$$z \longrightarrow z' = \frac{az + b}{cz + d} \qquad ad - bc = 1 \tag{6.172}$$

$$cz'z + z'd = az + b \tag{6.173}$$

$$z = \frac{b - z'd}{cz' - a} \tag{6.174}$$

$$z_i - z_j = \frac{b - z_i' d}{cz_i' - a} - \frac{b - z_j' d}{cz_j' - a} = \frac{(z_i' - z_j')}{(cz_i' - a)(cz_j' - a)} \tag{6.175}$$

$$A \sim \prod (z_i' - z_j')^{2\alpha' k_i \cdot k_j} \prod (cz_i' - a)^{2\alpha' k_i \sum_{j\neq i} k_j} \tag{6.176}$$

$$\sum_{j\neq i} k_j = -k_i \quad ; k_i^2 = -\frac{1}{\alpha'} \tag{6.177}$$

$$\therefore A \sim \prod |z_i' - z_j'|^{2\alpha' k_i \cdot k_j} \prod (cz_i' - a)^2 \tag{6.178}$$

$$\int dz_1 dz_2 ... dz_n \prod (cz_i' - a)^{-2} \prod |z_i' - z_j'|^2 \tag{6.179}$$

For 4 point case(for open strings)

$$Amp \sim \int_{-\infty}^{\infty} dz z^{2\alpha' k_1 \cdot k_2} (1-z)^{2\alpha' k_2 \cdot k_3} \delta^D(k_1 + k_2 + k_3 + k_4) = \int_{-\infty}^{\infty} + \int_0^1 + \int_1^{\infty} = I(t,u) + I(s,t) + I(s,u) \tag{6.180}$$

$$I(s,t) = \int_0^1 dz z^{-\alpha' s - 2} (1-z)^{-\alpha' t - 2} \qquad m^2 = -\frac{1}{\alpha'} \tag{6.181}$$

For closed strings

$$Amp \sim \int d^2z \, |z|^{-\alpha' s/2-4} \, |1-z|^{-\alpha' t/2-4} \tag{6.182}$$

z,$\bar{z}$ independent variable, deform contour until z,$\bar{z} \in R$

Fix $\bar{z}$ and integrate over z along C

1)$\bar{z} < 0$

$$\int_c dz = 0 \tag{6.183}$$

2)$\bar{z} > 1$

$$\int_c dz = 0 \tag{6.184}$$

3)$0 < \bar{z} < 1$

$$\int_1^\infty dz z^{-\alpha' s/4-2}(1-z)^{-\alpha' t/4-2} \tag{6.185}$$

$$I\left(\frac{s}{4}, \frac{t}{4}\right) = \int_0^1 d\bar{z}\bar{z}^{-\alpha' s/4-2}(1-\bar{z})^{-\alpha' t/4-2} \tag{6.186}$$

$$Amp \sim sin\left(\frac{\alpha' t\pi}{4}\right) I\left(\frac{t}{4}, \frac{u}{4}\right) I\left(\frac{s}{4}, \frac{t}{4}\right) \tag{6.187}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin(\pi x)} \tag{6.188}$$

$$x = \frac{-\alpha' t}{4} - 1 \qquad s+t+u = -\frac{16}{\alpha'} \tag{6.189}$$

$$Amp \sim \frac{\Gamma\left(\frac{-\alpha' s}{4}-1\right)\Gamma\left(\frac{-\alpha' t}{4}-1\right)\Gamma\left(\frac{-\alpha' u}{4}-1\right)}{\Gamma\left(\frac{-\alpha' s}{4}\frac{-\alpha' t}{4}-2\right)\Gamma\left(\frac{-\alpha' t}{4}\frac{-\alpha' u}{4}-2\right)\Gamma\left(\frac{-\alpha' s}{4}\frac{-\alpha' u}{4}-2\right)} \tag{6.190}$$

6.1.1 Moduli

We now examine a general way to construct 4-point amplitudes. As befor, 3 points on the z plane are fixed, at 0, 1, and $infty$, and the fourth point can be placed anywhere in the z plane. The amplitude is given by:

$$A = \int dz_1 \langle 0|V(0)V(1)V(z_1)V(\infty)|0\rangle \tag{6.191}$$

Where the V's are the single particle amplitudes.
Now as we have seen before, the z planes with the 4 points on it is conformally equivalent to a sphere with 4 punctures on it. The sphere has six isometries, which can be parametrized by three complex numbers - these are the generators of $SL(2,\mathcal{C})$ invariance, and leads to the fixing of 3 of the points.

The mapping on to the sphere is arbitrary - any two surfaces are related by a conformal transformation. Consider two spheres with 4 punctures in each - 3 fixed and the 4th at points z_1 on one and z_1'. The two spheres are conformally equivalent.
z_1 is called the *modulus*. The set of all z_1's (except points 0,1,∞) on all conformally equivalent spheres defines *moduli space*:

$$z_1 \in_{not\ 0,1,\infty} \mathcal{C} \tag{6.192}$$

Now the approach to wotking out general 4 point amplitudes is as follows: We take the original string interaction diagram and 'amputate' the 4 legs and map to the z plane - this will be the plane with holes where the points were.
Now take the 4 semi - infinite legs and collapse them telescopically in the z' plane to form a set of concentric circles emenating from the origin to a certain radius. We obtain 4 copies of this, one for each leg.
We now take the 4 'patches' created by the legs and 'glue them over each hole in the z plane mapped from the 'body' of the interaction. In doing this we are changing the modulus z_1' in the z' plane, but not the modulus z_1 in the z plane, which must remain in its' hole.
Recall:

$$V_i(z_i) =: e^{ik_i \cdot X(z_i)} : \tag{6.193}$$

There are 3 fixed z_i's: $V_i \to c\tilde{c}V_i$.
In our new setup, there is no difference between these three and the fourth point: We want

$$V_i \to c\tilde{c}V_i \qquad \forall\ i \tag{6.194}$$

So how we define the Amplitude? If we let it be

$$A \sim \langle 0|c\tilde{c}V_1(0)c\tilde{c}V_2(1)c\tilde{c}V_3(z_1)c\tilde{c}V_4(\infty)|0\rangle \tag{6.195}$$

Then nothing kills the extra $c\tilde{c}$ from V_1, so A = 0. We must therefor introduce a $b\tilde{b}$ term. How do we do this?

Beltrami Differentials

Changing z_1 between conformally inequivalent surfaces is not a conformally invariant transformation:

$$z_1 \to z_1 + \delta z_1 \tag{6.196}$$

gives

$$z' \to z' + \delta z_1 v^{z'} \tag{6.197}$$

$$\bar{z}' \to \bar{z}' + \delta z_1 v^{\bar{z}'} \tag{6.198}$$

Thus $v^z = v^z(z', \bar{z})$ and so this is not a conformal transformation ($v^z = v^z(z')$)
Let us instead try doing the transformation, and then patching in the legs: We define the beltrami differentials

$$\mu = \partial_{\bar{z}} v^z \neq 0 \tag{6.199}$$

and

$$\bar{\mu} = \partial_z v^{\bar{z}} \tag{6.200}$$

For the transformation to be conformal, $\mu = \bar{\mu} = 0$. So $\mu, \bar{\mu}$ encode the nonconformality of the transformation. It turns out that we must insert the quantity

$$B_1 = \frac{1}{2\pi} \int d^2z' (b\mu + \tilde{b}\bar{\mu}) \tag{6.201}$$

into the amplitude. The integral is over the disc formed in the z' plane by the legs. Now

$$\bar{B}_1 = B_1(z_1 \to \bar{z_1}) \tag{6.202}$$

We actually need $B_1 * \bar{B}_1$, but let's concentrate on B_1: using $\partial_{\bar{z}} b = 0$ and $\partial_z \tilde{b} = 0$, we can write

$$B_1 = \frac{1}{2\pi i} \int d^2z' [\partial_{\bar{z}}(bv^z) + \partial_z(\tilde{b}v^{\bar{z}})] \tag{6.203}$$

$$= \frac{1}{2\pi i} \int d^2z' \nabla \cdot (b\vec{v}) \tag{6.204}$$

Which by the divergence theorem is equal to

$$\frac{1}{2\pi i} \oint_C (dz' bv^z - dz' \tilde{b}v^{\bar{z}}) \tag{6.205}$$

The transformation we're looking at is $0 \to z_1$. The conformal transformation is

$$z = z' + z_1 \tag{6.206}$$

Now,

$$v^z = \frac{\partial z'}{\partial z_1} \tag{6.207}$$

In our case

$$0 = \frac{\partial z'}{\partial z_1} + 1 \tag{6.208}$$

Thus

$$v^z = -1 \tag{6.209}$$

so

$$B_1 = -\frac{1}{2\pi i} \oint_C dz' b \tag{6.210}$$

Where here $\delta \bar{z}_1 = 0, v^{\bar{z}} = 0$
Writing b as a series expansion

$$b(z) = \sum b_n z^{-n-2} \tag{6.211}$$

Then B_1 is simply residue of the pole at z^{-3}:

$$B_1 = -b_{-1} \tag{6.212}$$

If you recall, we had to insert b_{-1} into the vacuum $|0\rangle \to b_{-1}|0\rangle$ to make it $SL(2, \mathcal{C})$ invariant.
So let's insert B_{-1} into $c\tilde{c}V_3(z_1)$. First we note the OPE

$$b(z')c(z_1) \sim \frac{1}{z' - z_1} \tag{6.213}$$

so

$$\frac{1}{2\pi i} \oint_C dz' bc\tilde{c}V_3(z_1) = \tilde{c}V_3(z_1) \tag{6.214}$$

$\therefore B_1$ has kiled the c. Similarly, $\bar{B}_1$ kills $\tilde{c}$.

Now let's check BRST invariance:

$$\{Q_B, B_1\} = \frac{1}{2\pi i}\int d^2z'\{Q_B, b\}\mu + \quad barred\ piece \tag{6.215}$$

$$= \frac{1}{2\pi i}\int d^2z' T\mu + \quad piece\ with\ \bar{T} \tag{6.216}$$

$$= \frac{1}{2\pi i}\oint_C dz' T v^z + ... \tag{6.217}$$

Now the OPE

$$T(z')c\tilde{c}V_3(z_1) \sim \frac{h}{(z'-z_1)^2} + \frac{\partial(c\tilde{c}V_3)}{z'-z} + ... \tag{6.218}$$

so

$$\oint_C dz_1 T(z')c\tilde{c}V_3(z_1) \sim \frac{\partial(c\tilde{c}V_3)}{z'-z} \tag{6.219}$$

Then

$$\int dz_1 \frac{\partial(c\tilde{c}V_3)}{z'-z} = 0 \tag{6.220}$$

thus this scheme is BRST invariance, and we may use the following form for n-point Amplitude:

$$A \sim \int d(moduli)\langle \tilde{V}\tilde{V}\tilde{V}.....\tilde{V}\rangle \tag{6.221}$$

where

$$\tilde{V} = c\tilde{c}V \tag{6.222}$$

so now all $\tilde{V}$'s are on equal footing.